

A Study of Generating Functions Related to New Bernstein-Type Operators

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Abstract: The objective of this paper is to develop a generating function for a novel class of Bernstein type operators, as proposed by N. Deo et al. ([3]). This involves deriving functional equations and partial differentiation’s based on the underlying basis functions. In addition, we investigate the use of the Laplace transform to generate these new Bernstein basis functions.

Keywords: Bernstein Polynomials; Generating Functions; Functional Equations.

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I. INTRODUCTION AND BACKGROUND

The Bernstein polynomials play a crucial role in approximation theory and have substantial applications across disciplines such as mathematics and physics (see [6, 7, 9]). Their versatility extends to a variety of domains, encompassing computer graphics, computer-aided design, numerical analysis, and geometric modeling.

At its core, a Bernstein polynomial is a type of polynomial expression that is defined over a fixed interval [0, 1]. What makes Bernstein polynomials particularly intriguing is their ability to approximate any continuous function within the specified interval through a convex combination of control points. This property makes them essential tools for curve and surface representation in computer graphics and geometric modeling. Mathematically, the Bernstein operator can be expressed as follows:

Consider the function $(g(y))$ defined over the interval [a, b]. The Bernstein operator $B_{r1}^*[(g(y))]$ of degree $r1$ applied to $g(y)$ is defined as:

$$B_{r1}^*[(g(y))] = \sum_{c1=0}^{r1} g\left(\frac{c1}{r1}\right) \binom{r1}{c1} y^{c1} (1-y)^{r1-c1} \dots \dots \dots (1)$$

Where y lies in the interval [0, 1] and $\binom{r1}{c1}$ denotes the binomial coefficient, given by $\binom{r1}{c1} = \frac{r1!}{c1!(r1-c1)!}$
 $bc1, r1(y) = \binom{r1}{c1} y^{c1} (1-y)^{r1-c1}, c1 = 0, 1, 2, \dots, r1.$

Now we give the important properties of these polynomials.

- They are nonnegative over the interval [a,b].
- They are symmetric polynomials.
- Each polynomials has only one maximum over the interval [a,b].
- The set of these polynomials of deg $r1$ form a partition of unity.
- A Bernstein polynomial can always be written as a linear combination of polynomials of higher order.

The well-known Bernstein operators $B_{r1}^*[(g(y))]$ are available, if $g(y)$ is a function defined on [0, 1] and given in [1]. $bc1, r1$ are called the Bernstein basis polynomials there are $(r1 + 1)^{th}$ degree polynomials.

In the year 2008, N. Deo, M.A. Noor, and M.A. Siddiqui [3] introduced a novel and innovative mathematical concept known as the “ $Vr1$ Bernstein Type Special Operator.” This operator is a significant extension of the classical Bernstein operator and holds unique properties that distinguish it within the realm of numerical analysis and approximation theory.

$$Vr1 [g(y)] = \sum_{c1=0}^{r1} g\left(\frac{c1}{r1}\right) p(r1, c1)(y) \dots \dots \dots (2)$$

Where

$$p(r1, c1)(y) = \left(1 + \frac{1}{r1}\right)^{r1} \binom{r1}{c1} y^{c1} \left(\frac{r1}{r1+1} - y\right)^{r1-c1} \frac{r1}{r1+1} \geq y \text{ and } y \in \left[0, 1 - \frac{1}{1+r1}\right]$$

And $\binom{r1}{c1}$ denotes the binomial coefficient. This signifies a generalized variation of Bernstein operators. More precisely, when the parameter $r1$ attains a considerable

magnitude, the operators $Vr1$ transform back into the original Bernstein operators

Now, we rewrite $p_{r1,c1}(y)$ in the form of

$$p_{r1,c1}(y) = \left(1 + \frac{1}{r1}\right)^{c1} \binom{r1}{c1} y^{c1} \left(1 - \frac{r1 + 1}{r1} y\right)^{r1-c1}$$

$p_{r1,c1}$ are called the new kind of Bernstein basis polynomials there are $(r1 + 1)^{th}$ degree polynomials.

In the year 2010, the collaborative efforts of Mehmet Acikgoz and Serkan Araci [1] culminated in a significant contribution to the field of mathematics. Their work, which remains a cornerstone in the study of Bernstein polynomials and their associated properties, offers valuable insights into the realm of polynomial approximation and mathematical analysis. In 2013, Yilmaz Simsek's [10] exploration of Bernstein polynomials marked a pivotal contribution. His innovative work within this year unveiled new dimensions in polynomial approximation and mathematical analysis. Simsek's insights have since become a cornerstone for researchers and practitioners, enhancing the understanding and application of Bernstein polynomials in diverse fields.

Building upon these previous works, our paper focuses on constructing generating functions for a novel class of Bernstein type operators introduced in [3]. We explore several dimensions, encompassing topics like functional equations and partial differentiations expressed in relation to the basis functions. Moreover, we explore the application of the Laplace transform in conjunction with the generating functions associated with these recently introduced Bernstein basis functions. This exploration opens avenues to unveil the intricate relationships between Laplace transforms and the behavior of the specialized Bernstein functions, enriching our understanding of their mathematical properties and potential applications.

II. GENERATING FUNCTION

A multitude of strategies have been harnessed for the examination and formulation of Bernstein polynomials.

Proof:

$$\begin{aligned} G_{c1}(y, t) &= \sum_{c1=0}^{r1} p_{c1,r1}(y) \frac{t^{r1}}{(r1)!} \\ &= \sum_{c1=0}^{r1} \left(1 + \frac{1}{r1}\right)^{r1} \binom{r1}{c1} y^{c1} \left(\frac{r1}{1+r1} - y\right)^{r1-c1} \frac{t^{r1}}{(r1)!} \\ &= \sum_{c1=0}^{r1} \left(1 + \frac{1}{r1}\right)^{r1} \binom{r1}{c1} y^{c1} \left(\frac{r1}{1+r1} - y\right)^{r1-c1} t^{c1} \frac{t^{r1-c1}}{(r1)!} \\ &= \sum_{c1=0}^{r1} \left(1 + \frac{1}{r1}\right)^{r1} \binom{r1}{c1} \left(\frac{r1}{1+r1} - y\right)^{r1-c1} (yt)^{c1} \frac{t^{r1-c1}}{(r1)!} \end{aligned}$$

These encompass methodologies rooted in q-series, intricate complex functions, p-adic Volkenborn integrals, as well as an array of algorithms. For a comprehensive overview, references such as [4, 5, 11, 12, 13] offer valuable insights into these diverse approaches, shedding light on the breadth of techniques applied to understand and define these essential mathematical constructs.

Certain polynomials can be characterized by their generating functions. For instance, two well-known polynomials, namely, Bernoulli polynomials $Br1(y)$ and Euler polynomials $Er1(y)$, possess the following generating functions.

$$\begin{aligned} Br1(y) &= \frac{ye^{yt}}{e^t - 1} = \sum_{c1=0}^{r1} B_{c1} \frac{y^{r1-c1}}{(r1 - c1)!} t^{c1} \\ Er1(y) &= \frac{2^{r1}}{1} \sum_{c1=0}^{r1} \frac{y^{r1-c1}}{(r1 - c1)!} (-1)^{c1} \end{aligned}$$

cf.([2], [8], [14],[15]).

The generating function for the Bernstein polynomial $B_{r1}^*[(g(y))]$ can be represented as follows:

$$\frac{t^{ye(1-y)t}}{c1!} = \sum_{c1=0}^{r1} B_{c1,r1}(y) \frac{t^{r1}}{(r1)!}$$

The Bernstein polynomials, along with their numerous applications, have been subjects of extensive study and investigation by various mathematicians. In this work, we will delve into the generating function of a novel class of Bernstein polynomials, which takes the following form:

The new class of Bernstein polynomial, $(Vr1g)(y)$ having the following generating function.

$$G_{c1}(y, t) = \frac{(yt)^{c1}}{c1!} \frac{e^t}{e^{\frac{yt(r1+1)}{r1}}} \left(1 + \frac{1}{r1}\right)^{c1} \dots \dots \dots (3)$$

$$\begin{aligned}
 &= \sum_{c1=0}^{r1} \left(1 + \frac{1}{r1}\right)^{r1} \binom{r1}{c1} \left(\frac{tr1}{1+r1} - ty\right)^{r1-c1} \frac{(yt)^{c1}}{(r1)!} \\
 &= \sum_{c1=0}^{r1} \left(1 + \frac{1}{r1}\right)^{r1} \frac{r1!}{c1!(r1-c1)!} \left(\frac{tr1}{1+r1} - yt\right)^{r1-c1} \frac{(yt)^{c1}}{(r1)!} \\
 &= \sum_{c1=0}^{r1} \left(1 + \frac{1}{r1}\right)^{r1} \frac{(yt)^{c1}}{c1!(r1-c1)!} \left(\frac{tr1}{1+r1} - ty\right)^{r1-c1}.
 \end{aligned}$$

Upon performing the necessary calculations, we have successfully obtained the desired result.

$$G_{c1}(y, t) = \frac{(yt)^{c1}}{c1!} \frac{e^t}{e^{yt(r1+1)/r1}} \left(1 + \frac{1}{r1}\right)^{c1}$$

In equation (3), we present an alternative form of the generating function as follows:

$$\begin{aligned}
 \sum_{r1=c1}^{\infty} p(r1, c1)(y) \frac{t^{r1}}{(r1)!} e^{-t} &= \frac{(yt)^{c1}}{c1!} \frac{1}{e^{yt(r1+1)/r1}} \left(1 + \frac{1}{r1}\right)^{c1} \\
 \sum_{r1=c1}^{\infty} p(r1, c1)(y) \frac{t^{r1}}{(r1)!} e^{yt(r1+1)/r1} &= \frac{(yt)^{c1}}{c1!} \frac{e^t}{1} \left(1 + \frac{1}{r1}\right)^{c1}
 \end{aligned}$$

Using the generating function, we can express all new Bernstein polynomials as follows:

$$p_{0,0} = 1; p_{0,1} = 1 - 2y; p_{1,1} = 2y, p_{0,2} = 1 - 3y + \frac{9y^2}{4}, p_{1,2} = 3y - \frac{9y^2}{2}, p_{2,2} = 9y^2/4$$

$$p_{0,3} = 1 - 4y + 16y^2/3 - 64y^3/27, p_{1,3} = 4y - 32y^2/3 + 64y^3/9, p_{2,3} = 16y^2/3 - 64y^3/9, p_{3,3} = 64y^3/27$$

III. IDENTITIES FOR THE BASIS FUNCTION

In this section, we will provide the expressions for the derivatives of higher orders for the newly defined Bernstein basis functions. These equations serve as a gateway to deriving an array of identities specific to the basis functions. By delving into the intricacies of the generating functions, we unveil pivotal relationships and properties governing the behavior of these basis functions. This exploration fosters a profound grasp of their inherent characteristics and opens doors to their versatile applications in diverse mathematical contexts.

➤ Summations and Alternate Summations

Derived from the equation (3), we can deduce the subsequent equations:

$$\sum_{c1=0}^{\infty} G_{c1}(y, t) = e^t \dots \dots \dots (4)$$

$$\sum_{c1=0}^{\infty} (-1)^{c1} G_{c1}(y, t) = e^{t(1-2y\frac{r1+1}{r1})} \dots \dots \dots (5)$$

Summation of the novel category of Bernstein basis functions:

$$\sum_{c1=0}^{r1} p(r1, c1)(y) = 1$$

Proof: From (4)

$$\sum_{c1=0}^{\infty} G_{c1}(y, t) = e^t$$

$$\sum_{c1=0}^{\infty} G_{c1}(y, t) = \sum_{r1=0}^{\infty} \frac{t^{r1}}{(r1)!}$$

$$\sum_{r1=0}^{\infty} \sum_{c1=0}^{r1} p_{c1,r1}(y) \frac{t^{r1}}{(r1)!} = \sum_{r1=0}^{\infty} \frac{t^{r1}}{(r1)!}$$

By equating the coefficients of $\frac{t^{r1}}{(r1)!}$ on both sides of the mentioned equation, we arrive at the desired result.

➤ Derivative of the Generating Function

In this section, we will present the derivatives of higher orders for the newly defined Bernstein basis functions. To facilitate this analysis, we introduce the generating function, which will serve as a valuable tool in computing these higher order derivatives. With the help of this generating function, we can efficiently compute the desired higher order derivatives and explore their properties. These results will

enhance our understanding of the new type of Bernstein basis functions and their applications.

$$G_{c1}(y, t) = \frac{(yt)^{c1}}{c1!} \frac{e^t}{e^{yt(r1+1)/r1}} \left(1 + \frac{1}{r1}\right)^{c1}$$

Now, we put $G_{c1}(y, t) = f(t, y)h(t, y)$

Where

$$f(t, y) = \frac{(yt)^{c1}}{c1!} \left(1 + \frac{1}{r1}\right)^{c1} \text{ and } h(t, y) = \frac{e^t}{e^{yt(r1+1)/r1}}$$

Utilizing Leibniz’s formula for the n^{th} derivative with respect to ‘y’, we arrive at the subsequent higher-order partial differential equation:

$$\frac{\partial^n G_{c1}(y, t)}{\partial y^n} = \sum_{j=0}^n \binom{n}{j} \frac{\partial^j f(y, t)}{\partial y^j} \frac{\partial^{n-j} h(y, t)}{\partial y^{n-j}}$$

Thus, we can infer the subsequent theorem:

$$\frac{\partial^n G_{c1}(y, t)}{\partial y^n} = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \left(\frac{t(r1+1)}{r1}\right)^j G_{c1}(y, t)$$

Proof: Using Leibnitz’s formula

$$D^n(f; h) = \binom{n}{0} D^n(f; h) + \binom{n}{1} D^{n-1}(f) D^1(h) + \binom{n}{2} D^{n-2}(f) D^2(h) + \binom{n}{3} D^{n-3}(f) D^3(h) \dots + \binom{n}{n} (f) D^n(h)$$

$$D^n f(t; y) = \frac{(yt)^{c1-n}}{(c1-n)!} \left(1 + \frac{1}{r1}\right)^{c1}$$

$$D^n h(t; y) = (-1)^n \left(\frac{t(r1+1)}{r1}\right)^n \frac{e^t}{e^{yt(r1+1)/r1}}$$

Using above formula, we get our desire results.

$$\frac{\partial^n G_{c1}(y, t)}{\partial y^n} = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \left(\frac{t(r1+1)}{r1}\right)^j G_{c1-j}(y, t)$$

Substituting $n = 1$ into equation (6), we arrive at the following standard corollary.

➤ *Corollary*

Upon differentiating the generating function of Bernstein polynomials, as provided by the above theorem, with respect to y, we derive the following relationship:

$$\begin{aligned} \frac{\partial G_{c1}(y, t)}{\partial y} &= \frac{\partial}{\partial y} \frac{(yt)^{c1}}{c1!} \left(\frac{r1+1}{r1}\right)^{c1} \frac{e^t}{e^{yt(r1+1)/r1}} \\ &= \frac{c1(yt)^{c1-1} t \left(\frac{r1+1}{r1}\right)^{c1}}{c1!} \frac{e^t}{e^{yt(r1+1)/r1}} + \frac{(yt)^{c1}}{c1!} \left(\frac{r1+1}{r1}\right)^{c1+1} \frac{(-t)e^t}{e^{yt(r1+1)/r1}} \\ &= t(1 + 1/r1)[G_{c1-1}(y, t) - G_{c1}(y, t)] \end{aligned}$$

The property of differentiation inherent in the generating function for the new class of Bernstein polynomials facilitates the establishment of the subsequent recurrence relation. Through the process of taking derivatives of the generating function with respect to the variable y, we unveil a connection that articulates a Bernstein polynomial of degree n in relation to the Bernstein polynomials of lower degrees. This recurrence relation proves itself as an invaluable instrument, enabling the efficient computation and generation of an entire spectrum of Bernstein polynomials spanning diverse degrees.

$$\frac{1}{t} \frac{d}{dy} G_{c_1}(y, t) = (1 + 1/r_1)[G_{c_1-1}(y, t) - G_{c_1}(y, t)]$$

$$\frac{d}{dy} G_{c_1}(y, t) = t(1 + 1/r_1)[G_{c_1-1}(y, t) - G_{c_1}(y, t)]$$

Now, we will proceed to compute the higher-order derivatives of the basis functions.

The definition of the derivative for the new type of Bernstein polynomial is as follows:

$$\frac{d^n}{dy^n} p(r_1, c_1)(y) = \frac{(r_1 + 1)!}{(r_1 + 1 - n)!} \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} p(r_1 - n, c_1 - n)(y)$$

Proof:

$$p(r_1, c_1)(y) = \left(1 + \frac{1}{r_1}\right)^{c_1} \binom{r_1}{c_1} y^{c_1} \left(1 - \frac{r_1 + 1}{r_1} y\right)^{r_1 - c_1}$$

Differentiating with respect to 'y'

$$\begin{aligned} (p(r_1, c_1)(y))' &= \frac{d}{dy} \left(1 + \frac{1}{r_1}\right)^{c_1} \binom{r_1}{c_1} y^{c_1} \left(1 - \frac{r_1 + 1}{r_1} y\right)^{r_1 - c_1} \\ &= \left(1 + \frac{1}{r_1}\right)^{c_1} c_1 \binom{r_1}{c_1} y^{c_1-1} \left(1 - \frac{r_1 + 1}{r_1} y\right)^{r_1 - c_1} + \left(1 + \frac{1}{r_1}\right)^{c_1} \binom{r_1}{c_1} y^{c_1} \left(1 - \frac{r_1 + 1}{r_1} y\right)^{r_1 - c_1 - 1} (r_1 - c_1) \left(-\frac{r_1 + 1}{r_1}\right) \\ &= (r_1 + 1) \left(1 + \frac{1}{r_1}\right)^{c_1-1} \binom{r_1-1}{c_1-1} y^{c_1-1} \left(1 - \frac{r_1 + 1}{r_1} y\right)^{r_1-1-c_1+1} - (r_1 + 1) \left(1 + \frac{1}{r_1}\right)^{c_1} \binom{r_1-1}{c_1-1} y^{c_1} \left(1 - \frac{r_1 + 1}{r_1} y\right)^{r_1-1-c_1} \\ &= (r_1 + 1) [p_{r_1-1, c_1-1} - p_{r_1-1, c_1}] \end{aligned}$$

Through further differentiation, we arrive at the desired results.

$$= \frac{(yt)^{c_1}}{c_1!} \left(\frac{r_1 + 1}{r_1}\right)^{c_1} \frac{e^t}{e^{yt(r_1+1)/r_1}}$$

Investigating the Application of Laplace Transform in Conjunction with Generating Functions for the Novel Type of Bernstein Basis Functions Utilizing the Laplace transform in relation to the generating functions of the novel Bernstein basis functions provides opportunities to delve into their characteristics and properties within the frequency domain. This transformation offers a powerful tool to analyze the dynamics and characteristics of these functions in terms of complex exponentials, enabling insights into their responses to various inputs and stimuli. By harnessing the Laplace transform, we gain a fresh perspective on the behavior of the new Bernstein basis functions, facilitating applications in diverse fields such as signal processing, control systems, and differential equations. This connection between generating functions and the Laplace transform enriches our understanding and widens the scope of these basis functions' practical applications.

$$G_{c_1}(y, t) = \sum_{r_1=c_1}^{\infty} p_{r_1, c_1}(y) \frac{t^{r_1}}{(r_1)!}$$

Integrating with respect to t from 0 to ∞, we have

$$\begin{aligned} \sum_{r_1=c_1}^{\infty} p_{r_1, c_1}(y) \int_0^{\infty} e^{-t} \frac{t^{r_1}}{(r_1)!} dt \\ \int_0^{\infty} \frac{(yt)^{c_1}}{c_1!} \left(\frac{r_1+1}{r_1}\right)^{c_1} \frac{e^t dt}{e^{yt(r_1+1)/r_1}} \\ \sum_{r_1=c_1}^{\infty} p_{r_1, c_1}(y) = \frac{r_1}{(1 + r_1)y} \end{aligned}$$

IV. CONCLUSION

In this paper, we have been able to create a generating function for a new class of Bernstein-type operators, using the framework introduced by N. Deo et al. In addition, using functional equations and partial differentiation techniques for the basis functions, we were able to construct a mathematical structure for this new class of operators. Additionally, we demonstrated how the Laplace transform can be useful in this regard.

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