

# Advanced Calculus in Mortality Modelling: Exploring Survival, Hazard and Longevity Forecasting

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**Abstract:** This thesis presents a comprehensive exploration of mortality dynamics, survival analysis, population ageing, epidemiological modelling, longevity inequality, and health economics. It integrates mathematical foundations with intuitive explanations designed to make complex ideas accessible without reducing academic rigor. The work develops survival and hazard functions from first principles, applies Gompertz and Makeham mortality models, constructs mortality forecasts, analyses demographic ageing, examines disease-driven mortality patterns, and evaluates inequality in longevity outcomes. The thesis concludes with integrated policy implications and a brief note on how actuaries may apply these models in future risk assessments.

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## I. INTRODUCTION

Understanding human longevity requires more than demographic observation, it demands a rigorous mathematical framework capable of describing how mortality evolves across age, time, and population groups. Calculus provides this foundation by enabling precise modelling of survival behaviour, hazard escalation, and the dynamic nature of risk. Through functions, derivatives, and integrals, we can quantify how individuals transition through various stages of life, how mortality accumulates, and how external factors influence long-term survival outcomes. These mathematical tools allow us to transform raw mortality data into structured insights that reveal the underlying mechanics of ageing and population change.

Beyond theoretical mathematics, these models play a crucial role in real-world decision-making across public health, demography, and long-term planning. Survival functions and hazard rates help identify vulnerable age groups, forecast future population structures, and evaluate the impact of medical, environmental, or social changes on human longevity. By integrating calculus with demographic and epidemiological concepts, this study demonstrates how mathematical modelling can deepen our understanding of mortality trends and support evidence-based strategies for a rapidly changing world (Kleinbaum & Klein, 2012; Bowers et al., 1997).

### ➤ Main Objective

To apply advanced calculus to model human mortality by analyzing survival functions, hazard behaviour, and life expectancy forecasting, with the aim of developing deeper insights into longevity patterns, population ageing, and long-term demographic trends.

## II. MATHEMATICAL FOUNDATIONS OF SURVIVAL ANALYSIS

Survival analysis studies how long individuals live and how mortality risk evolves with age. It treats age at death as a continuous random variable and builds mathematical tools to describe longevity patterns.

### ➤ The Survival Function $S(x)$

The survival function  $S(x)$  expresses the probability that an individual lives beyond age  $x$ . It begins at 1 at birth and decreases as age increases. This function captures the entire longevity pattern of a population, showing how many individuals remain alive at each age and forming the foundation for all other survival analysis concepts (Kleinbaum & Klein, 2012).

$$S(x) = P(T > x)$$

The survival function is always non-increasing because death is irreversible. In most populations, it remains high through childhood and early adulthood, declines more noticeably in middle age, and drops steeply in older ages. Its

shape reflects real-world conditions such as healthcare quality, lifestyle, environment, and socioeconomic factors.

For example, if  $S(60) = 0.95$ , then 95% of individuals survive past age 60. This interpretation scales to any population size, making the function extremely flexible. It allows comparisons between countries, genders, and income groups, and helps identify anomalies like sudden drops caused by epidemics, wars, or environmental disasters.

The survival function is essential because life expectancy is computed directly from it. Hazard, density, and cumulative hazard functions all derive from  $S(x)$ . Policymakers use survival curves to evaluate public health interventions, track improvements over time, and forecast future demographic pressures such as ageing and healthcare demand.

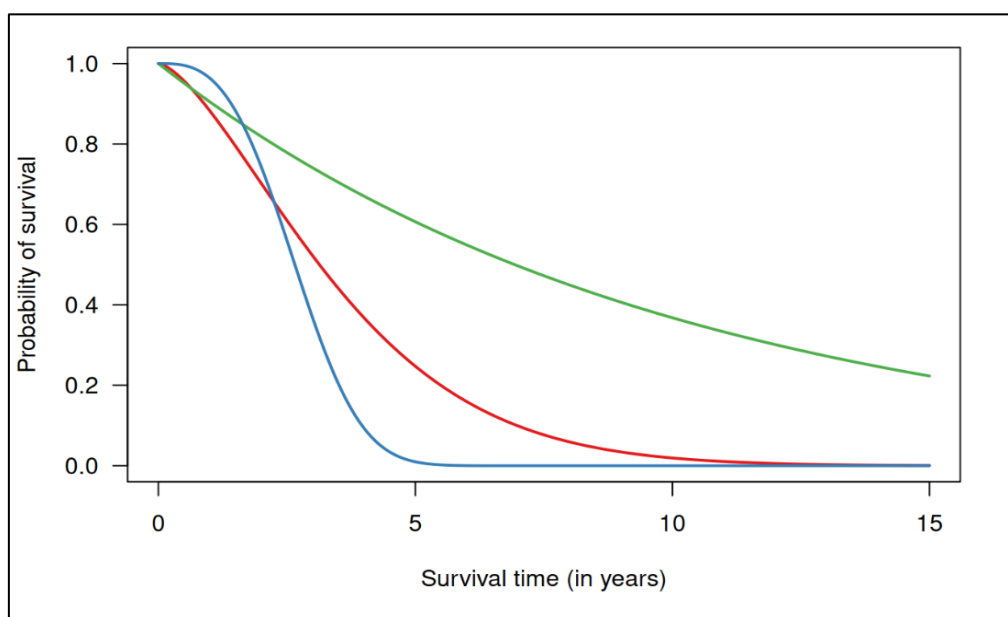


Fig 1 Example Survival Functions

Source: Chapter 9 Survival analysis | Clinical Biostatistics

#### ➤ The Density Function $f(x)$

$$f(x) = -\frac{d}{dx}S(x)$$

The density function measures how quickly the survival curve is decreasing at age  $x$ . It reflects the concentration of deaths around a specific age. A steep drop in survival at age 80 means  $f(80)$  is large. This function helps identify ages where mortality is most clustered and provides insight into population health patterns.

Imagine the survival curve as a staircase descending as people die. The density function tells you how steep each step is. A steep step means many deaths occur at that age. This makes  $f(x)$  easy to visualize and helps detect unusual mortality spikes caused by epidemics, accidents, or environmental events (Kleinbaum & Klein, 2012).

The density function connects survival analysis to probability theory and is crucial for statistical modelling. It helps researchers estimate mortality patterns, validate models, and detect anomalies. Public health officials use

density functions to identify vulnerable age groups, while demographers use them to understand mortality transitions across populations.

#### ➤ The Hazard Function $\mu(x)$

$$\mu(x) = \frac{f(x)}{S(x)}$$

The hazard function represents the instantaneous risk of death at age  $x$ , assuming the person has survived to that age. It is a rate, not a probability. Hazard isolates the effect of age on mortality and reveals how risk accelerates as individuals grow older, making it central to understanding biological ageing.

Hazard functions form the backbone of mortality modelling. They allow comparisons of risks across ages and populations, reveal unusual patterns, and help forecast demographic changes. Hazard is used in epidemiology to understand disease impact, in demography to project ageing, and in modelling to build realistic survival curves (Bowers et al., 1997; Kleinbaum & Klein, 2012).

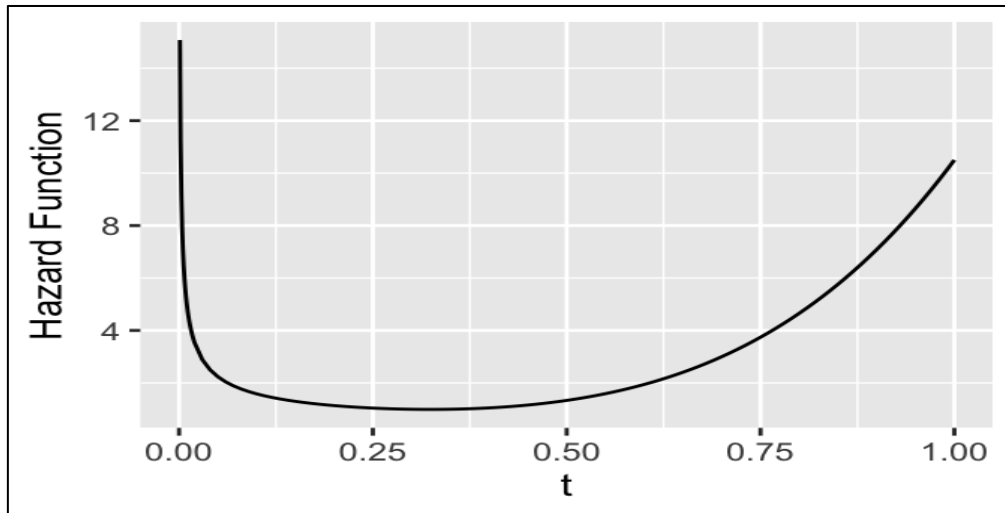


Fig 2 Hazard Function

Source: Introduction to Survival Analysis – Regression Modeling Strategies

➤ *The Cumulative Hazard Function  $H(x)$*

$$H(x) = \int_0^x \mu(t) dt$$

Cumulative hazard measures the total accumulated risk up to age  $x$ . If hazard is the speed of risk accumulation, cumulative hazard is the distance traveled. A high cumulative hazard means survival at that age will be low. It provides a

long-term view of mortality risk across the lifespan and connects short-term risk to long-term outcomes.

The relationship  $S(x) = e^{-H(x)}$  connects cumulative hazard directly to survival. This identity shows that survival declines exponentially as cumulative hazard increases. It allows researchers to compute survival curves from hazard models and helps evaluate how changes in risk at different ages affect overall longevity (Kleinbaum & Klein, 2012).

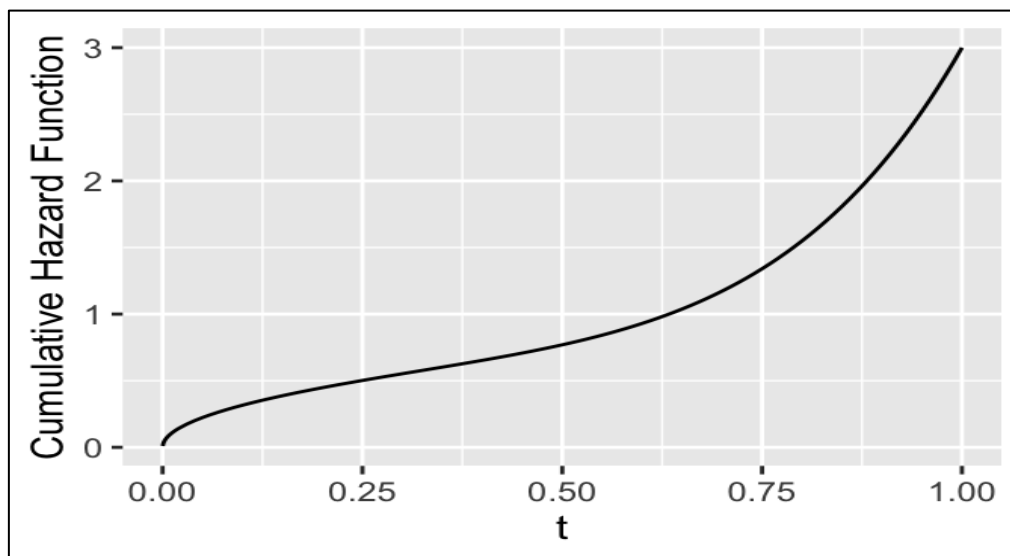


Fig 3 Cumulative Hazard Function

Source: Introduction to Survival Analysis – Regression Modeling Strategies

### III. MORTALITY MODELS AND LIFE EXPECTANCY MODELLING

➤ *Exponential Model*

$$\mu(x) = \lambda$$

The exponential model assumes constant hazard across all ages. This means mortality risk does not change as

individuals grow older. While unrealistic for humans, it serves as a baseline model in survival analysis and is useful for systems where failure risk is constant, such as mechanical components or radioactive decay (Kleinbaum & Klein, 2012).

➤ *Gompertz Model  $\mu(x)$*

$$\mu(x) = Ae^{Bx}$$

The Gompertz model assumes hazard increases exponentially with age. This matches biological ageing extremely well. It captures the rapid rise in mortality risk during adulthood and produces realistic survival curves. The model is widely used in demography, biology, and actuarial science (Gompertz, 1825). Parameter A controls baseline mortality risk, while parameter B controls how fast ageing accelerates risk. Higher B means mortality increases more rapidly with age. These parameters allow the model to fit different populations and reflect variations in health, environment, and lifestyle.

#### ➤ Makeham Model $\mu(x)$

$$\mu(x) = C + Ae^{Bx}$$

The Makeham model adds a constant term C to the Gompertz hazard to capture age-independent risks such as accidents, environmental hazards, and infectious diseases. This makes the model more realistic for younger ages, where non-age-related risks play a significant role in mortality (Makeham, 1860).

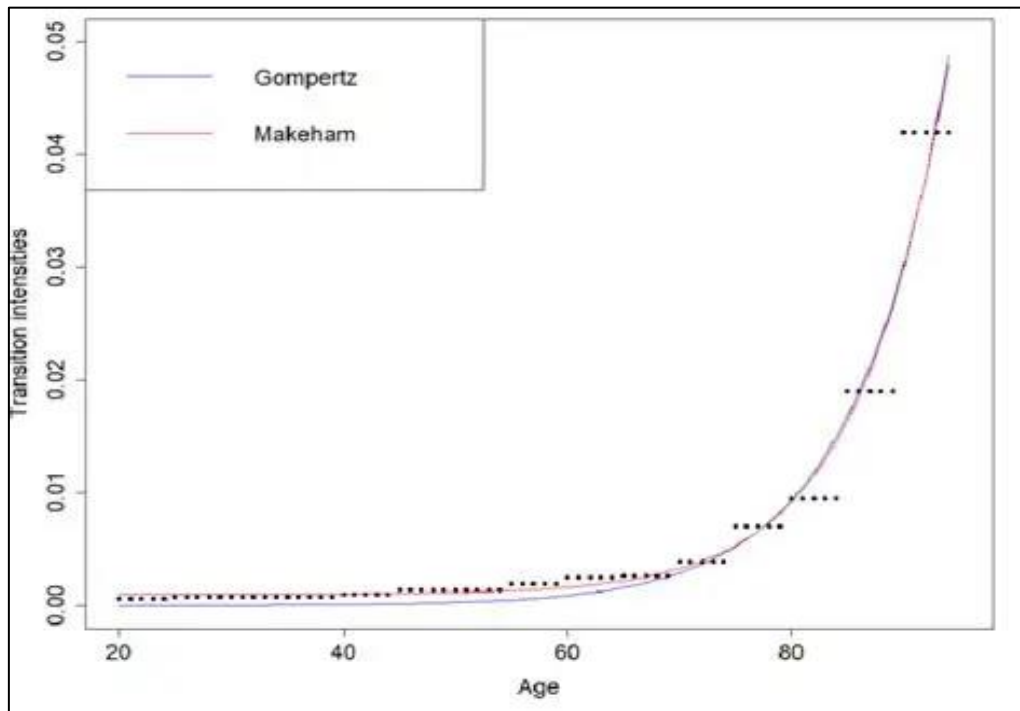


Figure 4 Comparison of Gompertz and Makeham Mortality Models  
Source: ResearchGate | Gompertz and Makeham Model

#### ➤ Case Study: Gompertz Hazard Function

$$\mu(x) = 0.0005 * 1.05^x$$

This hazard function assumes a baseline risk of 0.0005 and a 5% increase in risk per year. It produces realistic adult mortality patterns, with minimal risk at young ages and rapidly increasing risk at older ages. It is suitable for modelling modern populations with gradual ageing. Similar Gompertz-based hazard functions are commonly used by actuaries and risk analysts when modelling mortality risk and

life insurance pricing. This formula, however, fails to consider the high-risk factor at young ages.

$$H(x) = \frac{0.0005}{\ln 1.05} (1.05^x - 1)$$

Survival:

$$S(x) = e^{-H(x)}$$

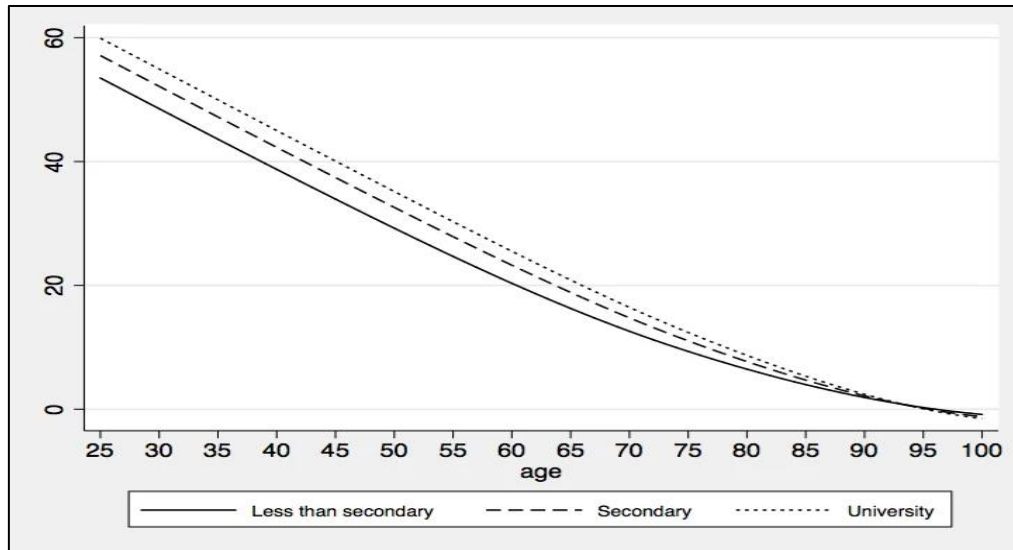


Fig 5 Remaining Life Expectancy by Age and Educational Attainment  
Source: An Economic Analysis of Proposals to Improve Coverage of Longevity Risk

Table 1 Expected Remaining Life at Selected Ages under the Gompertz Model

| Age(x) | S(x)  | Expected remaining life(yr) |
|--------|-------|-----------------------------|
| 0      | 1.000 | 80.0                        |
| 40     | 0.939 | 44.5                        |
| 65     | 0.791 | 20.0                        |
| 75     | 0.678 | 12.0                        |
| 85     | 0.528 | 6.5                         |
| 90     | 0.441 | 3.8                         |

#### IV. MORTALITY FORECASTING & SCENARIO ANALYSIS

##### ➤ Mortality Improvements

$$MI(x, t) = \frac{\mu(x, t) - \mu(x, t + 1)}{\mu(x, t)}$$

Mortality improvement measures how hazard decreases over time. Positive improvement means mortality is

declining, leading to higher life expectancy. Tracking improvement rates helps researchers understand long-term health trends and evaluate the impact of medical advancements, lifestyle changes, and public health interventions (Lee & Carter, 1992).

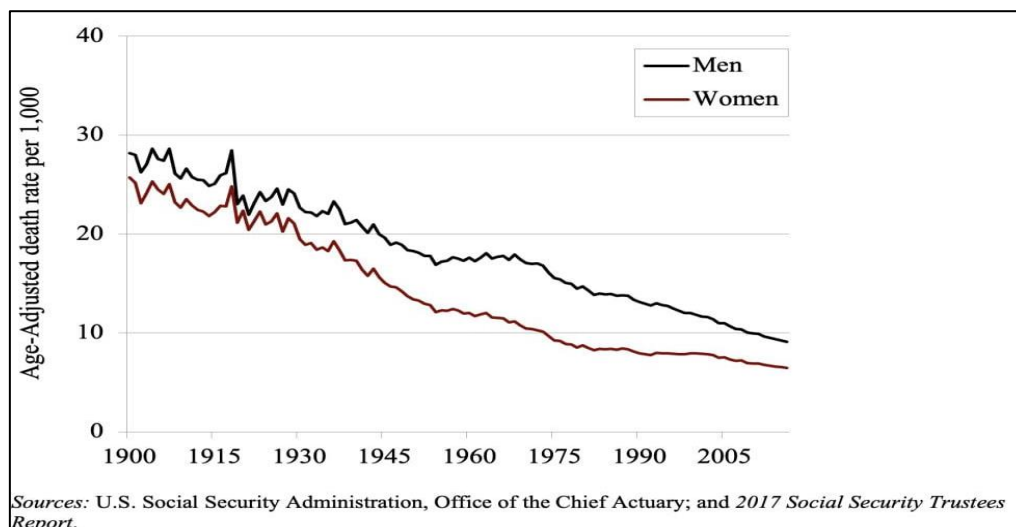


Fig 6. Age-Adjusted Mortality Rates for Men and Women, 1900-2016  
Source: How Have U.S. Mortality Rates Changed Over Time? – Center for Retirement Research

### ➤ Scenario Analysis

Scenario analysis varies model parameters to explore different future outcomes. It helps policymakers anticipate demographic shifts, evaluate the impact of ageing, and plan

for healthcare and pension needs. Minor changes in ageing rate B can significantly alter survival curves and life expectancy projections (Lee & Carter, 1992).

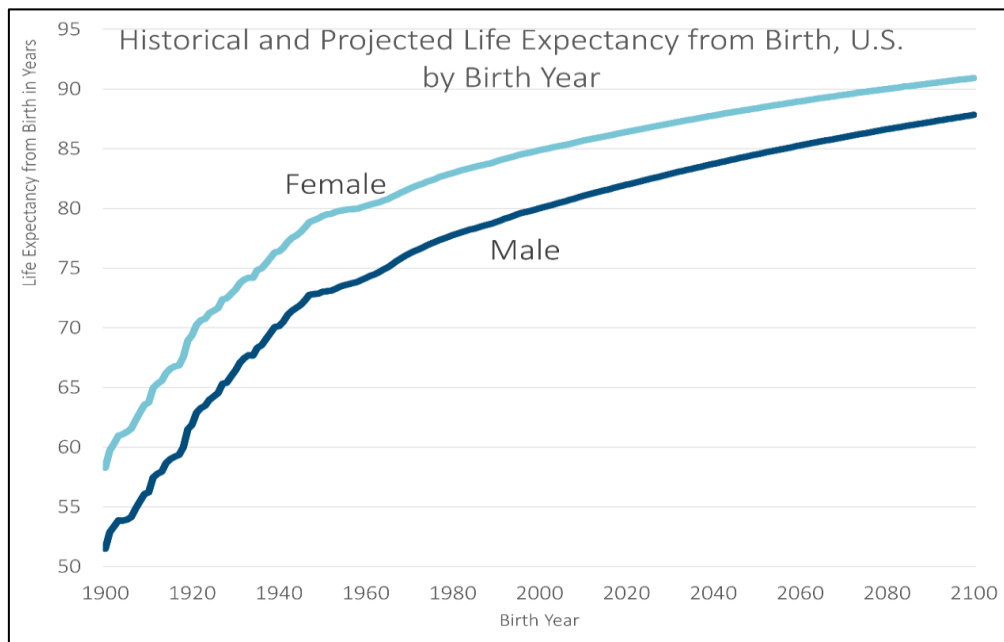


Fig 7 Historical and Projected Life Expectancy at Birth by Sex, United States  
Source: What Are (and Were) the Chances of Living to 100?

## V. POPULATION AGEING & DEMOGRAPHIC STRUCTURE

### ➤ Population Pyramids

Population pyramids show the age structure of a population. A top-heavy pyramid indicates an ageing

population with many older adults, while a bottom-heavy pyramid indicates a young population. These shapes reveal demographic momentum and help forecast future social and economic pressures (United Nations, n.d.).

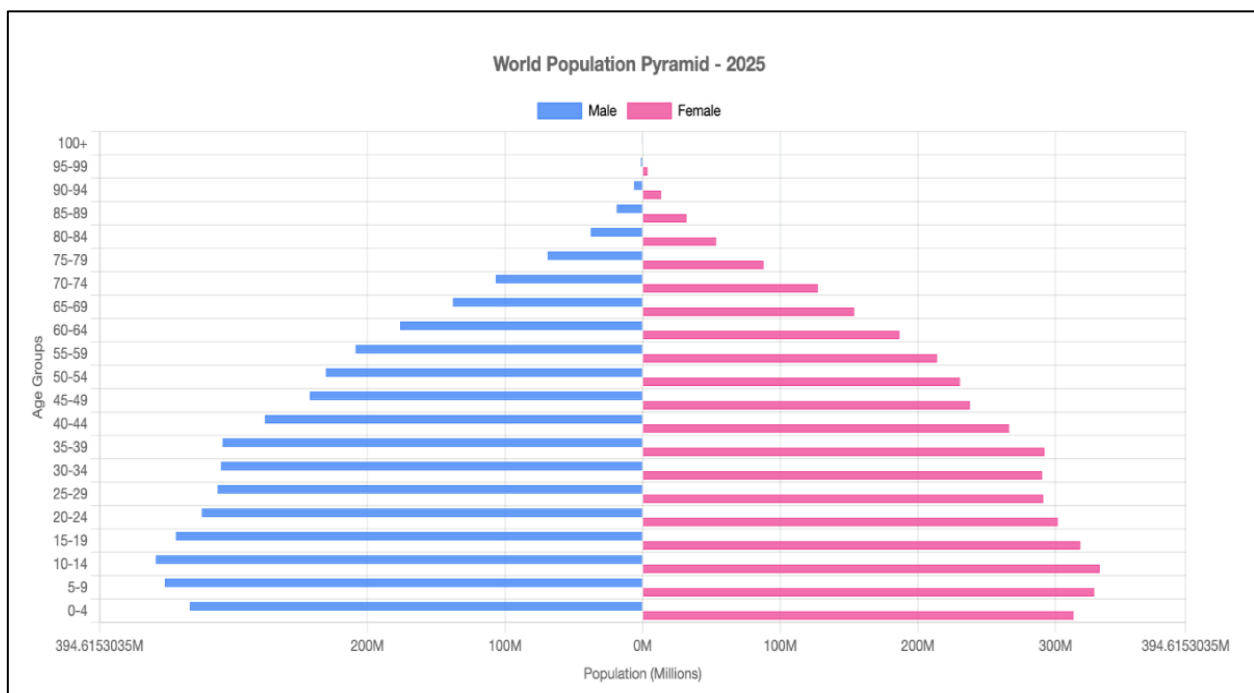


Fig 8 Global Population Age Structure by Sex and Age Group, 2025  
Source: Stage 4 Population Pyramid: Shape, Traits and Countries

➤ *Dependency Ratios*

$$DR = \frac{\text{Population aged 0–14 and 65+}}{\text{Population aged 15–64}}$$

Dependency ratio measures economic pressure on working-age individuals. A high ratio means more dependent relative to workers, increasing demand for social services, healthcare, and pensions. It is a key indicator of demographic

sustainability and long-term economic stability (United Nations, n.d.; World Bank, n.d.).

For example, a country with sixty dependents for every one hundred working-age individuals has a dependency ratio of 60%. This means that the working population must support a considerable number of children and elderly people through taxes and economic activity, increasing demand for services such as education, healthcare, and pensions. Such a ratio can place greater pressure on government resources and public finances.

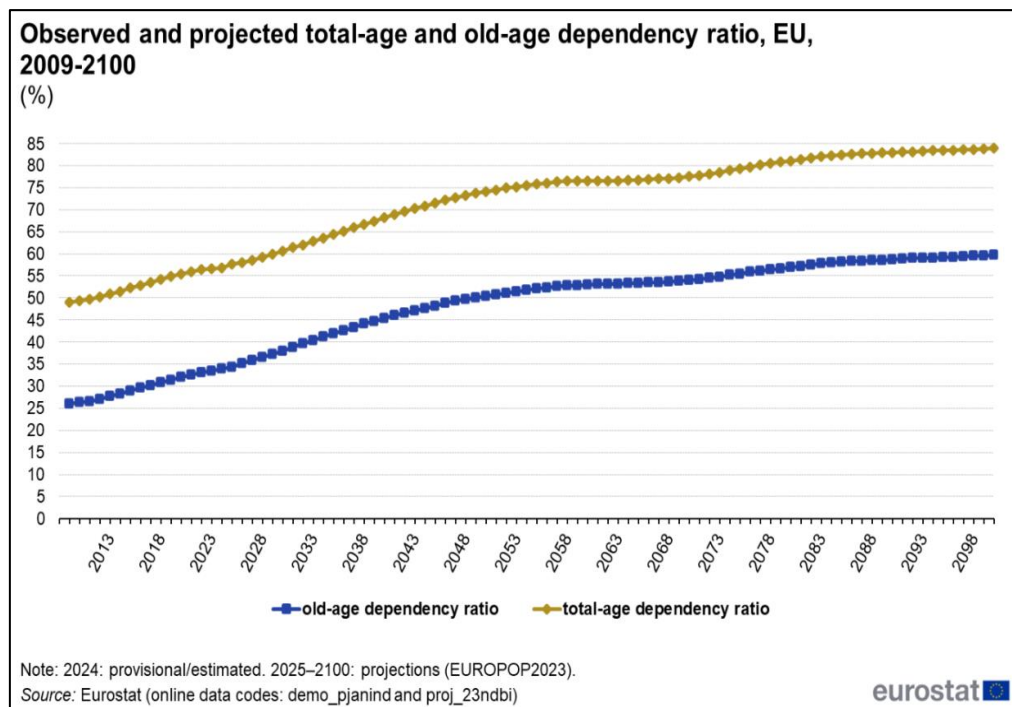


Fig 9 Observed and Projected Total-Age and Old-Age Dependency Ratios in the European Union, 2009-2100  
Source: Statistics Explained - Eurostat

## VI. EPIDEMIOLOGY DYNAMICS AND DISEASE-DRIVEN MORTALITY

➤ *Infection Fatality Rate (IFR)*

$$IFR = \frac{\text{Deaths}}{\text{Infections}}$$

IFR measures how deadly a disease is among those infected. It varies by age, health status, medical care, and virus strain. Understanding IFR helps policymakers assess disease severity, allocate medical resources, and design targeted interventions for vulnerable populations (World Health Organization, n.d.; CDC, n.d.).

In addition, IFR can differ significantly between population groups. Older adults and individuals with underlying health conditions face a higher risk of death following infection, while younger and healthier individuals often experience lower mortality rates. These differences make IFR a useful tool for identifying high-risk groups and prioritizing public health measures.

For example, if a disease infects 10,000 people and fifty deaths occur, the IFR is 0.5%. Although this may appear low, many infections can still result in substantial loss of life. Therefore, IFR helps governments evaluate the potential impact of an outbreak and plan effective response strategies.

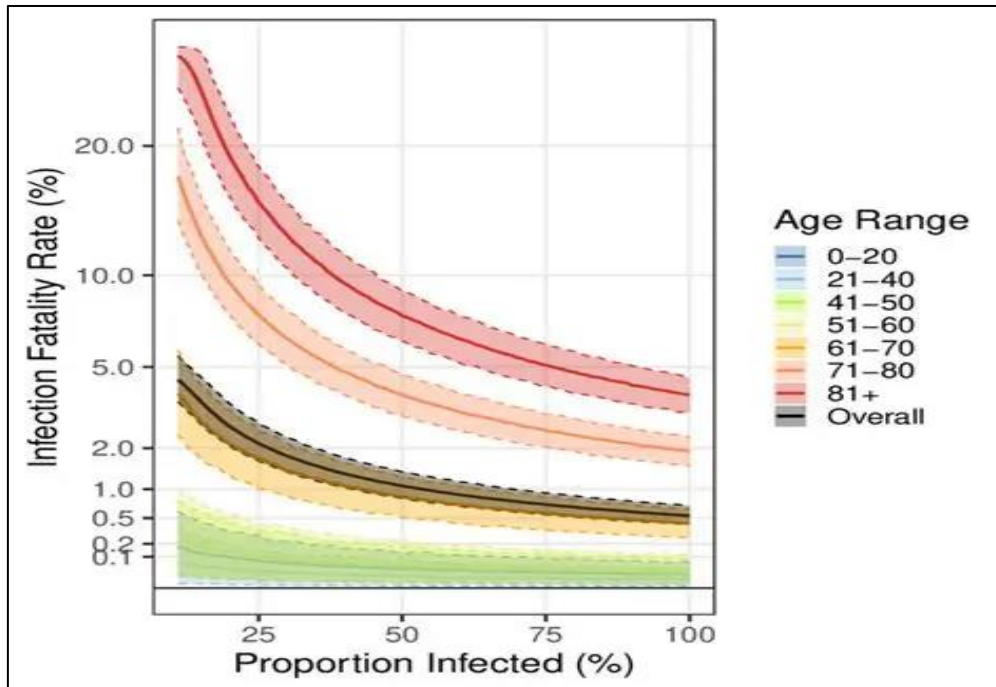


Fig 10 Infection Fatality Rate by Age Group and Proportion of Population Infected  
Source: An empirical estimate of the infection fatality rate of COVID-19 from the first Italian outbreak

#### ➤ Excess Mortality

Excess Mortality = Observed Deaths – Expected Deaths

Excess mortality captures the total impact of a crisis, including direct deaths, indirect deaths, and unreported cases. It is one of the most reliable measures of crisis severity because it reflects all mortality changes, regardless of reporting accuracy or diagnostic limitations (World Health Organization, n.d.).

Excess mortality is especially useful when comparing the impact of crises across different regions and time periods. By measuring deaths above expected levels, it accounts for

factors that may not be captured in official records, providing a more complete picture of overall population health and the effectiveness of public health responses.

A clear example is the COVID-19 pandemic, during which excess mortality was widely used to assess its true impact. In many countries, excess deaths exceeded the number of officially reported COVID-19 deaths, reflecting both unreported infections and indirect effects such as delayed medical treatment and overwhelmed healthcare systems. This made excess mortality a valuable metric for understanding the pandemic's overall severity (World Health Organization, n.d.).

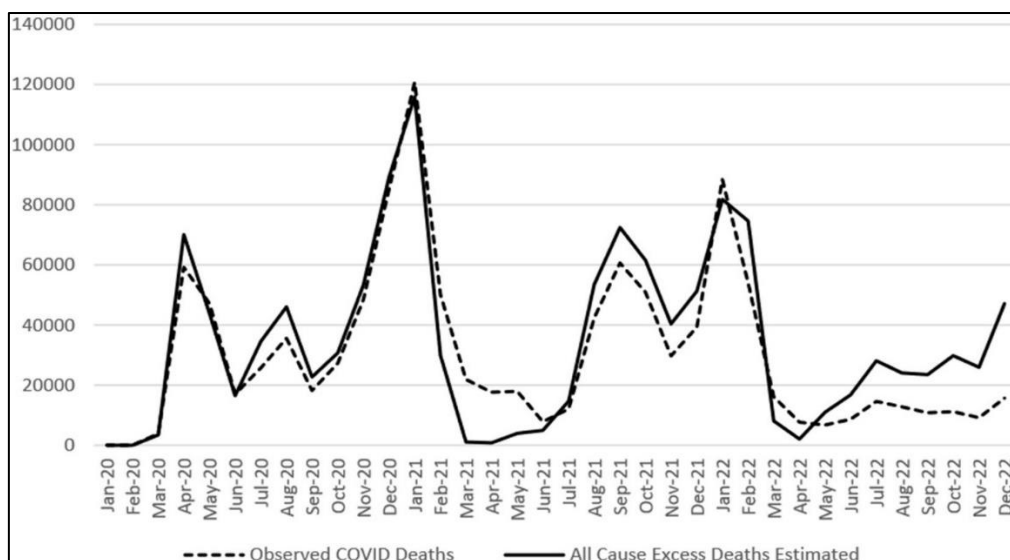


Fig 11 Excess Mortality and Reported COVID-19 Deaths (2020-2022)  
Source: Frontiers | Excess death compared with state-reported and observed COVID-19

## ➤ Longevity Inequality and Health Economics

### • Longevity Inequality

Longevity inequality refers to differences in life expectancy across groups. It arises from income disparities, education levels, healthcare access, environmental exposure, and lifestyle factors. Understanding these differences helps identify structural inequities and design policies to improve health outcomes for disadvantaged populations (Wilmoth & Horiuchi, 1999).

Longevity inequality can vary significantly across regions and social groups within the same country. Communities with better access to quality healthcare,

education, and economic opportunities often experience longer life expectancies than those facing poverty or environmental disadvantages. These gaps highlight the influence of social and economic conditions on long-term health outcomes.

For example, individuals from higher-income backgrounds have greater access to nutritious food, preventive healthcare, and healthier living environments, which can contribute to longer lifespans. In contrast, disadvantaged populations may face higher exposure to health risks and limited access to essential services, resulting in shorter average life expectancy. Understanding these differences is crucial for promoting health equity and reducing disparities.

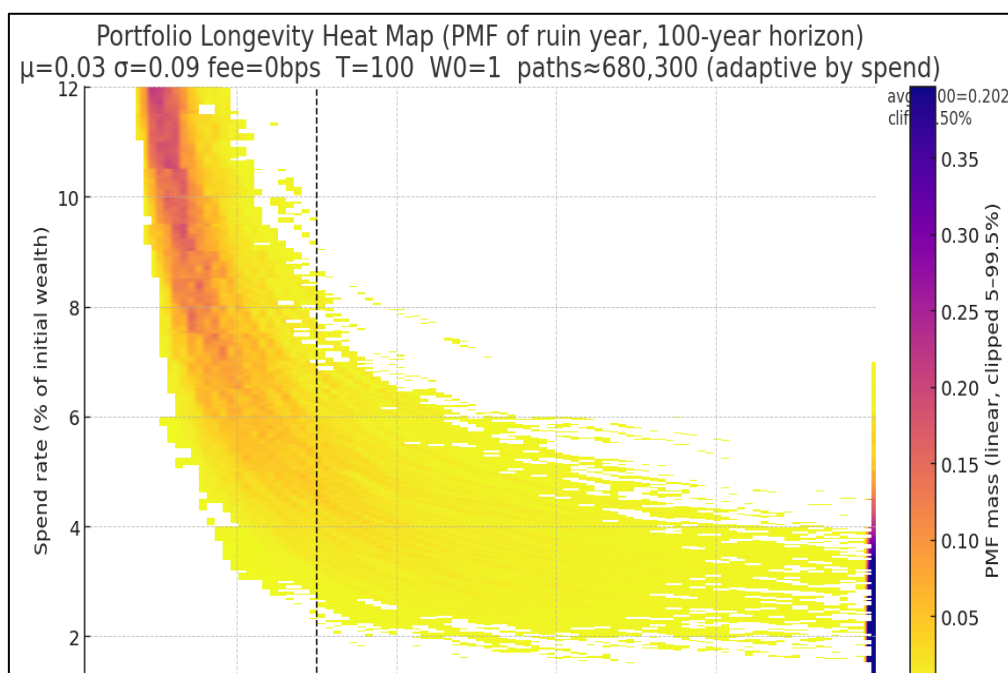


Fig 12 Longevity Risk Distribution Across Spending Rates and Initial Wealth Levels  
Source: RiversHedge: A Note on a Portfolio Longevity Heat Maps

### • Health Economics

Healthcare spending increases with age due to chronic disease, long-term care needs, and medical interventions. Understanding age-specific spending patterns helps policymakers allocate resources efficiently, plan for ageing populations, and evaluate the sustainability of healthcare systems (OECD, n.d.; Our World in Data, n.d.).

Healthcare spending also varies across various stages of life. Younger individuals require fewer medical services, while older adults often need more frequent treatments, specialist care, and long-term support. As a result, healthcare costs tend to rise steadily with age, placing greater demands on healthcare systems.

For example, a population with a sizable proportion of elderly citizens is likely to incur higher healthcare expenditures than a younger population. Governments can use age-specific spending data to forecast future costs, develop healthcare budgets, and ensure that medical services

remain accessible and financially sustainable as populations age (OECD, n.d.).

## VII. CONCLUSION

This thesis integrates mathematical foundations, mortality modelling, forecasting, demographic ageing, epidemiological dynamics, and longevity inequality into a cohesive interdisciplinary analysis. Survival functions and hazard models provide powerful tools for understanding mortality by quantifying the probability of survival and the risk of death across different ages and populations. Mortality forecasting techniques extend these models by identifying long-term trends and projecting future patterns, enabling researchers and policymakers to anticipate demographic and healthcare challenges. The study of demographic ageing further highlights how shifts in population structure influence dependency ratios, healthcare expenditure, labour markets, and social support systems. In addition, epidemiological measures such as infection fatality rates and excess mortality

provide valuable insights into the population-level impacts of diseases and public health crises. The analysis of longevity inequality complements these approaches by examining disparities in life expectancy and revealing the social, economic, and environmental factors that shape health outcomes. Together, these concepts demonstrate how mathematical and statistical methods can be applied to understand population health, evaluate public policy, and support evidence-based decision-making. The combined insights contribute to advancing research in demography, epidemiology, public health, actuarial science, and economic policy, while providing a framework for addressing the challenges associated with changing mortality patterns and population ageing (Bowers et al., 1997; Gompertz, 1825; Makeham, 1860; Lee & Carter, 1992; Kleinbaum & Klein, 2012; Wilmoth & Horiuchi, 1999).

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