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Study of Simplicial Methods with Stability

Ashaq Hussain Bhat[1]; Dr. R.S. Patel[2]

[1]Ph.D. (Mathematics) Research Scholar Department of Mathematics,
Govt. Autonomous P.G. College Satna (M.P.) India
[2]Research Supervisor Department of Mathematics,
Govt. Autonomous P.G. College Satna (M.P.) India

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Abstract: This paper begins with the introduction of the concepts of standard triangulation, essential set, approximate essential set etc. This study presents some new findings about simplicial algorithms that take into account the continuities of approximate fixed point sets. The subsistence of finite essential connected components in approximate fixed point sets by vector-valued labels is obtained by proving the upper semi-continuity of a set-valued mapping of approximate fixed points with the help of simplicial methods that are vector-valued; examples are provided to demonstrate how this differs significantly from the property for integer-valued labeling simplicial methods. It is also demonstrated that essential sets exist by concentrating on both domain and function perturbations. Finally, the paper is concluded in section 5.

Keywords: *Triangulation; Simplex; Stability; Approximate Fixed Point.*

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I. INTRODUCTION

In the fields of mathematics and economics, fixed point theorems have significant implications. The well-known Brouwer fixed point theorem [5] is essential to many existence problems and has spurred a surge of equilibria discovery and other uses, including such as computer science [12], network problems [8-10], approximation theory [11], the Nash equilibrium [6], the general equilibrium [7], etc.

Computation of brouwer fixed point by different algorithms is an important field.

As is widely famous, Sperner's lemma evolved into a straightforward method for demonstrating the existence of Brouwer type fixed points. Sperner's lemma continues to be the foundation for simplicial algorithms following Scarf's [13] outstanding work, including the restart algorithms [16,18], Kuhn's algorithm [14,15], variable dimension algorithm [19] and homotopy algorithms [20,21]. Finding a complete labeled sub-simplex in a simplex for the approximation of a fixed point is a familiar practice for simplicial algorithms. Vector-valued and integer-valued are two popular labels. A complete vector-valued sub-simplex to fixed points is known to have a better approximation degree than the other given a set grid size. We will see whether the stability of these algorithms differ. Also we will focus on whether functions or simplices be perturbed without affecting a complete labeled sub-simplex.

There has been a lot of interest in fixed point stability. Essential sets of fixed points and essential components were presented[23,24] following the groundbreaking work for essential fixed points of continuous functions (Brouwers fixed points) in [22]. Minimal essential sets appear to be reasonable choices from the standpoint of stability, since they are the analogues of singletons [25]. Numerous problems, including coincidence points [26, 27], fixed points [28], KKM points [29, 30], game equilibrium points [31–35], maximal elements [36], variational relation problems [37,39], and many other, were analyzed using essential stabilities that are associated with lower semi-continuity.

II. PRELIMINARIES

Consider S be an m-simplex in $R^{m+1}$with vertices $v^1, v^2, \dots, v^{m+1}$. With $I_k = \{1, 2, \dots., k\}$ and uniform metric,$C(S)$ be the space of continuous mappings g on S. Denote ith unit vector of R^{m+1} by $e(i), i = 1, 2, \dots, m$, and the $(m+1)$-vector$(1,1, \dots, 1)^T$is denoted by e.

A few definitions pertaining to simplicial fixed point algorithms are recalled. If we have the grid size $\frac{1}{p}$, then the standard triangulation of S is the collection of all sub-simplices $\sigma(y^1, \pi)$ with vertices $y^1, y^2, \dots, y^{m+1}$ in S such that:

- each element of y^1 is a multiple of $\frac{1}{p}$;
- $\pi = (\pi_1, \dots, \pi_m)$ is a permutation of the members of I_m;

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- $y^{i+1} = y^i + \frac{v(\pi_{i+1})-v(\pi_i)}{p}$, where $v(i) = v^i, \forall i \in I_m$.

This must be noted that the mesh of the standard triangulation of S having the grid size $\frac{1}{p}$ is $\frac{\sqrt{m+1}}{p}$ or $\frac{\sqrt{m}}{p}$ if m is odd or even, respectively. For a mapping $g \in C(S)$, a point z in S is labeled an integer $l(z) \in I_{m+1}$where $l(z) = i$ if

$$i = min\{j | g_j(z) - z_j = {}_{h\in I_{m+1}}^{min}(f_h(z) - z_h)|\}.$$

Explicitly, if $f(z) = z$ and $z_1 = 0$, then allocate the label of z as the first index i such that $z_i > 0, i \in I_{m+1}$,We call $l: S \to I_{m+1}$a standard integer-valued labeling mapping. According to Sperner's lemma, when we have the mesh of a standard triangulation on S, there is at least a sub-simplex having complete integer labels (a fully labeled simplex, where the labels of the sub-simplex's vertices are completely distinct).

If the $(m + 1)$-vector $L(z)$, where $L(z) = - g(z) + z + e$, is received by a point z in S,

$L: S \to R^{m+1}$is referred to as a standard vector-valued labeling function in this context. A sub-simplex $\sigma(y^1, y^2, \ldots, y^{m+1})$ with vector-valued labels for a triangulation of S is complete if $\sum_{i=1}^{m+1} \lambda_i L(y_i) = e$ has a solution $\lambda^* = (\lambda_1, \lambda_2, \ldots, \lambda_{m+1})$ with $\lambda^* \in R_+^{m+1}$.

Given a grid size $\frac{1}{p}$, the assemblage of all the sub-simplices having complete integer-valued (vector-valued) labels in S is represented by $G(g,p)(G'(g,p))$, for each $g \in C(S)$. Let us now define aset-valued mapping from $C(S)$ to S with $G(G'): C(S) \to 2^S$, additionally, for notations to be convenient, we write $G(g)$ as $G(g,p)$. Note that each $z \in G(g,p), (G'(g,p))$ is an approximation of fixed points of g on S. It is possible to decompose $G(g,p), (G'(g,p))$ as $\cup_{i\in V} C_i$with $c_i \cap c_j = \emptyset$ for any $i \neq j$, and $c_i, \forall i \in \wedge$, is a connected element based on connectedness.

The upcoming example demonstrates that a set-valued function G is not upper semi-continuous on $C(S)$, and there are notable distinctions between G and G' with regard to semi-continuities; further outcomes we will show in Section 3.

- **Example 2.1:** Consider S as a standard simplex in R^2. The identity of a map $g \in C(S)$ is $g(z) = z, \forall z \in S$. For the integer labels of the sub-simplicies of the triangulation having the grid size $\frac{1}{p}$, given that $\frac{1}{p} = \frac{1}{2}$,we obtain

$$l(z) = \begin{cases} 2, & z = (0,1), \\ 1, & z = (1,0), \\ 1, & z = \left(\frac{1}{2}, \frac{1}{2}\right), \end{cases}$$

Here, we can check that $G(g,p) = \left\{(z_1, 1-z_1) | 0 \leq z_1 \leq \frac{1}{2}\right\}$. For each $m = 1, 2, \ldots$, define $g^m \in C(S)$, satisfying

$$f^m(z_1, z_2) = \left((z_1)^{\frac{m}{1+m}}, 1 - (z_1)^{\frac{m}{1+m}}\right)$$

Then, the corresponding integer labels using g^m for each $m = 1, 2, \ldots$ is same as

$$l(z) = \begin{cases} 2, & z = (0,1), \\ 1, & z = (1,0), \\ 2, & z = \left(\frac{1}{2}, \frac{1}{2}\right), \end{cases}$$

It can be calculated that $G(g^m, p) = \left\{(z_1, 1 - z_1) \mid \frac{1}{2} \leq z_1 \leq 1\right\}$. Clearly, for a sufficiently small openset U with $G(g,p) \subset$ U, we have $G(g^m, p) \not\subset U$ however close g^m is to g. Consequently, G is not upper semi-continuous on $C(S)$, therefore, the graph of G is not closed also. Obviously, G is not lower semi-continuous either.

Let us denote the fixed point set of g on S for each $g \in C(S)$ by Fix(g). The following definitions take into consideration a type of illustration for stability of $G'$and subsets of Fix(g) as shown in Example 2.1 regarding G.

- **Definition 2.1:** If for any open set U we have U $\supset e(g)$,there exists an open neighbourhood $O(g)$ of g in $C(S)$ such that $G'(g', p) \cap$ U $\neq \emptyset, \forall g' \in O(g)$, then a closed subset $e(g)$ of $G'(g,p)$,for each $g \in C(S)$, with the grid size $\frac{1}{p}$ is said to be an essential set with respect to $C(S)$ .If a connected component $C \subset G'(g,p)$ is an essential set, C is said to be an essential connected component of $G'(g,p)$ with respect to $C(S)$.
- **Definition 2.2:** Let $g \in C(S)$, a closed subset of $Fix(g)$ be $e(g)$. $e(g)$ is said to be an approximate essential set if for each ϵ neighbourhood, $B(e(g), \epsilon)$ of $e(g)$ there exists a $\kappa > 0$ such that, for each $g' \in C(S)$ with $\|g - g'\| < \kappa$, we have N a number, such that $G'(g', \text{p}) \cap B(e(g), \epsilon) \neq \emptyset \; \forall p >$ N.
- **Lemma 2.1:** Let E be a Baire space, Y be a metric space and $G : E \to 2^Y$ be a mapping with compact values that is upper semi-continuous. Hence, there exists a dense residual subset Q of E in such a way G is lower semi-continuous at each $z \in Q$. ([40]).

III. RESULTS OF STABILITY WHEN FUNCTIONS ARE PERTURBED

- **Theorem 3.1:** Given a grid size $\frac{1}{p}$ and a triangulation of S comprising vertices $v^1, v^2, \ldots, v^{m+1}$, the graph of the set-valued mapping G',

$GrG' = \{(g, z) \mid g \in C(S), z \in G'(g, p)\}$, is closed.

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- **Proof:** Suppose $(g^n, z^n) \in GrG'$ with $(g^n, z^n) \to (g^0, z^0), n = 1,2, \ldots$. Clearly $(g^0, z^0) \in C(S) \times S$. With vector-valued labels in S we must show that z^0 is a point of a complete sub-simplex α_{g_0}. Since $(g^n, z^n) \in GrG'$, for each $n = 1,2, \ldots,$ there exists a complete labeled sub-simplex α_{g_n} such that $z^n \in \alpha_{g^n} \subset G'(g^n, p) \subset S,$ hence, denote α_{g^n} as $\alpha_{g^n}(y_n^1, y_n^2, \ldots, y_n^{m+1}) = \alpha_{g^n}(y_n^1, \pi_n)$.

Since $\{\pi_n^1\}$ belongs to the finite set I_{m+1}, $\{\pi_n^1\}$ has a convergence subsequence $\{\pi_{nk}^1\}$, such that $\pi_{ni}^1 = \pi_{nj}^1$ for very large i and j where $i \neq j$. For convenience of notation, we may also find such a convergence subsequence for $\{\pi_{nk}^2\}$, that is represented by $\{\pi_{nk}^2\}$. This approach can be unified as one $\{\pi^i\}$ with $\pi^i \neq \pi^j, \forall i \neq j,$ i.e $\alpha_{g^{nk}}(y_{nk}^1, \pi_{nk}) = \alpha_{g^{nk}}(y_{nk}^1, \pi)$ and will then yield a convergence subsequence $\{\pi_{nk}^i\}$ of $\{\pi_n^i\}$. Since $\{y_{nk}^1\} \subset Z$, then we have a sequence, which is its convergence subsequence, without losing generality,we also indicate it by $\{y_{nk}^1\}$ with $\{y_{nk}^1\} \longrightarrow y_0^1, k$ tends to infinity. Through the selection of a few real numbers $q_{nk}^i, i \in I_{m+1}$, we can write $\alpha_{g^{nk}}(y_{nk}^1, \pi)$, as

$$y_{nk}^1 = (q_{nk}^1, q_{nk}^2, \ldots, q_{nk}^{m+1})/p$$

And

$$y_{nk}^{i+1} = y_{nk}^i + \left(v^{\pi^{i+1}} - v^{\pi^i}\right)/p, \forall i| \in I_{m.}$$

Then, we get a point y_0^i such that $y_{nk}^i \longrightarrow y_0^i \in S$ for each $i \in I_{m+1}$. That is, $\alpha(y_0^1, \pi) = \alpha(y_0^1, y_0^2 \ldots, y_0^{m+1})$ is obviously a sub-simplex in the triangulation of S under the grid size $\frac{1}{p}$. Remember, $(g^{nk}, z^{nk}) \in GrG'$ with $(g^{nk}, z^{nk}) \to (g^0, z^0)$ as $k \to \infty$. Since $\alpha(y_{nk}^1, y_{nk}^2, \ldots, y_{nk}^{m+1})$ with vector valued labels is a complete sub-simplex there exists a non-negative vector $(\lambda_{nk}^1, \lambda_{nk}^2, \ldots, \lambda_{nk}^{m+1})$ such that

$$\sum_{i=1}^{m+1} \lambda_{nk}^i\left(-g^{nk}(y_{nk}^i) + y_{nk}^i + e\right) = e. \tag{1}$$

We have convergence subsequences $\{\lambda_{nkj}^i\}$ of $\{\lambda_{nk}^i\}$ with $\lambda_{nkj}^i \to \lambda_o^i \geq 0\ (j \to \infty), \forall i \in I_{m+1}$. Now substitute nk with nkj in equation (1), as $j \to \infty$, we have

$$\sum_{i=1}^{m+1} \lambda_0^i\left(-g^0(y_0^i) + y_0^i + e\right) = e. \tag{2}$$

Therefore, $\alpha_{g^0} = \alpha(y_0^1, y_0^2, \ldots, y_0^{m+1})$ with vector-valued labels is a complete sub-simplex. Also, we have $z^0 \in \alpha_{g^0}$. Since $z^{nkj} \in \alpha_{g^{nkj}}, \exists\ \beta_{nkj}^i \geq 0$ such that

$$z^{nkj} = \sum_{i=1}^{m+1} \beta_{nkj}^i\, y_{nkj}^i \tag{3}$$

With $\sum_{i=1}^{m+1} \beta_{nkj}^i = 1$. Without losing generality, suppose that β_{nkj}^i is convergent with the limit β_0^i, that is, $\beta_{nkj}^i \to \beta_0^i (j \to \infty)$. Then, as $(j \to \infty)$ for equation (3), we have $z^0 = \sum_{i=1}^{m+1} \beta_0^i\, y_0^i \in \alpha_{g^0}$.

We have the following direct corollary From Theorem 3.1.

- **Corollary 3.1** The set-valued mapping G' is upper semi-continuous on $C(S)$ given a triangulation of S with a grid size of $1/p$.

G' is not lower semi-continuous on $C(S)$, as demonstrated by the example that follows.

- **Example 3.1.** Let a standard simplex in R^2 be S, $g \in C(S)$ be an identity mapping i.e $g(z) = z, \forall\, z \in S$, For the vector-valued labels of the sub-simplices of the triangulation with g, for each point in grid $z = \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right), L(z) = (1, 1)$, when the grid size $\frac{1}{p}$ with $\frac{1}{p} = \frac{1}{4}$.

We have, the sub-simplex

$\alpha = \{(z_1, z_2) \in S : 1/4 \leq z_1 \leq 1/2\}$ is complete and $\alpha \subset G'(g\ , p)$.

We choose a point $\bar{z} = (3/8, 5/8) \in \alpha$. For each $m = 1,2, \ldots,$ define $g^m \in C(S)$ such that

$$g^m(z_1, z_2) = \left((z_1)^{\frac{m+1}{m}}, 1 - (z_1)^{\frac{m+1}{m}}\right)$$

So, for each $m = 1,2, \ldots,$ the vector-valued labels for the sub-simplex α using g^m is

$$L(z) = \begin{cases} \left(\frac{5}{4} - \left(\frac{1}{4}\right)^{\frac{m+1}{m}}, \left(\frac{1}{4}\right)^{\frac{m+1}{m}} + \frac{3}{4}\right) = (r, s), & z = (1/4, 3/4), \\ \left(\frac{3}{2} - \left(\frac{1}{2}\right)^{\frac{m+1}{m}}, \left(\frac{1}{2}\right)^{\frac{m+1}{m}} + \frac{1}{2}\right) = (t, u), & z = (1/2, 1/2), \end{cases}$$

So, the right-hand side of equation

$$\begin{bmatrix} r & t \\ s & u \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{ru - ts} \begin{bmatrix} u - s \\ r - t \end{bmatrix}$$

Is the solution $(\lambda_1^*, \lambda_2^*)$ of the equations $\lambda_1 L\left(\frac{1}{4}, \frac{3}{4}\right) + \lambda_2 L\left(\frac{1}{2}, \frac{1}{2}\right) = e$, then $\lambda_2^* = \frac{r-s}{ru-st}$, by a direct calculation, for each $m = 1, 2, \ldots,$ we get, $r - s = \frac{4^{\frac{1}{m}} - 1}{2.4^{\frac{1}{m}}} > 0,$ while $ru - ts = \frac{\left(2 - 2^{\frac{1}{m}}\right) 2^{\frac{1}{m}} - 1}{2.4^{\frac{1}{m}}} < 0.$ Then $\lambda_2^* < 0, \forall\, m = 1,2, \ldots$ .Thus, one can see that α is not complete by labeling the sub-simplex α using g^m. Hence, for very small open neighborhood U of $\bar{z}$, we get $G'(g_m, p) \cap U = \emptyset$ for each $m = 1, 2, \ldots,$ that is, G' is not lower semi-continuous on $C(S)$.

The set-valued mapping G' with $G' : C(S) \to 2^s$ is upper semi-continuous according to Theorem 3.1. If we have a grid size $\frac{1}{p}$, it is clear, each point in $G'(g^0, p)$ is

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essential if the set-valued mapping G' is lower semi-continuous at a point g^0. Therefore, the following generic stability result can be obtained using Fort's lemma (Lemma 2.1) and Definition 2. 1.

- **Corollary 3.2** We always have a dense residual set H in $C(S)$ such that for each $g \in C(S)$, given a grid size $\frac{1}{p}$, each point in $G'(g,p)$ is essential with respect to $C(S)$.
- **Theorem 3.2:** For each $g \in C(S), \exists$ finite essential connected components in $G'(g,p)$ with respect to $C(S)$, if we have a triangulation of S having grid size $\frac{1}{p}$.
- **Proof:** The set-valued mapping G' is upper semi-continuous on S, as stated in Theorem 3.1. Then, with respect to $C(S)$ the set $G'(g,p)$, itself is anessential set. Suppose the collection of all essential sets in $G'(g,p)$ be denoted by θ. Keep in mind that the intersection of any descending chain in θ having the specified inclusion order serves as a lower bound. Consequently, an essential set in $G'(g,p)$ is a minimal element $e(g)$ in θ. Therefore, by definition 2.1, it is evident that every connected component C with $C \supset e(g)$ is an essential connected component. The next challenge is to demonstrate that each $e(g)$ is connected.

Otherwise, let $e(g) = D^1 \cup D^2$. Two open sets U^1 and U^2 can be used to separate nonessential closed sets D^1 and D^2 with $D^i \subset U^i$, $i = 1,2$. For each $i = 1,2$ and $\varepsilon > 0$, there exists an open set $W^i$and $g^i \in C(S)$ with $D^i \subset W^i \subset \overline{W}^i \subset U^i$ such that $\|g - g^i\| < \frac{\varepsilon}{3}$ but $G'(g^i,p) \cap W^i = \emptyset$; in the interim, for any $g' \in C(S)$ with $\|g' - g\| < \varepsilon$, we have $G'(g',p) \cap (W^1 \cup W^2) \neq \emptyset$. Construct a new $g' \in C(S)$ by defining.

$$g'(z) = \lambda(z)g^1(z) + \big(1 - \lambda(z)\big)g^2(z)\ \forall z \in S,$$

Where, $\lambda(z) = \frac{d(z,\overline{W}^2)}{d(z,\overline{W}^1)+d(z,\overline{W}^2)}$. $\|g' - g\| < \epsilon$ can be checked routinely, this means that there is at least a point z such that $z \in G'(g',p) \cap (W^1 \cup W^2)$. For each $i = 1,2$, if $z \in W^i$, such that $g'(z) = g^i(z)$, then the labels for the sub-simplice's vertices in W^i using g^i or g are same. Therefore, $G'(g^i,p) \cap W^i = G'(g',p) \cap W^i$, from which we can deduce the fact that $(z \notin G'(g',p)$, is a contradiction.

Finally, the result is derived from the finiteness of the complete labeled simplex in S. As $p \to \infty$ the following result shows that essential connected components under the grid size $\frac{1}{p}$ could be very close to an approximate fixed point set when $p \to \infty$.

- **Theorem 3.3:** For each grid size $\frac{1}{p}$, we have a continuous function $g \in C(S)$, let $C^p \subset G'(g,p)$ with respect to $C(S)$ be an essential connected component, there exists a subsequence $\{C^{pk}\}$ of $\{C^p\}$ with $C^{pk} \xrightarrow{h} C^0$ and C^0 is an approximate essential connected set in Fix(g), where h is the Hausdorff metric induced by the Euclidean metric on R^{m+1}.

- **Proof:** we have $\{C^p\}$ is a sequence in $L(S)$, where $L(S)$ is the collection of nonempty compact subsets of S. Since S is compact so there is a subsequence $\{C^{pk}\}$ of $\{C^p\}$ with the limit $C^0 \in L(S)$. For convenience, we denote the subsequence just as $\{C^p\}$. For each $z^0 \in C^0$, there is a sequence $\{z^p\}$ with $z^p \in C^p$ and $z^p \to z^0$. For each $\varepsilon > 0$, there exists a number N since g is continuous, such that, for each sub-simplex α_g in the triangulation of S under the grid size $\frac{1}{N}$, we have

$$\max_{i \in I_{m+1}}\{|g_i(z) - g_i(y)|\} < \frac{1}{N} < \frac{\varepsilon}{3\sqrt{m+1}}\ , \forall z, y \in \alpha_g.$$

For each $p > N$, since $z^p \in C^p \subset G'(g,p)$, we have $\|g(z^p) - z^p\| < \frac{\varepsilon}{3}$. Next, we can identify a sufficiently large p such that:

$$\|g(z^0) - z^0\| \le \|g(z^0) - g(z^p)\| + \|g(z^p) - z^p\| + \|z^p - z^0\|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ holds.}$$

Therefore, we claim that $z^0 \in Fix(g)$. hence, $C^0 \subset Fix(g)$.

Let us suppose that C^0 is not connected, henceC^0 can be split as two disjoint compact sets like $C^0 = C' \cup C''$ with two open sets W' and W'' such that $C' \subset W'$, $C'' \subset W''$ and $W' \cap W'' = \emptyset$. From compactness of C' and C'', we have two open sets U' and U'' such that $C' \subset U' \subset \overline{U}' \subset W'$ and $C'' \subset U'' \subset \overline{U}'' \subset W''$. Since C^p is connected, we get $C^p \subset U'$ or $C^p \subset U''$ as p sufficiently large. So, the limit of C^p is in W' or W'', that contradicts the fact that $C^p \xrightarrow{h} C'UC''$ and $C' \subset W'$, $C'' \subset W''$, and $W' \cap W'' = \emptyset$. Therefore, C^0 is connected.

Lastly, we demonstrate that C^0 is an approximate essential set of Fix(g). Otherwise, then there exists $\bar{\varepsilon} > 0$ and $g^j (j = 1,2, \dots)$ with $g^j \to g$, such that for each number p, $G'(g^j,p) \cap B(C^0,\bar{\varepsilon}) = \emptyset$, $j = 1,2, \dots$. Since $C^p \xrightarrow{h} C^0$, there is a number N such that $C^p \subset B(C^0,\bar{\varepsilon})$ when $p \ge N$. Since C^N is essential, for the open set $B(C^0,\varepsilon)$, there is a $\kappa > 0$ such that for any g' with $\|g - g'\| < \kappa$, we have $G(g',N) \cap B(C^0,\varepsilon) \neq \emptyset$. From the fact that $g^j \to g$, for sufficiently large j, we have $G(g^j,N) \cap B(C^0,\varepsilon) \neq \emptyset$, a contradiction.

IV. RESULTS OF STABILITY WHEN SIMPLICES AND FUNCTIONS ARE PERTURBED

To have an analysis of the perturbation of domains, let $Z \subset R^{m+1}$ be a m dimensional compact set, T be the set of all m-simplex in Z. Take any two $S_1(v_1^2, v_1^2, \dots, v_1^{m+1})$ and $S_2(v_2^1, v_2^2, \dots, v_2^{m+1})$ in T, define

$$\eta(S_1, S_2) = \min_{\pi} \sum_{k=0}^{m+1} \left\|v_1^k - v_2^{\pi k}\right\|$$

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- **Lemma 4.1** η is a metric on T.
- **Proof:**
- For any $S_1(v_1^1, v_1^2, \dots, v_1^{m+1})$, $S_2(v_2^1, v_2^2, \dots, v_2^{m+1}) \in T$, we have $\eta(S_1,S_2)= \eta(S_2,S_1)$.

Let $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_{m+1})$ match $\eta(S_1,S_2)$. We have

$$\eta(S_1,S_2) = \sum_{k=0}^{m+1}\left\|v_1^k - v_2^{\bar{\pi}k}\right\|. \text{ Then}$$

$$\eta(S_2,S_1) = \frac{min}{\pi}\sum_{k=0}^{m+1}\left\|v_2^{\bar{\pi}k} - v_1^{\pi k}\right\|$$

$$=\sum_{k=0}^{m+1}\left\|v_2^{\bar{\pi}k} - v_1^k\right\| = \eta(S_1,S_2).$$

- For any $S_1(v_1^1, v_1^2, \dots, v_1^{m+1}), S_2(v_2^1, v_2^2, \dots, v_2^{m+1}) \in T$,

We have $\eta(S_1,S_2) = 0 \Leftrightarrow S_1 = S_2$.

We need only the proof of the necessity

From the definition of η. Let $\eta(S_1,S_2) = 0$, then there

Exists $\bar{\pi}$ such that $\sum_{k=0}^{m+1}\left\|v_1^k - v_2^{\bar{\pi}k}\right\| = 0$, which means

That $\left\|v_1^k - v_2^{\bar{\pi}k}\right\| = 0, \forall k \in I_{m+1}$. That is $S_1 = S_2$.

- For any $S_1(v_1^1, v_1^2, \dots, v_1^{m+1})$, $S(v^1, v^2, \dots, v^{m+1})$, $S(v_2^1, v_2^2, \dots, v_2^{m+1}) \in T$, we have$\eta(S_1,S_2)$ =≤ $\eta(S_1, S)$ + $\eta(S, S_2)$. Let $\eta(S_1, S)$ =$\sum_{k=0}^{m+1}\left\|v_1^k - v^{\bar{\pi}k}\right\|$. Then we have

$$\eta(S_1,S_2) = \frac{min}{\pi}\sum_{k=0}^{m+1}\left\|v_1^k - v_2^{\pi k}\right\|$$

$$\leq \frac{min}{\pi}\left(\sum_{k=0}^{m+1}\left\|v_1^k - v^{\bar{\pi}k}\right\| + \sum_{k=0}^{m+1}\left\|v^{\bar{\pi}k} - v_2^{\pi k}\right\|\right)$$

$$= \sum_{k=0}^{m+1}\left\|v_1^k - v^{\bar{\pi}k}\right\| + \frac{min}{\pi}\sum_{k=0}^{m+1}\left\|v^{\bar{\pi}k} - v_2^{\pi k}\right\|$$

$$= \eta(S_1,S) + \frac{min}{\pi}\sum_{k=0}^{m+1}\left\|v^k - v_2^{\pi k}\right\|$$

$$= \eta(S_1,S) + \eta(S,S_2).$$

We want to restrict domains to escape from a domain perturbed in a large-scale range with regard to a stability analysis of approximate fixed points. Let Δ be a subset of a compact set Z in R^{m+1} with m dimensions. Let $T' \subset T$ satisfy $T' = \{S \in T: \Delta \subset S \subset Z\}$.

- **Lemma 4.2:** Prove that the metric space (T',η) is complete.
- **Proof:** Choose $\{S_n(v_n^1, v_n^2, \dots, v_n^{m+1})\}$ a Cauchy sequence in T'. Therefore, for each $\varepsilon > 0, \exists$, a number N such that $\eta(S_s,S_t) < \varepsilon$ for any $s,t > N$. It is possible to assume that $\eta(S_s,S_t) = \sum_{k=0}^{m+1}\left\|V_s^k - V_t^k\right\|$ without losing generality . Therefore, $\{V_n^k\}$ is a Cauchy sequence with the limit $v_0^k, \forall k \in I_{m+1}$. Denote the simplex $S_0(v_0^1, v_0^2, \dots, v_0^{m+1})$ by S_0. Then we have $\eta(S_n,S_0) \to 0$. Since $\Delta \subset \cap_{n=1}^{\infty} S_n \subset Z$, it follows that $\Delta \subset S_0 \subset Z$, hence S_0 is an m-simplex in T'.

Consider Q as the set of pairs (g,S) such that$Q = \{(g\ ,S) \in C(Z) \times T' :\ g\ (z) \in S, \forall z \in S\}$

Let us now define the metric d between two $u_1 = (g_1,S_1)$ and $u_2 = (g_2,S_2)$ in T'as

$$d(u_1,u_2) = \frac{max}{z \in Z}\|g_1(z) - g_2(z)\| + \eta(S_1,S_2).$$

We establish a set-valued mapping R from Q to Z having a grid size of $1/p$. For each $u = (g,S) \in Q$, let $R(u,p)$ be the collection of all sub-simplices with complete vector-valued labels with the mapping g in the triangulation of S having the grid size of $1/p$.

We take into account the essential stability of approximation fixed points under function and domain perturbations, just like in Definition 2.1.

- **Definition 4.1:** If we have the grid size $\frac{1}{p}$, for each $u = (g\ ,S) \in Q$, we call a closed subset $e(g\ )$ in $R(u,p)$ an essential set with respect to Q if, for any open set U with $U \supset e(g)$, there existsan open $O(u)$ of u in Q such that $U \cap R(u',p) \neq \emptyset, \forall u' \in O(u)$. A minimal essential set with respect to Q is a minimal element in the collection of essential sets in $R(u,p)$ (arranged by set inclusion).
- **Theorem 4.1:** If we have a grid size $\frac{1}{p}$, and a continuous mapping $g \in C(Z)$, then the graph of the set-valued function R, $Gr\ R = \{(u,z)\,|\,u \in Q, z \in R(u,p)\}$, is closed.
- **Proof:** Let $\{(u_n,z_n)\} \subset Gr\ R$ with $(u_n,z_n) \to (u_0,z_0)$ where $u_n = (g_n,S_n), u_0 = (g_0,S_0)$ and S_n is the simplex with $v_n^1, v_n^2, \dots\dots, v_n^{m+1}$ as its vertices for each $n = 1,2 \dots$. Since $(u_n,z_n) \in Gr\ R$, there exists a complete sub-simplex $\alpha_{g_n}(y_n^1, y_n^2, \dots, y_n^{m+1})$ with vector-valued labels such that $z_n \in \alpha_{gn} \subset R(u_n,p) \subset S_n, n = 1,2, \dots$.

Denote $\alpha_{g_n}(y_n^1, y_n^2, \dots, y_n^{m+1})$as $\alpha_{gn}(y_n^1, \pi_n)$ Just like Theorem 3.1, there exists a subsequence $\{nk\}\ of\ \{n\}$ and a permutation π such that $\alpha_{gnk}(y_{nk}^1, \pi_{nk}) = \alpha_{g_{nk}}(y_{mk}^1, \pi)$. We have a convergent subsequence of $\{y_{nk}^1\} \subset Z$, which is also denoted by $\{y_{nk}^1\}$ with $y_{nk}^1 \to y_0^1(k \to \infty)$. So far, for each nk, by choosing some real numbers $q_{nk}^i(i \in I_{m+1})$ with $q_{nk}^i \to q_0^i(k \to \infty)$, the sub-simplex $\alpha_{g_{nk}}(y_{nk}^1, \pi)$ can be written as

$$y_{nk}^1 = (q_{nk}^1, q_{nk}^2, \dots, q_{nk,}^{m+1})/p \text{ and}$$

$$y_{nk}^{i+1} = y_{nk}^i + (v_{nk}^{\pi^{i+1}} - v_{nk}^{\pi^i})/p, \forall i \in I_m.$$

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Since, $u_n \xrightarrow{d} u_0$, which means that $S_n \xrightarrow{\eta} S_0 \in T'$, then, by the definition of η, we have $v_{nk}^{\pi^i} \to v_0^{\pi^i}, \forall i \in I_{m+1}$.

Nothing that $y_{nk}^1 \to y_0^1$, we have $\alpha g_{nk}(y_{nk}^1, \pi) \xrightarrow{\eta} \alpha(y_0^1, \pi)$ as $k \to \infty$. Clearly, $\alpha_{g0}(y_0^1, \pi)$ is a simplex in the triangulation of $S_0(v_0^1, v_0^2, \dots, v_0^{m+1})$ with the grid size $\frac{1}{p}$.To conclude the proof that $z_0 \in \alpha(y_0^1, \pi)$ and $\alpha(y_0^1, \pi)$ is a complete sub-simplex with vector- valued labels by function g_0, we can adopt the corresponding part of Theorem 3.1.

R is upper semi-continuous on Q according to Theorem 4.1. We arrive at the following conclusion after proving theorem 3.2 for the existence of minimal element of essential sets.

- **Theorem 4.2:** For each $u = (g, S) \in Q$, there exists a minimal essential set in $\mathrm{R(u, p)}$ with respect to Q, when we have a triangulation of S with a grid size $\frac{1}{\mathrm{p}}$.

V. CONCLUSION

The steadiness of approximate fixed point sets utilizing simplicial methods under perturbation of the related functions and domains is addressed by applying the key stabilities. We demonstrate that simplicial methods being vector-valued and integer-valued differ significantly. Using vector-valued labeling, it is proved that a set-valued function for approximate fixed points is upper semi-continuous.. For vector-labeled simplicial approaches, it is also proved that approximate fixed point sets have finite essential connected components.

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