Anisotropic (N+1)-dimensional generalizations of Korteweg-de Vries, Kadomtsev-Petviashvili and Boussinesq equations via Hirota Bilinear Forms

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Abstract: This research paper deals with various $(N+1)$-dimensional generalizations and the overall study of the KdV, KP and Boussinesq equations. The generalizations are constructed systematically with each of the lower dimensional cases explored appropriately. The Hirota bilinear forms of the constructed generalized partial differential equations are then derived utilizing key bilinear form identities. Along with the bilinear form representations, one-soliton based dispersion relations and two-soliton interaction coefficients are derived systematically. Anisotropic generalizations of the CBS equation and the variable coefficient KP equation are also deliberated which collectively provide a unified framework applicable across numerous disciplines of nonlinear sciences.

Keywords: *Korteweg-de Vries (KdV) equation, Kadomtsev-Petviashvili (KP) equation, Boussinesq equation, Hirota Bilinear Forms, Higher Order KdV, Higher Order KP, Higher Order Boussinesq equations, Dispersion Relations, Solitonic Solutions, Cole-Hopf Substitutions, Partial differential equations.*

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I. INTRODUCTION

Hirota bilinear forms are of great importance in numerous fields of science including applied mathematics, physics, nonlinear sciences, oceanography etc. The reason being Hirota bilinear forms admit solitonic solutions which become quite important to understand the overall nature of the partial differential equation and Hirota bilinear forms can also be used to develop lump solutions. Famous equations such as the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) have simplistic Hirota bilinear forms which help in modelling 1-soliton solutions (1SS), 2-soliton solutions (2SS) etc. Hirota Bilinear transform can be defined as follows assuming the functions bilinearized are $f(x,t)$ and $g(x,t)$:

$$D_x^m D_t^n (f \cdot g) = \lim_{X \to x, T \to t} \left((\partial_x - \partial_X)^m (\partial_t - \partial_T)^n f(x,t) g(X,T) \right)$$

➢ *Few Important Properties Related to Bilinear Transforms are as follows Assuming m, n are Positive Integers:*

- $D_x^m D_t^n (f \cdot g) = (-1)^{m+n} D_x^m D_t^n (g \cdot f)$
- $D_x^{2n-1} (f \cdot f) = 0$
- $D_x^m D_t^n (\alpha f \cdot \beta g) = \alpha\beta D_x^m D_t^n (f \cdot g)$ (where α, β are constants)
- $D_x^m D_t^n (f \cdot 1) = \partial_x^m \partial_t^n f(x,t)$

➢ *Important Bilinear form Results Include:*

- $D_x D_t (f \cdot f) = 2(f_{xt} f - f_x f_t)$
- $D_x^2 (f \cdot f) = 2(f_{xx} f - f_x^2)$
- $D_x^3 D_t (f \cdot f) = 2\left(f_{(3x)t} f - f_t f_{(3x)} - 3 f_{xxt} f_x + 3 f_{xt} f_{xx}\right)$
- $D_x^4 (f \cdot f) = 2\left(f_{(4x)} f - 4 f_{(3x)} f_x + 3 f_{xx}^2\right)$

Where $f_{(px)} = \frac{\partial^p f}{\partial x^p}$ assuming p is a positive integer. By letting $f(x,t) = e^{w(x,t)}$, another important list of results is produced which is used extensively throughout this article:

- $\frac{D_x D_t (f \cdot f)}{2f^2} = w_{xt}$
- $\frac{D_x^2 (f \cdot f)}{2f^2} = w_{xx}$
- $\frac{D_x^3 D_t (f \cdot f)}{2f^2} = w_{(3x)t} + 6 w_{xx} w_{xt}$
- $\frac{D_x^4 (f \cdot f)}{2f^2} = w_{(4x)} + 6 w_{xx}^2$

Now, to understand the important results coming from theory of solitons, we assume the initial PDE to be given as $Q(u, u_x, u_t, \cdots) = 0$ and the substitution taken here to be $u = R\frac{\partial^n}{\partial x^n}(ln\, f)$ where n is some positive integer and R is some constant. This leads to the following construction of the Hirota bilinear form:

$$P(D_x, D_t)(f \cdot f) = 0$$

Assuming P denotes some even polynomial. With the help of the perturbation technique $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$ where $f_1 = e^\eta + e^\xi + \cdots$ and $f_2 = Ae^{\eta+\xi} + \cdots$, we obtain an important result in one-soliton solution case (1SS) as follows:

$$P(D_x, D_t)(e^\eta \cdot 1) = 0$$

By utilizing $\eta = px + qt$, $P(D_x, D_t) = \sum_i a_i D_x^{b_i} D_t^{c_i}$ where b_i, c_i are some positive integers and the properties mentioned above, we obtain:

$$P(p, q) = 0$$

This equation leads to an important result known as the dispersion relation, which allows q to be written in terms of p. The same result can be scaled to $(N + 1)$-variables for $\eta = p_1 x + \sum_{i=2}^N p_i x_i + qt$:

$$P(p_i, q) = 0\,;\, 1 \le i \le N$$

The one-soliton solution (1SS) is ultimately given as:

$$u = R(ln(1 + e^\eta))_{(nx)}$$

An important result obtained from two-soliton solution case (2SS) is the value of the interaction coefficient A as $f = 1 + e^\eta + e^\xi + Ae^{\eta+\xi}$:

$$P(D_x, D_t)\left(Ae^{\eta+\xi} \cdot 1 + e^\eta \cdot e^\xi\right) = 0$$

By additionally assuming $\xi = kx + lt$, we obtain:

$$A = -\frac{P(p - k, q - l)}{P(p + k, q + l)}$$

The above result can be scaled to $(N + 1)$-variables for $\xi = k_1 x + \sum_{i=2}^N k_i x_i + lt$ and $\eta = p_1 x + \sum_{i=2}^N p_i x_i + qt$:

$$A = -\frac{P(p_i - k_i, q - l)}{P(p_i + k_i, q + l)}\,;\, 1 \le i \le N$$

Whereas two-soliton solution (2SS) is expressed as follows:

$$u = R\left(ln\left(1 + e^\eta + e^\xi + Ae^{\eta+\xi}\right)\right)_{(nx)}$$

While discussing the anisotropic extensions of the KdV, KP and Boussinesq equations, the primary focus will lie upon exploring the bilinear form of each extension, building the adequate dispersion relation and finding the value of the interaction coefficient A as discussed above.

II. MAIN RESULTS AND GENERALIZATIONS

Starting off this section by discussing the (1+1)-KdV equation:

$$u_t + u_{xxx} + 3(u^2)_x = 0$$

To bilinearize this equation, we utilize the Cole-Hopf substitution given as $u = 2(ln\, f)_{xx} = 2w_{xx}$:

$$w_{xxt} + w_{(5x)} + 6(w_{xx}^2)_x = 0$$

$$\int \left(w_{xxt} + w_{(5x)} + 6(w_{xx}^2)_x\right) dx = C(t)$$

Assuming rapidly decaying boundary conditions because of which the integration constant nullifies, and we obtain the following simplified equation:

$$w_{xt} + w_{(4x)} + 6w_{xx}^2 = 0 \tag{1}$$

We now recall the earlier prescribed Hirota identities:

- $\frac{D_x D_t (f \cdot f)}{2f^2} = w_{xt}$
- $\frac{D_x^4 (f \cdot f)}{2f^2} = w_{(4x)} + 6w_{xx}^2$

Thus, we obtain the well-known bilinearized form of the (1+1)-KdV equation.

$$(D_x D_t + D_x^4)(f \cdot f) = 0 \tag{2}$$

Now, we implement the perturbation technique to extrapolate f as follows which in turn allows us to construct dispersion relations and soliton-based solutions:

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots$$

For the dispersion relation, we set $f = 1 + \varepsilon f_1$ where $f_1 = e^\eta$ such that $\eta = px + qt$ leading to:

$$(D_x D_t + D_x^4)\left((1 + \varepsilon f_1) \cdot (1 + \varepsilon f_1)\right) = 0$$

$$\varepsilon(D_x D_t + D_x^4)(1 \cdot f_1 + f_1 \cdot 1) = 0$$

$$2\varepsilon(D_x D_t + D_x^4)(f_1 \cdot 1) = 0$$

The simplifications became possible as $P(D_x, D_t) = D_x D_t + D_x^4$ is an even polynomial and changing the order of the bilinearized functions will have no effect. Finally, the required dispersion relation being:

$$q = -p^3 \tag{3}$$

Thus, a single-soliton solution for the (1+1)-KdV is given as such where $\eta = px - p^3 t$:

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$$u = 2(ln(1+e^{\eta}))_{xx} = 2\left(\frac{e^{\eta}}{1+e^{\eta}}\right)_x = \frac{2e^{\eta}}{(1+e^{\eta})^2}$$

In case of a two-soliton solution, we set $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2$ where $f_1 = e^{\eta} + e^{\xi}$ and $f_2 = Ae^{\eta+\xi}$ such that $\xi = kx + lt$ leading to the following:

$$(D_x D_t + D_x^4)\big((1 + \varepsilon f_1 + \varepsilon^2 f_2) \cdot (1 + \varepsilon f_1 + \varepsilon^2 f_2)\big) = 0$$

$$2\varepsilon^2 (D_x D_t + D_x^4)(f_2 \cdot 1 + f_1 \cdot f_1) = 0$$

Upon substituting $f_1 = e^{\eta} + e^{\xi}$ and $f_2 = Ae^{\eta+\xi}$, we can simplify further and obtain an algebraic expression in terms of A :

$$(D_x D_t + D_x^4)\big(Ae^{\eta+\xi} \cdot 1 + e^{\eta} \cdot e^{\xi}\big) = 0$$

$$A\big((p+k)(q+l) + (p+k)^4\big) + (p-k)(q-l) + (p-k)^4 = 0$$

Upon rearranging the terms and substituting $q = -p^3$ and $l = -k^3$, we get:

$$A = \left(\frac{p-k}{p+k}\right)^2 \tag{4}$$

Therefore, a two-soliton solution here is given as:

$$u = 2\left(ln\left(1 + e^{\eta} + e^{\xi} + \left(\frac{p-k}{p+k}\right)^2 e^{\eta+\xi}\right)\right)_{xx} \tag{5}$$

Now that the (1+1)-KdV is thoroughly discussed, we discuss the anisotropic extension of the KdV to (2+1)-dimensions. The consensus that will be used in the entirety of this paper will be transforming pure derivative terms $u_{(nx)}$ to $u_{((n-1)x)y}$ and to rewrite the advection term u^2 as $u\partial_x^{-1}(u_y)$. This allows us to rewrite the (1+1)-KdV as follows using the independent variables x, y, t:

$$u_t + u_{xxy} + 3\left(u\partial_x^{-1}(u_y)\right)_x = 0$$

Implementing the Cole-Hopf substitution yet again where $u = 2(ln\, f)_{xx} = 2w_{xx}$:

$$w_{xxt} + w_{(4x)y} + 6\big(w_{xx}w_{xy}\big)_x = 0$$

$$\int \left(w_{xxt} + w_{(4x)y} + 6\big(w_{xx}w_{xy}\big)_x\right) dx = C(t)$$

Once again, assuming rapidly decaying boundary conditions because of which the integration constant nullifies and we obtain:

$$w_{xt} + w_{(3x)y} + 6w_{xx}w_{xy} = 0$$

Recall Hirota bilinear identity $\frac{D_x^3 D_y(f \cdot f)}{2f^2} = w_{(3x)y} + 6w_{xx}w_{xy}$ leading to the bilinear form of the (2+1)-KdV:

$$\big(D_x D_t + D_x^3 D_y\big)(f \cdot f) = 0 \tag{6}$$

An important feature to notice here is diagonalizing the (2+1)-KdV i.e. substituting $y = x$ results in the (1+1)-KdV. To calculate the dispersion relation, we set $f_1 = e^{\eta}$ where $\eta = px + ry + qt$:

$$\big(D_x D_t + D_x^3 D_y\big)(e^{\eta} \cdot 1) = 0$$

$$pq + p^3 r = 0$$

The obtained dispersion relation is $q = -p^2 r$. For the two-soliton case, the value of coefficient A is given as follows as we set $\xi = kx + my + lt$ and assume $P\big(D_x, D_y, D_t\big) = D_x D_t + D_x^3 D_y$:

$$A = -\frac{P(p-k, r-m, k^2 m - p^2 r)}{P(p+k, r+m, -k^2 m - p^2 r)} \tag{7}$$

Now, to extend the same idea to (3+1)-dimensions we essentially add the dispersion and advection terms involving z:

$$u_t + u_{xxy} + u_{xxz} + 3\left(u\partial_x^{-1}\big(u_y + u_z\big)\right)_x = 0$$

Using the Cole-Hopf substitution and then integrating once on both sides with respect to x while ignoring the integration constant results in:

$$w_{xt} + w_{(3x)y} + w_{(3x)z} + 6w_{xx}w_{xy} + 6w_{xx}w_{xz} = 0$$

Thus, the required Hirota bilinear form of the (3+1)-KdV is given as:

$$\big(D_x D_t + D_x^3 D_y + D_x^3 D_z\big)(f \cdot f) = 0 \tag{8}$$

Here, the dispersion relation is found by assuming $\eta = px + ry + sz + qt$:

$$\big(D_x D_t + D_x^3 D_y + D_x^3 D_z\big)(e^{\eta} \cdot 1) = 0$$

$$pq + p^3 r + p^3 s = 0$$

The obtained dispersion relation is $q = -p^2(r+s)$. Likewise, to calculate A we choose $\xi = kx + my + nz + lt$ and assume $P\big(D_x, D_y, D_z, D_t\big) = D_x D_t + D_x^3 D_y + D_x^3 D_z$:

$$A = -\frac{P\big(p-k, r-m, s-n, k^2(m+n) - p^2(r+s)\big)}{P\big(p+k, r+m, s+n, -k^2(m+n) - p^2(r+s)\big)} \tag{9}$$

One can formulate a $(N+1)$-KdV by introducing the independent variables $x, x_2, \ldots, x_N, t$ and adding up dispersion and advection terms from all spatial axes as follows:

$$u_t + \sum_{i=2}^{N} u_{xxx_i} + 3\left(u\partial_x^{-1}\left(\sum_{i=2}^{N} u_{x_i}\right)\right)_x = 0$$

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Now, we use new notations such as $\nabla u = (u_x, u_{x_2}, \dots, u_{x_N})$ and $\mathbf{I} = (0,1,1,\dots)$ to rewrite the summation term as follows:

$$\mathbf{I} \cdot \nabla u = \sum_{i=2}^{N} u_{x_i}$$

Where "$\cdot$" here denotes the usual dot product. By letting $\mathcal{L}(u) = \mathbf{I} \cdot \nabla u$, we reformulate the generalized anisotropic $(N+1)$-KdV:

$$u_t + \mathcal{L}_{xx}(u) + 3\left(u\partial_x^{-1}(\mathcal{L}(u))\right)_x = 0$$

Using the Cole-Hopf substitution and assuming standard rapidly decaying conditions:

$$w_{xxt} + \mathcal{L}_{(4x)}(w) + 6\big(w_{xx}\mathcal{L}_x(w)\big)_x = 0$$

$$\int \left(w_{xxt} + \mathcal{L}_{(4x)}(w) + 6\big(w_{xx}\mathcal{L}_x(w)\big)_x\right) dx = 0$$

$$w_{xt} + \mathcal{L}_{(3x)}(w) + 6w_{xx}\mathcal{L}_x(w) = 0$$

An important property to realize here being:

$$\frac{D_x^3 D_{x_i}(f \cdot f)}{2f^2} = w_{(3x)x_i} + 6w_{xx}w_{xx_i}$$

Where $2 \le i \le N$. This property naturally implies the following:

$$\frac{D_x^3 \mathcal{L}(D)(f \cdot f)}{2f^2} = \mathcal{L}_{(3x)}(w) + 6w_{xx}\mathcal{L}_x(w)$$

Such that $\mathcal{L}(D) = \mathbf{I} \cdot \nabla(D) = \sum_{i=2}^{N} D_{x_i}$. Thus, the required Hirota bilinear form of the anisotropic $(N+1)$-KdV is given as:

$$\big(D_x D_t + D_x^3 \mathcal{L}(D)\big)(f \cdot f) = 0 \tag{10}$$

To find the dispersion relation, we set $\eta = p_1 x + \sum_{i=2}^{N} p_i x_i + qt$ resulting in the following:

$$\big(D_x D_t + D_x^3 \mathcal{L}(D)\big)(e^{\eta} \cdot 1) = 0$$

$$p_1 q + p_1^3 \sum_{i=2}^{N} p_i = 0$$

The dispersion relation is found to be $q = -p_1^2 \sum_{i=2}^{N} p_i$. Whereas to find the coefficient A we set $\xi = k_1 x + \sum_{i=2}^{N} k_i x_i + lt$ to figure:

$$\big(D_x D_t + D_x^3 \mathcal{L}(D)\big)\big(e^{\eta} \cdot e^{\xi} + Ae^{\eta+\xi} \cdot 1\big) = 0$$

Let $P\big(D_x, D_{x_2}, \dots, D_{x_N}, D_t\big) = D_x D_t + D_x^3 \mathcal{L}(D)$ where P is an even polynomial leading to the following:

$$A = -\frac{P\left(p_i - k_i, k_1^2 \sum_{i=2}^{N} k_i - p_1^2 \sum_{i=2}^{N} p_i\right)}{P\left(p_i + k_i, -k_1^2 \sum_{i=2}^{N} k_i - p_1^2 \sum_{i=2}^{N} p_i\right)}\ ;\ 1 \le i \le N \tag{11}$$

Moving forward, we now discuss the anisotropic extensions of the KP equation. A standard (2+1)-KP equation is given as:

$$(u_t + u_{xxx} + 3(u^2)_x)_x + u_{yy} = 0$$

To find the bilinear form of the KP equation, we utilize the Cole-Hopf substitution as per usual and simplify:

$$w_{(3x)t} + w_{(6x)} + 6(w_{xx}^2)_{xx} + w_{xxyy} = 0$$

Integrating on both sides with respect to x twice and ignoring the occurring integration constants as per rapidly decreasing boundary conditions:

$$\int \left(w_{(3x)t} + w_{(6x)} + 6(w_{xx}^2)_{xx} + w_{xxyy}\right)(dx)^2 = 0$$

$$w_{xt} + w_{(4x)} + 6w_{xx}^2 + w_{yy} = 0$$

Upon using the standard bilinear form identities, we arrive at the required result:

$$\big(D_x D_t + D_x^4 + D_y^2\big)(f \cdot f) = 0 \tag{12}$$

We now construct the dispersion relation by using $\eta = px + ry + qt$.

$$\big(D_x D_t + D_x^4 + D_y^2\big)(e^{\eta} \cdot 1) = 0$$

$$pq + p^4 + r^2 = 0$$

The dispersion relation is given as $q = -p^3 - \frac{r^2}{p}$. To find A we assume $P\big(D_x, D_y, D_t\big) = D_x D_t + D_x^4 + D_y^2$ and $\xi = kx + my + lt$:

$$A = -\frac{P\left(p-k, q-m, k^3 + \frac{m^2}{k} - p^3 - \frac{r^2}{p}\right)}{P\left(p+k, q+m, -k^3 - \frac{m^2}{k} - p^3 - \frac{r^2}{p}\right)} \tag{13}$$

One standard $(N+1)$-KP extension is by utilizing (1+1)-KdV and adding it up with quadratic derivatives of every other spatial variable. This can be given as follows assuming the independent variables are $x, x_2, \dots, x_N, t$:

$$(u_t + u_{xxx} + 3(u^2)_x)_x + \sum_{i=2}^{N} u_{x_i x_i} = 0$$

But do notice that by using $\nabla^2 u = (u_{xx}, u_{x_2 x_2}, \dots, u_{x_N x_N})$ and $\mathbf{I} = (0,1,1,\dots)$, we have:

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$$\mathbf{I} \cdot \nabla^2 u = \sum_{i=2}^{N} u_{x_i x_i}$$

By letting $\mathcal{L}^{(2)}(u) = \mathbf{I} \cdot \nabla^2 u$, we restate the $(N+1)$-KP extension:

$$(u_t + u_{xxx} + 3(u^2)_x)_x + \mathcal{L}^{(2)}(u) = 0$$

Upon using the Cole-Hopf substitution and simplifying further by integrating twice with respect to x gives us:

$$w_{xt} + w_{(4x)} + 6w_{xx}^2 + \mathcal{L}^{(2)}(w) = 0$$

The following property must be known where $\mathcal{L}^{(2)}(D) = \mathbf{I} \cdot \nabla^2(D) = \sum_{i=2}^{N} D_{x_i x_i}$:

$$\frac{\mathcal{L}^{(2)}(D)(f \cdot f)}{2f^2} = \mathcal{L}^{(2)}(w)$$

Thus, the Hirota bilinear form in this case is given as:

$$\left(D_x D_t + D_x^4 + \mathcal{L}^{(2)}(D)\right)(f \cdot f) = 0 \tag{14}$$

For the dispersion relation, we set $\eta = p_1 x + \sum_{i=2}^{N} p_i x_i + qt$:

$$\left(D_x D_t + D_x^4 + \mathcal{L}^{(2)}(D)\right)(e^{\eta} \cdot 1) = 0$$

$$p_1 q + p_1^4 + \sum_{i=2}^{N} p_i^2 = 0$$

Therefore, the dispersion relation is $q = -p_1^3 - \sum_{i=2}^{N} \frac{p_i^2}{p_1}$. To find A in two-soliton case, we say $P\left(D_x, D_{x_2}, \ldots, D_{x_N}, D_t\right) = D_x D_t + D_x^4 + \mathcal{L}^{(2)}(D)$ and $\xi = k_1 x + \sum_{i=2}^{N} k_i x_i + lt$:

$$A = -\frac{P\left(p_i - k_i, k_1^3 + \sum_{i=2}^{N} \frac{k_i^2}{k_1} - p_1^3 - \sum_{i=2}^{N} \frac{p_i^2}{p_1}\right)}{P\left(p_i + k_i, -k_1^3 - \sum_{i=2}^{N} \frac{k_i^2}{k_1} - p_1^3 - \sum_{i=2}^{N} \frac{p_i^2}{p_1}\right)}\ ;\ 1 \leq i \leq N \tag{15}$$

Another generalization of the $(N+1)$-KP equation comes from utilizing anisotropic $(N+1)$-KdV and sum of all quadratic derivative terms excluding x. The assumed independent variables are $x, x_2, \ldots, x_N, t$:

$$\left(u_t + \mathcal{L}_{xx}(u) + 3\left(u \partial_x^{-1}\left(\mathcal{L}(u)\right)\right)_x\right)_x + \mathcal{L}^{(2)}(u) = 0$$

Upon using the Cole-Hopf substitution and simplifying by integrating on both sides with respect to x and ignoring the occurring integration constants, we have:

$$w_{xt} + \mathcal{L}_{(3x)}(w) + 6w_{xx}\mathcal{L}_x(w) + \mathcal{L}^{(2)}(w) = 0$$

By utilizing the earlier discussed properties, we have the required Hirota bilinear form:

$$\left(D_x D_t + D_x^3 \mathcal{L}(D) + \mathcal{L}^{(2)}(D)\right)(f \cdot f) = 0 \tag{16}$$

The dispersion relation is calculated by setting $\eta = p_1 x + \sum_{i=2}^{N} p_i x_i + qt$:

$$\left(D_x D_t + D_x^3 \mathcal{L}(D) + \mathcal{L}^{(2)}(D)\right)(e^{\eta} \cdot 1) = 0$$

$$p_1 q + \sum_{i=2}^{N} (p_1^3 p_i + p_i^2) = 0$$

Thus, the dispersion relation being $q = -\sum_{i=2}^{N} \left(p_1^2 p_i + \frac{p_i^2}{p_1}\right)$. Whereas the value of A in case of two-soliton solution is computed by using $P\left(D_x, D_{x_2}, \ldots, D_{x_N}, D_t\right) = D_x D_t + D_x^3 \mathcal{L}(D) + \mathcal{L}^{(2)}(D)$ and $\xi = k_1 x + \sum_{i=2}^{N} k_i x_i + lt$:

$$A = -\frac{P\left(p_i - k_i, \sum_{i=2}^{N}\left(k_1^2 k_i + \frac{k_i^2}{k_1} - p_1^2 p_i - \frac{p_i^2}{p_1}\right)\right)}{P\left(p_i + k_i, \sum_{i=2}^{N}\left(-k_1^2 k_i - \frac{k_i^2}{k_1} - p_1^2 p_i - \frac{p_i^2}{p_1}\right)\right)}\ ;\ 1 \leq i \leq N \tag{17}$$

Another important hierarchy of generalizations comes from the Boussinesq equations. A standard (1+1)-Boussinesq equation is defined as follows:

$$u_{tt} - u_{xx} - u_{(4x)} - 3(u^2)_{xx} = 0$$

As per usual, Cole-Hopf substitution is implemented to yield the following:

$$w_{xxtt} - w_{(4x)} - w_{(6x)} - 6(w_{xx}^2)_{xx} = 0$$

Integrating on both sides with respect to x twice and ignoring the occurring integration constants.

$$\int \left(w_{xxtt} - w_{(4x)} - w_{(6x)} - 6(w_{xx}^2)_{xx}\right)(dx)^2 = 0$$

$$w_{tt} - w_{xx} - w_{(4x)} - 6w_{xx}^2 = 0$$

Thus, the required Hirota bilinear form for the (1+1)-Boussinesq equation is given as:

$$(D_t^2 - D_x^2 - D_x^4)(f \cdot f) = 0 \tag{18}$$

The dispersion relation here is found by choosing $\eta = px + qt$:

$$(D_t^2 - D_x^2 - D_x^4)(e^{\eta} \cdot 1) = 0$$

$$q^2 - p^2 - p^4 = 0$$

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The dispersion relation is found to be $q = \sqrt{p^2 + p^4}$. Whereas the value of A is given as such where $P(D_x, D_t) = D_t^2 - D_x^2 - D_x^4$ and $\xi = kx + lt$:

$$A = -\frac{P\left(p-k, \sqrt{p^2+p^4} - \sqrt{k^2+k^4}\right)}{P\left(p+k, \sqrt{p^2+p^4} + \sqrt{k^2+k^4}\right)} \tag{19}$$

Where the two-soliton solution is given as $u = 2\left(ln\left(1 + e^{\eta} + e^{\xi} + Ae^{\eta+\xi}\right)\right)_{xx}$. To construct an appropriate anisotropic (2+1)-Boussinesq equation, we rewrite u_{xx} as u_{xy}, $u_{(4x)}$ as $u_{(3x)y}$ and u^2 as $u\partial_x^{-1}(u_y)$:

$$u_{tt} - u_{xy} - u_{(3x)y} - 3\left(u\partial_x^{-1}(u_y)\right)_x = 0$$

On applying Cole-Hopf substitution and integrating twice with respect to x while ignoring the integration constants, we have:

$$w_{xxtt} - w_{(3x)y} - w_{(5x)y} - 6\left(w_{xx}w_{xy}\right)_{xx} = 0$$

$$\int \left(w_{xxtt} - w_{(3x)y} - w_{(5x)y} - 6\left(w_{xx}w_{xy}\right)_{xx}\right)(dx)^2 = 0$$

$$w_{tt} - w_{xy} - w_{(3x)y} - 6w_{xx}w_{xy} = 0$$

By using the earlier prescribed Hirota identities, we have:

$$\left(D_t^2 - D_xD_y - D_x^3D_y\right)(f \cdot f) = 0 \tag{20}$$

The dispersion relation is calculated by choosing $\eta = px + ry + qt$:

$$\left(D_t^2 - D_xD_y - D_x^3D_y\right)(e^{\eta} \cdot 1) = 0$$

$$q^2 - pr - p^3r = 0$$

The obtained dispersion relation is $q = \sqrt{pr + p^3r}$. Whereas the value of A is given as such where $P\left(D_x, D_y, D_t\right) = D_t^2 - D_xD_y - D_x^3D_y$ and $\xi = kx + my + lt$:

$$A = -\frac{P\left(p-k, r-m, \sqrt{pr+p^3r} - \sqrt{km+k^3m}\right)}{P\left(p+k, r+m, \sqrt{pr+p^3r} + \sqrt{km+k^3m}\right)} \tag{21}$$

The anisotropic $(N+1)$-Boussinesq equation is formulated as follows assuming the independent variables to be $x, x_2, \ldots, x_N, t$:

$$u_{tt} - \mathcal{L}_x(u) - \mathcal{L}_{xxx}(u) - 3\left(u\partial_x^{-1}(\mathcal{L}(u))\right)_{xx} = 0$$

Using Cole-Hopf substitution and integrating on both sides with respect to x while ignoring the occurring integration constants.

$$w_{xxtt} - \mathcal{L}_{xxx}(w) - \mathcal{L}_{(5x)}(w) - 6\left(w_{xx}\mathcal{L}_x(w)\right)_{xx} = 0$$

$$\int \left(w_{xxtt} - \mathcal{L}_{xxx}(w) - \mathcal{L}_{(5x)}(w) - 6\left(w_{xx}\mathcal{L}_x(w)\right)_{xx}\right)(dx)^2 = 0$$

$$w_{tt} - \mathcal{L}_x(w) - \mathcal{L}_{xxx}(w) - 6\left(w_{xx}\mathcal{L}_x(w)\right) = 0$$

The Hirota bilinear form of the anisotropic $(N+1)$-Boussinesq equation is given as:

$$\left(D_t^2 - D_x\mathcal{L}(D) - D_x^3\mathcal{L}(D)\right)(f \cdot f) = 0 \tag{22}$$

For the dispersion relation we choose $\eta = p_1x + \sum_{i=2}^{N} p_ix_i + qt$:

$$\left(D_t^2 - D_x\mathcal{L}(D) - D_x^3\mathcal{L}(D)\right)(e^{\eta} \cdot 1) = 0$$

$$q^2 - \sum_{i=2}^{N}\left(p_1p_i + p_1^3p_i\right) = 0$$

The obtained dispersion relation is $q = \sqrt{\sum_{i=2}^{N}(p_1 + p_1^3)p_i}$. Likewise, the value of A is computed by assuming $P\left(D_x, D_{x_2}, \ldots, D_{x_N}, D_t\right) = D_t^2 - D_x\mathcal{L}(D) - D_x^3\mathcal{L}(D)$ and $\xi = k_1x + \sum_{i=2}^{N} k_ix_i + lt$.

$$A = -\frac{P\left(p_i - k_i, \sqrt{\sum_{i=2}^{N}(p_1+p_1^3)p_i} - \sqrt{\sum_{i=2}^{N}(k_1+k_1^3)k_i}\right)}{P\left(p_i + k_i, \sqrt{\sum_{i=2}^{N}(p_1+p_1^3)p_i} + \sqrt{\sum_{i=2}^{N}(k_1+k_1^3)k_i}\right)}\ ;\ 1 \le i \le N \tag{23}$$

III. EXAMPLES AND APPLICATIONS

This established framework can be extended to a myriad of PDEs and one such example is the (2+1)-Calogero-Bogoyavlenskii-Schiff (CBS) equation which can also be anisotropically extended to $(N+1)$-dimensions. A (2+1)-CBS equation is defined as:

$$u_{(3x)y} + 6u_xu_{xy} + 6u_{xx}u_y - u_{tt} = 0$$

It is important to notice that the above PDE can also be restated as follows:

$$u_{(3x)y} + 6\left(u_xu_y\right)_x - u_{tt} = 0$$

The substitution taken here will be of the form $u = (ln\, f)_x = w_x$ and then both sides will be integrated with respect to x while assuming the integration constant to be zero.

$$w_{(4x)y} + 6\left(w_{xx}w_{xy}\right)_x - w_{xtt} = 0$$

$$\int \left(w_{(4x)y} + 6\left(w_{xx}w_{xy}\right)_x - w_{xtt}\right)dx = 0$$

$$w_{(3x)y} + 6w_{xx}w_{xy} - w_{tt} = 0$$

Thus, the required Hirota bilinear form of the (2+1)-CBS is given as:

$$\left(D_x^3 D_y - D_t^2\right)(f \cdot f) = 0 \tag{24}$$

Dispersion relation and value of A are listed as follows assuming $\eta = px + ry + qt$, $\xi = kx + my + lt$ and $P\left(D_x, D_y, D_t\right) = D_x^3 D_y - D_t^2$:

- $q = \sqrt{p^3 r}$
- $A = -\frac{P\left(p-k, r-m, \sqrt{p^3 r} - \sqrt{k^3 m}\right)}{P\left(p+k, r+m, \sqrt{p^3 r} + \sqrt{k^3 m}\right)}$

Similarly, to construct an anisotropic (3+1)-CBS we simply add the dispersion and advection terms based on the spatial variable z:

$$u_{(3x)y} + u_{(3x)z} + 6\left(u_x\left(u_y + u_z\right)\right)_x - u_{tt} = 0$$

By taking the substitution $u = (\ln f)_x = w_x$ and upon integration on both sides with respect to x while assuming the integration constant to be zero, we find the required Hirota bilinear form:

$$w_{(3x)y} + w_{(3x)z} + 6\left(w_{xx}\left(w_{xy} + w_{xz}\right)\right) - w_{tt} = 0$$

$$\left(D_x^3 D_y + D_x^3 D_z - D_t^2\right)(f \cdot f) = 0 \tag{25}$$

The dispersion relation and value of A are listed as follows assuming $\eta = px + ry + sz + qt$, $\xi = kx + my + nz + lt$ and $P\left(D_x, D_y, D_z, D_t\right) = D_x^3 D_y + D_x^3 D_z - D_t^2$:

- $q = \sqrt{p^3(r+s)}$
- $A = -\frac{P\left(p-k, r-m, s-n, \sqrt{p^3(r+s)} - \sqrt{k^3(m+n)}\right)}{P\left(p+k, r+m, s+n, \sqrt{p^3(r+s)} + \sqrt{k^3(m+n)}\right)}$

Finally, we construct anisotropic $(N+1)$-CBS as follows assuming $\mathcal{L}(u) = \mathbf{I} \cdot \nabla u = \sum_{i=2}^{N} u_{x_i}$ and the independent variables are $x, x_2, \ldots, x_N, t$:

$$\mathcal{L}_{xxx}(u) + 6\left(u_x \mathcal{L}(u)\right)_x - u_{tt} = 0$$

Taking the substitution $u = (\ln f)_x = w_x$ and integrating on both sides with respect to x while ignoring the integration constant.

$$\mathcal{L}_{(4x)}(w) + 6\left(w_{xx}\mathcal{L}_x(w)\right)_x - w_{xtt} = 0$$

$$\int \left(\mathcal{L}_{(4x)}(w) + 6\left(w_{xx}\mathcal{L}_x(w)\right)_x - w_{xtt}\right) dx = 0$$

$$\mathcal{L}_{xxx}(w) + 6 w_{xx}\mathcal{L}_x(w) - w_{tt} = 0$$

Therefore, the Hirota bilinear form for $(N+1)$-CBS equation is given as follows where $\mathcal{L}(D) = \mathbf{I} \cdot \nabla D = \sum_{i=2}^{N} D_{x_i}$:

$$\left(D_x^3 \mathcal{L}(D) - D_t^2\right)(f \cdot f) = 0 \tag{26}$$

Now, the dispersion relation is computed by letting $\eta = p_1 x + \sum_{i=2}^{N} p_i x_i + qt$:

$$\left(D_x^3 \mathcal{L}(D) - D_t^2\right)(e^{\eta} \cdot 1) = 0$$

$$p_1^3 \sum_{i=2}^{N} p_i - q^2 = 0$$

The dispersion relation is found to be $q = \sqrt{p_1^3 \sum_{i=2}^{N} p_i}$. To find the value of A we simply assume $\xi = k_1 x + \sum_{i=2}^{N} k_i x_i + lt$:

$$A = -\frac{P\left(p_i - k_i, \sqrt{p_1^3 \sum_{i=2}^{N} p_i} - \sqrt{k_1^3 \sum_{i=2}^{N} k_i}\right)}{P\left(p_i + k_i, \sqrt{p_1^3 \sum_{i=2}^{N} p_i} + \sqrt{k_1^3 \sum_{i=2}^{N} k_i}\right)}\ ;\ 1 \le i \le N \tag{27}$$

Next up, we will be discussing variable coefficient KP equation and its anisotropic generalizations. But first let us define variable coefficient (2+1)-KP equation:

$$\left(u_t + u_{xxx} + 3(u^2)_x\right)_x + F(t)u_{yy} = 0$$

The bilinear form can be constructed with ease by using Cole-Hopf substitution and integrating twice on both sides with respect to x while ignoring the integration constants.

$$w_{(3x)t} + w_{(6x)} + 6(w_{xx}^2)_{xx} + F(t)w_{xxyy} = 0$$

$$\int \left(w_{(3x)t} + w_{(6x)} + 6(w_{xx}^2)_{xx} + F(t)w_{xxyy}\right)(dx)^2 = 0$$

$$w_{xt} + w_{(4x)} + 6w_{xx}^2 + F(t)w_{yy} = 0$$

The constructed Hirota bilinear form of the variable coefficient (2+1)-KP equation is stated as:

$$\left(D_x D_t + D_x^4 + F(t)D_y^2\right)(f \cdot f) = 0 \tag{28}$$

The dispersion relation is obtained by choosing $\eta = px + ry + q(t)$ where q is a function in t:

$$\left(D_x D_t + D_x^4 + F(t)D_y^2\right)(e^{\eta} \cdot 1) = 0$$

$$pq'(t) + p^4 + F(t)r^2 = 0$$

The dispersion relation is found to be as follows assuming C is an arbitrary constant of integration:

$$q(t) = -p^3 t - \frac{r^2}{p}\int F(t)\, dt + C \tag{29}$$

Now, variable coefficient $(N+1)$-KP equation is constructed as follows assuming the independent variables to be $x, x_2, \ldots, x_N, t$:

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$$(u_t + u_{xxx} + 3(u^2)_x)_x + \sum_{i=2}^{N} F_i(t) u_{x_i x_i} = 0$$

By using Cole-Hopf substitution and then integrating twice on both sides with respect to x while ignoring the integration constants, we have:

$$w_{(3x)t} + w_{(6x)} + 6(w_{xx}^2)_{xx} + \sum_{i=2}^{N} F_i(t) w_{xxx_i x_i} = 0$$

$$\int \left(w_{(3x)t} + w_{(6x)} + 6(w_{xx}^2)_{xx} + \sum_{i=2}^{N} F_i(t) w_{xxx_i x_i} \right) (dx)^2 = 0$$

$$w_{xt} + w_{(4x)} + 6w_{xx}^2 + \sum_{i=2}^{N} F_i(t) w_{x_i x_i} = 0$$

The Hirota bilinear form of variable coefficient $(N+1)$-KP equation is:

$$\left(D_x D_t + D_x^4 + \sum\nolimits_{i=2}^{N} F_i(t) D_{x_i}^2 \right) (f \cdot f) = 0 \tag{30}$$

The dispersion relation is calculated by choosing $\eta = p_1 x + \sum_{i=2}^{N} p_i x_i + q(t)$ and is given as:

$$q(t) = -p_1^3 t - \frac{1}{p_1} \int \sum\nolimits_{i=2}^{N} p_i^2 F_i(t)\, dt + C \tag{31}$$

With this the overall material for this paper remains concluded.

IV. CONCLUSIONS

Bilinear forms of the KdV equation were calculated in (1+1), (2+1), (3+1) and $(N+1)$-dimensions were computed by using the Cole-Hopf substitutions and appropriate dispersion relations along with relevant coefficient values were also derived. Bilinear forms of (2+1)-KP equation, generalized $(N+1)$-KP equation as well as $(N+1)$-KP equation constructed using $(N+1)$-KdV were computed accordingly using Cole-Hopf substitutions and relevant dispersion relations were also derived. Bilinear forms of the (1+1), (2+1) and $(N+1)$-Boussinesq equation were computed systematically along with the appropriate dispersion relations and coefficient values.

➢ *Declaration of Competing Interests*

The author hereby declares that there are no existing competing interests or any arrangements as such that could potentially influence the work presented in this paper.

➢ *Data Availability*

No additional data was used or analysed in the course of this systematic study.

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