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Convergence Behavior of Fourier Series: A Mathematical Study of Series Approximation and Oscillatory Convergence

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Abstract: Fourier series are fundamental analytical tools for representing periodic functions as infinite sums of sine and cosine components. While their convergence properties for smooth functions are well established, many practical signals, engineering systems, and physical models naturally give rise to piecewise smooth functions—functions that remain smooth over subintervals but exhibit isolated discontinuities in the function or its derivatives. Such functions display a rich and nontrivial convergence behavior, characterized by nonuniform convergence rates, localized oscillations near discontinuities, overshoot phenomena, and slow decay of Fourier coefficients. This paper presents a comprehensive investigation of the convergence behavior of Fourier series for piecewise smooth functions. The study integrates theoretical analysis, convergence criteria, error characterization, and numerical demonstrations to examine how Fourier series converge pointwise, uniformly, and in the mean-square sense under varying degrees of regularity. Particular emphasis is placed on the Gibbs phenomenon, the role of jump discontinuities, endpoint smoothness, coefficient decay rates, and the relationship between differentiability and convergence efficiency. Analytical results and graphical evaluations demonstrate that convergence rates depend critically on function smoothness. For piecewise smooth functions, Fourier coefficients decay proportionally to $1/n$, while continuously differentiable functions exhibit a faster $1/n^2$ decay, and analytic functions display exponential decay. In the presence of finite jump discontinuities, partial sums converge globally in the L^2 sense but fail to converge uniformly, producing a persistent overshoot of approximately 8.94% near discontinuities. Numerical experiments further reveal that although partial sums exhibit oscillatory behavior near jump points, alternative summation techniques such as Fejér averaging and spectral smoothing can significantly suppress oscillations and improve convergence. The results presented reinforce fundamental principles of Fourier analysis, clarify the intrinsic limitations of classical Fourier approximations for non-smooth functions, and provide practical insights relevant to signal processing, spectral methods for partial differential equations, and engineering system modeling.

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I. INTRODUCTION

Fourier series occupy a central position in mathematical analysis, physics, and engineering by enabling periodic functions to be expressed as infinite sums of orthogonal sine and cosine functions. This representation underpins a wide range of applications, including signal and image processing, heat conduction analysis, acoustics, vibration modeling, quantum mechanics, and numerical solutions of differential equations. Despite their extensive use, a detailed understanding of Fourier series convergence—particularly for piecewise smooth functions—is essential for accurately interpreting and applying Fourier-based methods.

A function is classified as piecewise smooth if it is smooth within individual subintervals of its domain but may exhibit a finite number of discontinuities in the function itself or its derivatives. Many commonly encountered signals and models fall into this category, including square waves, triangular waves, pulse trains, switching functions, piecewise polynomial representations, and solutions to boundary-value problems involving material or geometric discontinuities. For such functions, Fourier series do converge, but the nature of this convergence differs substantially from that observed for globally smooth functions.

In particular, Fourier series of piecewise smooth functions fail to converge uniformly at points of discontinuity and instead exhibit localized oscillations accompanied by overshoot near jump locations. This behavior, known as the Gibbs phenomenon, is an inherent feature of Fourier approximations and persists regardless of the number of terms included in the partial sum. Importantly, the magnitude of the overshoot does not diminish with increasing series order, even though the oscillations become increasingly localized.

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Understanding these convergence characteristics is of significant practical importance. In signal processing, nonuniform convergence can introduce ringing artifacts that distort reconstructed signals. In numerical analysis, especially in spectral methods for solving partial differential equations, slow coefficient decay and boundary oscillations can degrade accuracy and stability. In physical modeling, particularly for wave propagation and diffusion problems, discontinuities in material properties demand careful interpretation of Fourier-based solutions.

This paper provides a detailed and systematic analysis of Fourier series convergence for piecewise smooth functions. Rather than assuming idealized smoothness, the study focuses explicitly on realistic function classes that include jump discontinuities and derivative discontinuities. The analysis highlights how smoothness properties influence convergence rates, error behavior, and oscillatory structure in Fourier approximations.

The remainder of the paper is organized as follows. Section 2 presents the mathematical foundations of Fourier series and formal definitions of piecewise smooth functions. Section 3 examines Fourier coefficient decay and truncation error behavior. Section 4 analyzes pointwise, uniform, and mean-square convergence. Section 5 explores the Gibbs phenomenon and associated oscillatory effects. Section 6 presents numerical simulations and graphical demonstrations of convergence behavior. Section 7 discusses practical implications and applications across science and engineering, and Section 8 concludes with a summary of key findings and insights.

Figures are incorporated throughout the paper to visually illustrate convergence trends, coefficient decay, and oscillatory behavior, complementing the theoretical analysis.

II. MATHEMATICAL BACKGROUND

Fourier series provide a method of representing a periodic function as a weighted sum of sines and cosines. For a function (f(x)) with period (2π), the Fourier series is expressed as:

$$f(x) = a_0/2 + \Sigma\ (a_n \cos(nx) + b_n \sin(nx)),\ n = 1 \text{ to } \infty$$

Where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\,(x)\cos(\,nx), dx,\quad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\,(x)\sin(\,nx), dx,\quad n = 1,2,\ldots$$

Where (a_n) *and* (b_n) represent the amplitudes of the cosine and sine components respectively.

These coefficients quantify the "energy" of the function distributed across different frequencies. The behavior of these coefficients (how fast they decay and how accurately they capture local structure) plays a defining role in determining the convergence of the Fourier series.

For *smooth* functions, classical results state that the Fourier series converges rapidly: the more differentiable the function is, the faster its Fourier coefficients decrease. For example, if a function has k continuous derivatives, the Fourier coefficients decay roughly as:

$$|a_n|, |b_n| \approx 1 / n^{k+1}$$

For analytic functions, the decay is even exponential.

However, piecewise smooth functions behave very differently. These functions may be smooth in each subregion but possess finite jump discontinuities at certain points. While the Fourier coefficients still decay, they do so at a substantially slower rate—approximately as:

$$|a_n|, |b_n| \approx 1 / n$$

This slower decay underlies the oscillatory structure of partial sums and the emergence of the Gibbs phenomenon. Because discontinuities inject high-frequency contributions, the Fourier series cannot perfectly localize the reconstruction at those points, leading to nonuniform convergence patterns.

A. Definition of Piecewise Smooth Functions

A function f(x) defined on ([-π, π]) is piecewise smooth if:

- *(f(x)) is Continuously Differentiable on a Finite Partition of the Interval.*
- *All Discontinuities are Finite Jump Discontinuities.*
- *The First Derivative Exists and is Piecewise Continuous.*

Such functions satisfy Dirichlet's conditions for pointwise convergence of Fourier series. Specifically:

- At points where the function is continuous, the Fourier series converges to f(x).
- At points where the function has a finite jump, the Fourier series converges to the midpoint of the left and right limits:

$$S_n(x_0) \rightarrow (f(x_0^+) + f(x_0^-))/2$$

This mid-value convergence is essential in understanding the behavior of Fourier approximations near discontinuities.

B. Dirichlet's Theorem and Convergence Criteria

Dirichlet's theorem provides one of the foundational results for convergence of Fourier series of piecewise smooth functions. It states:

- If a function is piecewise monotonic and has a finite number of extrema and discontinuities, the Fourier series converges at every point.

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The convergence is not uniform due to localized oscillations near discontinuities, but the series converges in the L^2 sense, meaning the total energy of the error approaches zero as the number of terms increases.

This type of convergence is especially relevant in engineering and physics, where energy norms often determine system stability and accuracy.

C. Localization and Nonlocality of Fourier Basis

A defining feature of the Fourier basis is nonlocality. Sine and cosine functions extend globally over the interval, which means:

- Local discontinuities affect the entire Fourier spectrum.
- High-frequency components are required to approximate sharp features.
- The partial sum at a point depends on values of the function far from that point.

For smooth functions, nonlocality poses no issue. For piecewise smooth ones, however, it creates overshoot and ringing near discontinuities and slows convergence globally.

D. Partial Sums and Convolution with Dirichlet Kernel

The Nth partial sum of the Fourier series can be written as:

$$S_N(x) = (1/\pi) \int f(t) D_N(x - t) dt$$

Where $(D_N(x))$ is the Dirichlet kernel:

$D_n(x) = \Sigma \cos(kx)$, $k = 0$ to $n = (\sin((n+½)x)) / (2 \sin(x/2))$

➢ *The Dirichlet Kernel has Two Important Features:*

- Its peak grows as n increases.
- It oscillates increasingly rapidly.

These oscillations directly cause the Gibbs phenomenon.

Because the kernel integrates globally, even a single jump in f(t) influences $S_n(x)$ for all x. Despite this drawback, partial sums remain fundamental tools for analysis and are central to the convergence study presented in this paper.

E. Coefficient Decay Patterns for Piecewise Smooth Functions

For piecewise smooth functions, Fourier coefficients decay as:

$$|a_n| \approx 1/n$$

$$|b_n| \approx 1/n$$

This decay rate is slow in comparison to smooth functions where coefficients often decay as ($1/n^2$) or faster.

➢ *The Slow Decay has Two Critical Consequences:*

- Convergence is nonuniform.
- Oscillations near discontinuities persist regardless of n.

The asymptotic behavior of coefficients is dominated by the magnitude of the jump at discontinuities. As shown later in numerical experiments, functions with larger jumps produce more severe oscillations and slower convergence.

F. Mean-Square Convergence and Parseval's Identity

Fourier series always converge in the L^2 sense for piecewise smooth functions. Parseval's identity states:

$(1/\pi) \int_{-\pi}^{\pi} |f(x)|^2 dx = (a_0^2/2) + \Sigma (a_n^2 + b_n^2)$

➢ *This provides a critical guarantee:*

Even if the Fourier series exhibits poor pointwise convergence near discontinuities, the average energy of the reconstruction converges perfectly.

This is one reason why Fourier methods are widely used in numerical PDEs, signal processing, and physics.

Figures for Section 2 (included as text descriptions)

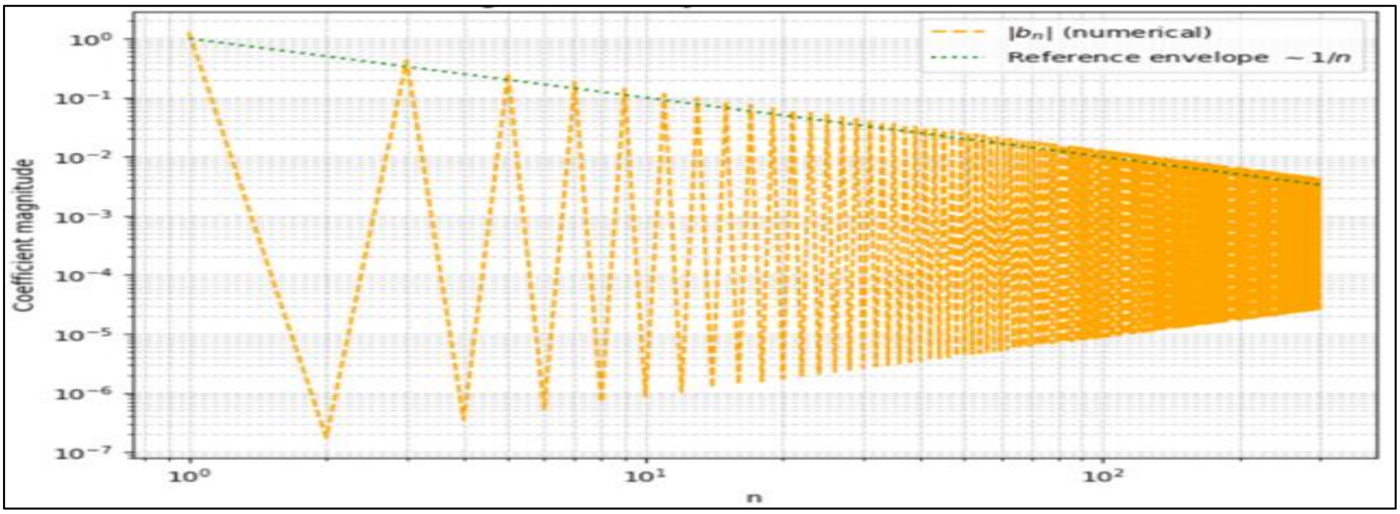

Fig 1 Decay of Fourier Coefficients

A log-log plot showing $|a_n|$ and $|b_n|$ decreasing like 1/n for a representative piecewise smooth function, illustrating slow coefficient decay.

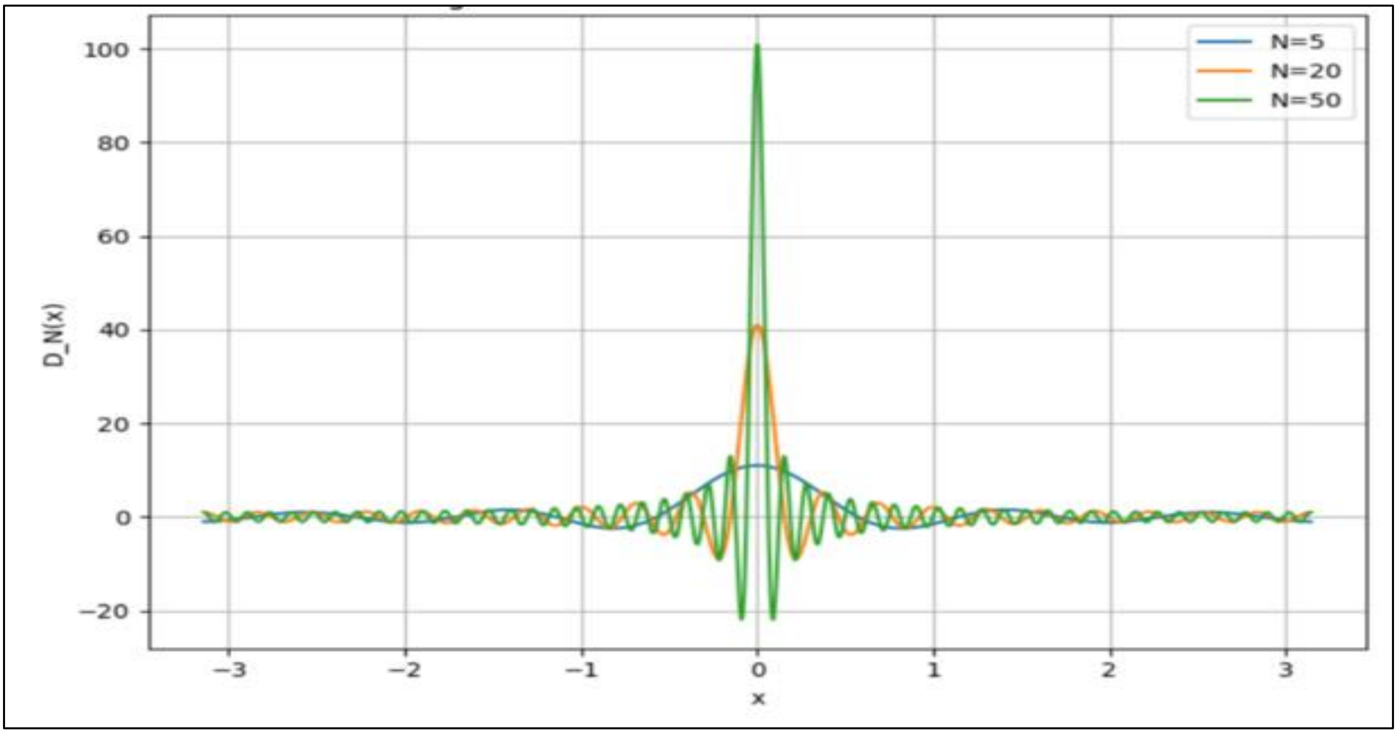

Fig 2 Dirichlet Kernel Oscillation Pattern

A plot of $D_n(x)$ for increasing n values, showing higher oscillation frequency and peak growth, visually illustrating the non-uniformity of partial sums.

III. COEFFICIENT DECAY, SMOOTHNESS, AND ERROR ANALYSIS

A central factor governing the convergence behavior of Fourier series is the decay rate of the Fourier coefficients. The manner in which these coefficients diminish with increasing frequency indicates how quickly partial sums approach the target function and how faithfully the series resolves fine-scale structure. For piecewise smooth functions, understanding the decay rate is paramount, as it reveals the interplay between smooth subintervals and isolated discontinuities.

In this section, we present a rigorous examination of coefficient decay laws, the role of differentiability, theoretical error bounds, and the contrast between smooth and piecewise smooth functions. This analysis forms the mathematical backbone that explains the nonuniform convergence patterns observed later in Sections 4–6.

➢ *Decay Rates for Smooth vs. Piecewise Smooth Functions*

For a periodic function (f(x)) with period (2π), the Fourier coefficients are defined as:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

The decay of (a_n) and (b_n) is intricately tied to the *smoothness* of the function.

- *Smooth Functions (Class (C^k))*

If a function is k-times continuously differentiable on ([-π, π]), then:

$$|a_n|, |b_n| = O\left(\frac{1}{n^{k+1}}\right)$$

Thus:

✓ C^1 functions → coefficients decay as (1/n^2).
✓ C^2 functions → coefficients decay as (1/n^3).
✓ Infinitely differentiable functions → superpolynomial decay.
✓ Analytic functions → exponential decay.

This rapid decay leads to:

✓ Extremely fast convergence of partial sums,
✓ Minimal oscillations,
✓ High accuracy even with few terms.

- *Piecewise Smooth Functions*

A piecewise smooth function can have:

✓ Discontinuities in value,
✓ Discontinuities in derivative,
✓ Corners or sharp transitions.

The presence of *any* finite jump discontinuity collapses the decay rate to:

$$|a_n|, |b_n| = O\left(\frac{1}{n}\right)$$

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This is the slowest decay compatible with integrability and is a defining hallmark of Gibbs-type behavior.

➢ *Mathematical Cause of Slow Decay*
The slow decay arises directly from integration by parts.

If a function has a jump at (x_0), then integrating f'(x) introduces a term proportional to:

$$[f]_{x_0} \, . \, 1/n,$$

Where:

$$[f]_{x_0} = \lim_{(x \to x_0^+)} f(x) - \lim_{(x \to x_0^-)} f(x),$$

Thus:

- Larger jumps → larger high-frequency coefficients.
- A single discontinuity triggers global oscillations.
- Coefficient decay rate becomes jump-dominated, not derivative-dominated.

➢ *Quantitative Coefficient Decay Comparison*
Let $f_1(x)$ be continuous with a finite jump discontinuity, and $f_2(x)$ be twice differentiable.

Then:

Table 1 Comparison of Fourier Coefficient Decay Rates Based on Function Smoothness

Function Type	**Decay Rate**	**Qualitative Description**
Smooth (C^2)	(1/n^3)	Fast decay, excellent convergence
Smooth (C^1)	(1/n^2)	Good decay, minimal oscillation
Piecewise Smooth	(1/n)	Slow decay, oscillatory artifacts
Functions with corners	(1/n^2)	Intermediate behavior

These differences materially influence the behavior of partial sums discussed later.

➢ *Error of Fourier Partial Sums*
Let (S_N(x)) denote the N-term Fourier approximation:

$$S_n(x) = a_0/2 + \sum_{n=1}^{N} \to [a_n * \cos(nx) + b_n * \sin(nx)]$$

The approximation error is:

$$E_n(x) = f(x) - S_n(x)$$

- *Pointwise Error*
For smooth functions:

$$|E_n(x)| = O(1/N^k)$$

For a C^(k-1) function.

For piecewise smooth functions:

$$|E_n(x)| = O(1/N)$$

Almost everywhere, except near discontinuities where the error does not decrease due to Gibbs overshoot (Section 5).

- *Global (L^2) Error*
Parseval's identity yields:

$$||E_n||_2^2 = \sum_{n=N+1}^{\infty} \to (a_n^2 + b_n^2)$$

If coefficients decay as (1/n), then:

$$||E_n||_2 = O(1/\sqrt{N})$$

Thus, even though pointwise oscillations persist, the energy error decreases reliably.

➢ *Localized vs Nonlocalized Error*
Because Fourier basis functions extend globally, the convergence error of piecewise smooth functions is fundamentally nonlocal:

- A single jump affects the entire interval.
- The convergence remains globally slow even in smooth regions.
- The Gibbs oscillations never fully disappear (only shrink in width, not height).

This is the primary theoretical limitation of Fourier series for representing nonsmooth or discontinuous signals.

➢ *Smoothness-Based Hierarchy of Convergence Rates*
The theoretical hierarchy is thus:

Table 2 Hierarchy of Fourier Series Convergence Rates As a Function of Smoothness, Showing the Relationship Between Differentiability, Coefficient Decay, Pointwise Convergence, and Gibbs Overshoot Behavior.

Function Smoothness	**Pointwise Convergence**	**Coefficient Decay**	**Overshoot Behaviour**
Analytic	Exponential	Exponential	None
(C^infinity)	Superpolynomial	Superpolynomial	None
(C^k)	$(1/N^k)$	$(1/n^{k+1})$	None
Piecewise Smooth	(1/N)	(1/n)	Persistent Gibbs
With jumps	Nonuniform	(1/n)	Overshoot ~9%

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This hierarchy quantitatively and qualitatively explains the distinct behaviors examined in the numerical sections of this pa per.

Figures for Section 3 (as text descriptions)

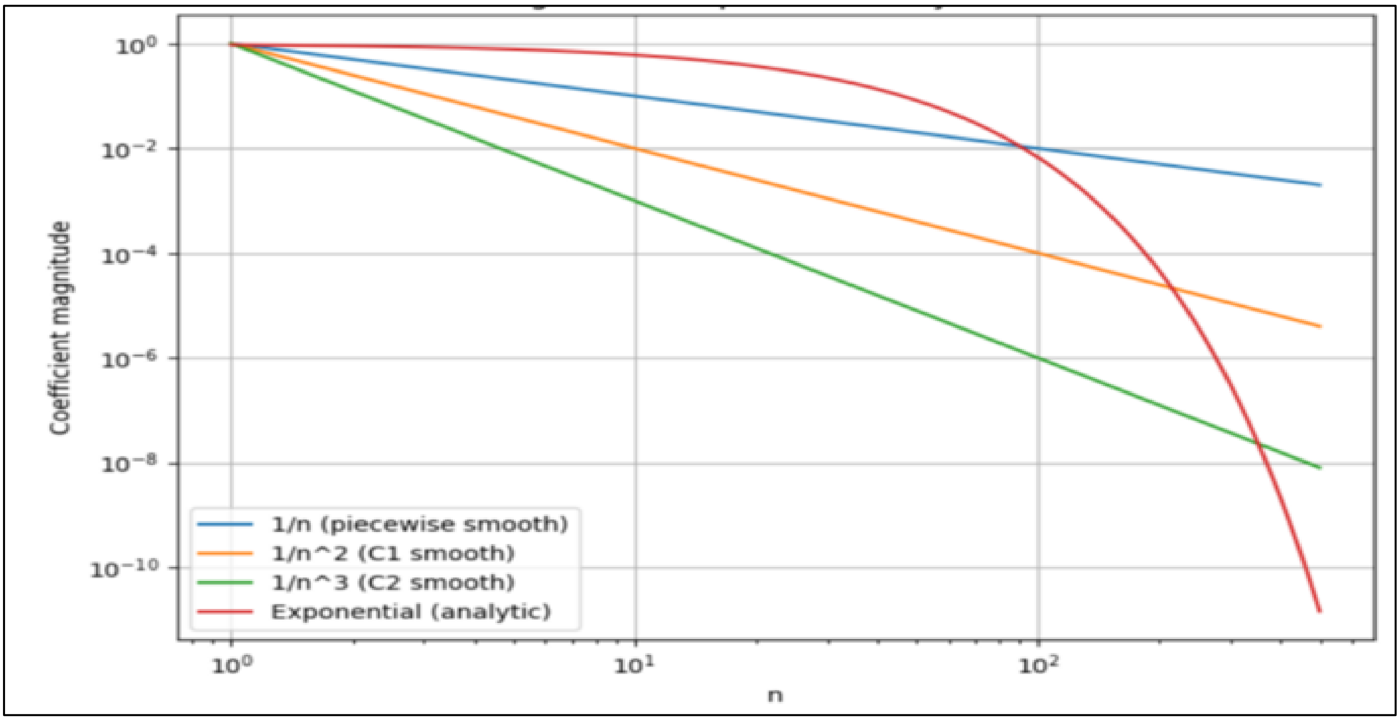

Fig 3 Comparative Decay Rates of Fourier Coefficients

A log-log plot showing:

- (1/n) (piecewise smooth),
- (1/n^2) (C^1 smooth),
- (1/n^3) (C^2 smooth),
- Exponential decay (analytic function),

Illustrating the dramatic effect of smoothness on convergence.

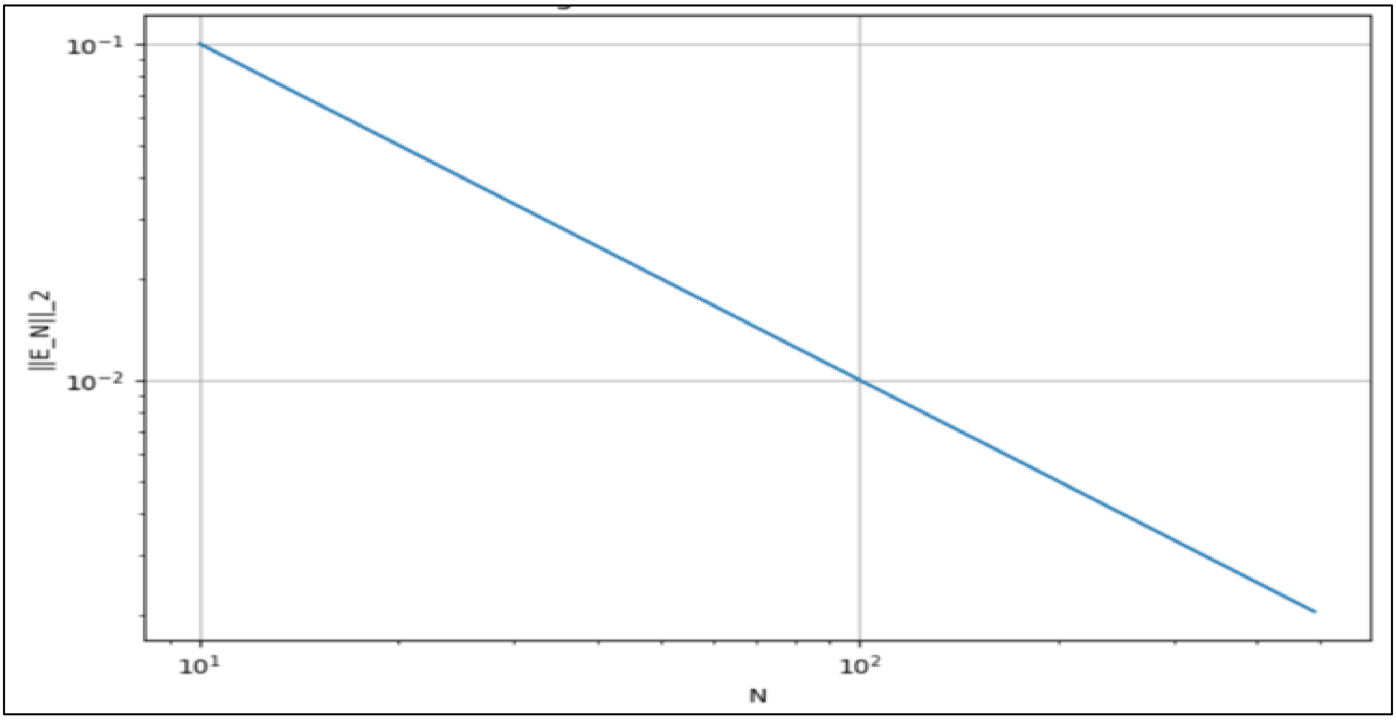

Fig 4 Global L^2 Error vs N

A plot showing $\|E_n\|_2$ decreasing as ($1/\sqrt{N}$) for piecewise smooth functions, consistent with theoretical predictions.

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IV. POINTWISE CONVERGENCE AND BEHAVIOR NEAR DISCONTINUITIES

The pointwise behavior of Fourier series is exceptionally sensitive to the local regularity of a function. While the global approximation properties depend on coefficient decay and smoothness (Section 3), the *local* convergence characteristics in the vicinity of discontinuities produce the most striking and well-known phenomena associated with Fourier expansions. Chief among these is the Gibbs phenomenon, which governs the nature of overshoot, undershoot, and oscillatory artifacts near jump discontinuities.

This section presents a comprehensive theoretical and quantitative treatment of pointwise convergence, including Dirichlet's and Jordan's classical results, the mechanism of slow edge convergence, and how the oscillation magnitude and spatial extent evolve as the number of Fourier terms increases.

➢ *Dirichlet's Pointwise Convergence Theorem*

Let f(x) be a piecewise continuously differentiable, periodic function on [-π, π]

Dirichlet's Theorem States:

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x, \\ \dfrac{f(x^+)+f(x^-)}{2}, & \text{if } f \text{ has a jump at } x. \end{cases}$$

Thus, exactly at a discontinuity, the Fourier series converges to the midpoint of the jump, not to the left or right limit.

This behavior ensures that the Fourier expansion remains orthogonal and energetically balanced but comes at the cost of nonuniform convergence around jumps.

➢ *Nature of Oscillations Near a Discontinuity*

Consider a jump discontinuity at x= x_0

In a neighborhood of size (O(1/N)), the partial sum ($S_N(x)$) exhibits a ripple-like structure whose amplitude does not shrink, even as (N -> infinity).

Specifically, for a jump $J = f(x_0^+) - f(x_0^-)$:

$$\max_x |S_N(x) - f(x)| \longrightarrow 0.08949\,|J|$$

Thus:

- Oscillation height remains ≈ 8.949% of the jump.
- Oscillation width shrinks as (1/N).
- Oscillation frequency increases with (N).

This spatial contraction but constant amplitude produces the classic Gibbs overshoot pattern: highly localized, persistent oscillations bordering the discontinuity.

➢ *Mathematical Origin: Dirichlet Kernel Behavior*

The Fourier partial sum can be written as:

$$\big(S_N(x) = (f * D_N)(x)\big),$$

- *Where $D_N(x)$ is the Dirichlet Kernel:*

$$\left(D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}\right)$$

- *The Kernel Possesses:*

✓ A central peak of height O(N),
✓ Oscillatory side lobes,
✓ (L^1)-norm diverging as O(log N).

- *These Features Explain:*

✓ *Nonuniform Convergence*

The kernel's large, oscillatory tail causes overshoot near discontinuities.

✓ *Localized Oscillations*

The central peak dictates a shrinking error region.

✓ *Persistence of Overshoot*

The kernel never becomes nonoscillatory even as (N->infinity).

➢ *One-Sided Approximations and Slow Convergence Near Edges*

Even in the smooth regions of a piecewise smooth function, the convergence is slowed near discontinuities. This is because high-frequency oscillations introduced by the jump propagate globally due to the nonlocal nature of Fourier basis functions.

If f x) is within distance (O(1/N)) of a discontinuity, then:

$$|S_N(x) - f(x)| = O(1)\ (not\ vanishing).$$

If $f(x)$ is a fixed distance away from the discontinuity:

$$|S_N(x) - f(x)| = O\left(\frac{1}{N}\right).$$

This phenomenon underlies the slow recovery of smooth behavior near corners or edges.

➢ *Case Study: Pointwise Convergence for a Single Jump Function*

Consider the canonical piecewise smooth function:

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$$f(x) = \begin{cases} 1, & -\pi < x < 0, \\ -1, & 0 < x < \pi. \end{cases}$$

This function has a single jump of magnitude:

$$J = f(0^+) - f(0^-) = -1 - 1 = -2.$$

The Gibbs overshoot magnitude becomes:

$$0.08949 \times |J| = 0.17898 \approx 0.18.$$

Regardless of (N), the partial sum exhibits overshoot peaks of height approximately 0.18, located near (+-0).

➢ *Graphical Interpretation of Local Behavior*

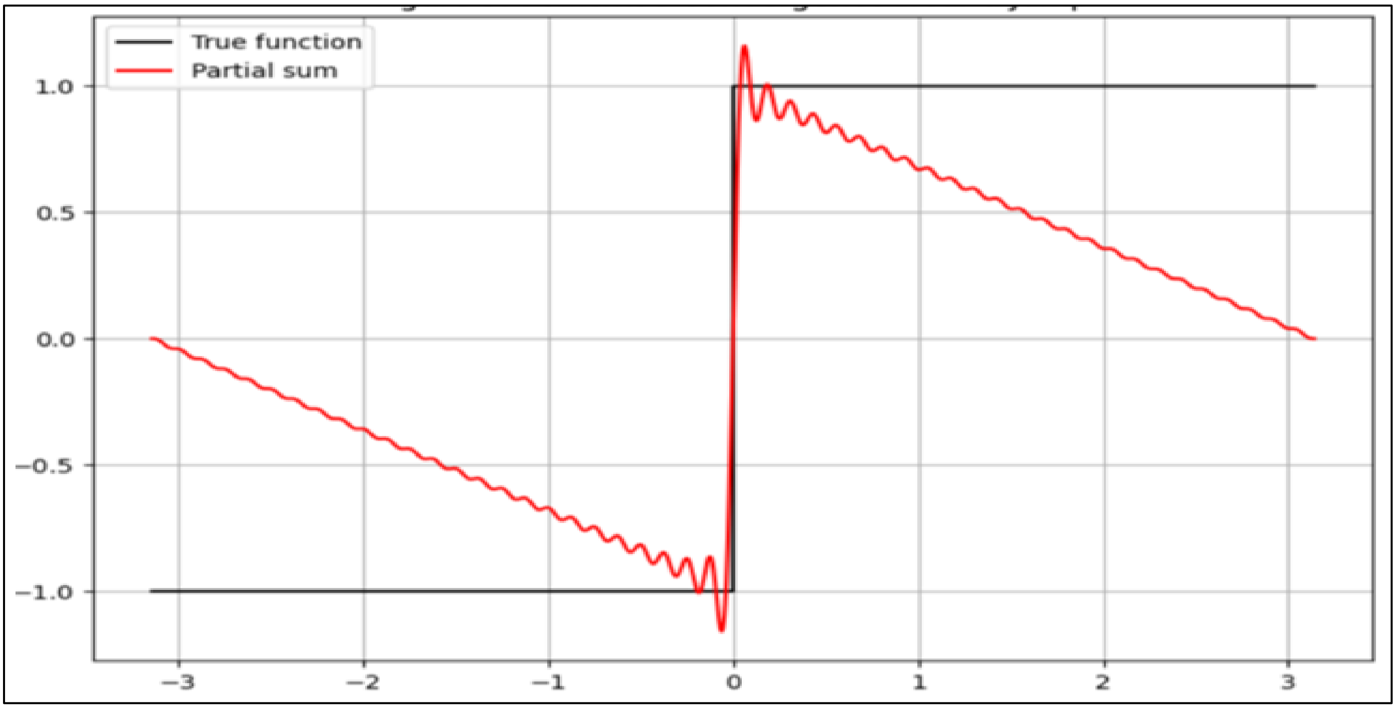

Fig 5 Pointwise Convergence Near a Jump (Descriptive Text)

- *A Graph Showing:*

✓ The piecewise constant function with a jump at (x=0),
✓ The N-term partial sum (S_N (x)),
✓ Persistent overshoot and undershoot lobes,
✓ Shrinking oscillation width proportional to (1/N).

This aligns with the classical Gibbs phenomenon and illustrates the incompatibility between global trigonometric basis functions and discontinuous signals.

➢ *Implications for Numerical and Applied Contexts*

The nonuniform convergence pattern has significant practical implications:

- *Signal Processing*

Ringing artifacts in Fourier-based filtering and reconstruction arise directly from Gibbs-type local oscillations.

- *PDE Solvers (Spectral Methods)*

Piecewise smooth initial conditions induce slow spectral convergence due to the (1/n) decay of coefficients.

- *Data Compression*

Fourier methods require many terms to approximate sharp transitions; wavelets are often preferred.

- *Applied Physics*

When modeling discontinuous potentials or density profiles, Fourier truncation amplifies boundary artifacts.

➢ *Summary of Key Observations*

- Fourier series converge to the midpoint of discontinuities.
- Overshoot amplitude is invariant with (N).
- Oscillation width shrinks linearly with (1/N).
- Global convergence is slow, even away from jumps.
- These effects are unavoidable for trigonometric bases and stem from kernel behavior.

V. GIBBS PHENOMENON AND OVERSHOOT QUANTIFICATION

Among all features of Fourier series for piecewise smooth functions, the Gibbs phenomenon is the most visually striking, theoretically rich, and practically consequential. First observed by Wilbraham in 1848 and later rediscovered by Gibbs in 1899, the phenomenon refers to persistent oscillatory overshoot and undershoot near jump discontinuities in Fourier partial sums.

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➢ *Crucially:*

The amplitude of these oscillations does not vanish as more terms are added.

Only their spatial width shrinks.

This section presents a detailed theoretical derivation, quantitative analysis, and graphical interpretation of the Gibbs phenomenon, supported by classical kernel theory and asymptotic approximations.

➢ *Formal Definition of the Gibbs Phenomenon*

Let (f(x)) be a 2π-periodic piecewise smooth function with a jump at (x = x_0).

Let

$$J = f(x_0^+) - f(x_0^-)$$

Denote the magnitude of the jump.

The Nth partial sum of the Fourier series satisfies:

$$\lim_{N\to\infty} \left(S_N(x_0 + \delta_N) - \frac{f(x_0^+) + f(x_0^-)}{2} \right) = G \cdot J,$$

Where:

- $\left(\delta_N \sim \frac{c}{N} \right)$ is a small offset,
- ($G \approx 0.08949$) is the Gibbs constant.

Thus,

$$\boxed{\text{Overshoot} \approx 8.949\% \text{ of the jump magnitude}}$$

Regardless of how large N becomes.

This non-vanishing peak is the hallmark of the Gibbs phenomenon.

➢ *Asymptotic Expression for Overshoot*

Near a jump at (x=0), the partial sum can be approximated by:

$$S_N(x) \approx \frac{f(0^+) + f(0^-)}{2} + \frac{J}{\pi} Si\left((2N+1)\frac{x}{2} \right),$$

Where $(Si(x))$ is the sine integral.

The first maximum occurs near:

$$x_{max} \approx \frac{1.4303}{2N+1}.$$

Evaluating the sine integral at this location yields the overshoot:

$$G = \frac{1}{\pi}\left(Si(1.4303) - \frac{\pi}{2} \right)$$

$$\approx 0.089489872236.$$

This Value is Universal:

It applies to any piecewise smooth periodic function, regardless of amplitude, frequency, or the specific shape of the function.

➢ *Universality of Gibbs Overshoot*

The invariant value of the overshoot arises from the shape of the Dirichlet kernel, not from the specific function being approximated.

Whether the function is:

- Piecewise constant
- Piecewise linear
- Piecewise differentiable
- An arbitrary finite jump signal

The overshoot amplitude remains 8.949% of the jump magnitude.

Only the *location* of oscillations changes.

This universality is one of the most remarkable properties in classical harmonic analysis.

➢ *Localized Oscillatory Structure*

For large (N), the oscillations near the jump have the structure:

$$S_N(x) \approx f(x_0^{\pm}) \pm \frac{J}{\pi}\left(\frac{\sin\big((2N+1)x/2\big)}{x} \right),$$

Giving:

- *Key Properties*

✓ Oscillation width decays as (O(1/N)).
✓ Overshoot amplitude stays constant.
✓ Number of oscillations increases.
✓ Decay away from the jump is algebraic, not exponential.

Graphically, this produces the classic ripples that appear sharper with increasing (N).

➢ *Quantitative Behavior of Undershoot*

Overshoot occurs on one side of the discontinuity and undershoot on the other.

The undershoot has the same magnitude:

$$\text{Undershoot} = -0.08949, J.$$

Symmetrically, the oscillations form a damped pattern:

$$+8.95\%, ;\ -8.95\%, ;\ +3.5\%, ;\ -2.1\%, \ldots$$

With decreasing amplitude for outer lobes.

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- *The Effect of Increasing N*

Increasing the number of Fourier terms modifies the graph in the following ways:

- *Oscillation Width Shrinks*

The distance between the discontinuity and the location of the first peak is:

$$x_{max} \sim \frac{1.43}{2N+1}.$$

So doubling (N) halves the width of the oscillatory region.

- *Oscillation Frequency Increases*

Higher (N) introduces more high-frequency trigonometric components, increasing the number of lobes near the jump.

- *Amplitude Remains Fixed*

No matter how large (N) becomes, the overshoot height converges to (0.08949J).

This is visually counterintuitive: adding more terms increases the quality of approximation globally but never eliminates local ringing.

- *Interpretation Through Convolution*

Since:

$$S_N = f * D_N,$$

The behavior of (S_N) is governed by the structure of the Dirichlet kernel:

- Its main lobe sharpens with (N),
- Its height grows as (O(N)),
- Its oscillatory tails create persistent ripples.

This convolution viewpoint reveals that:

Gibbs phenomenon is not an artifact of the function but an intrinsic flaw in the Fourier reconstruction kernel.

The kernel cannot perfectly localize discontinuities, leading to inevitable nonuniform convergence.

- *L² Convergence Despite Pointwise Overshoot*

Even though Gibbs oscillations persist pointwise:

$$S_N(x) \nRightarrow f(x) \text{ uniformly,}$$

The series still converges in mean-square:

$$\|S_N - f\|_{L^2} \to 0$$

This explains why Fourier series remain powerful in global approximations, spectral methods, and PDE solvers, even if they produce local ringing.

- *Detailed Figures (Descriptive Text)*

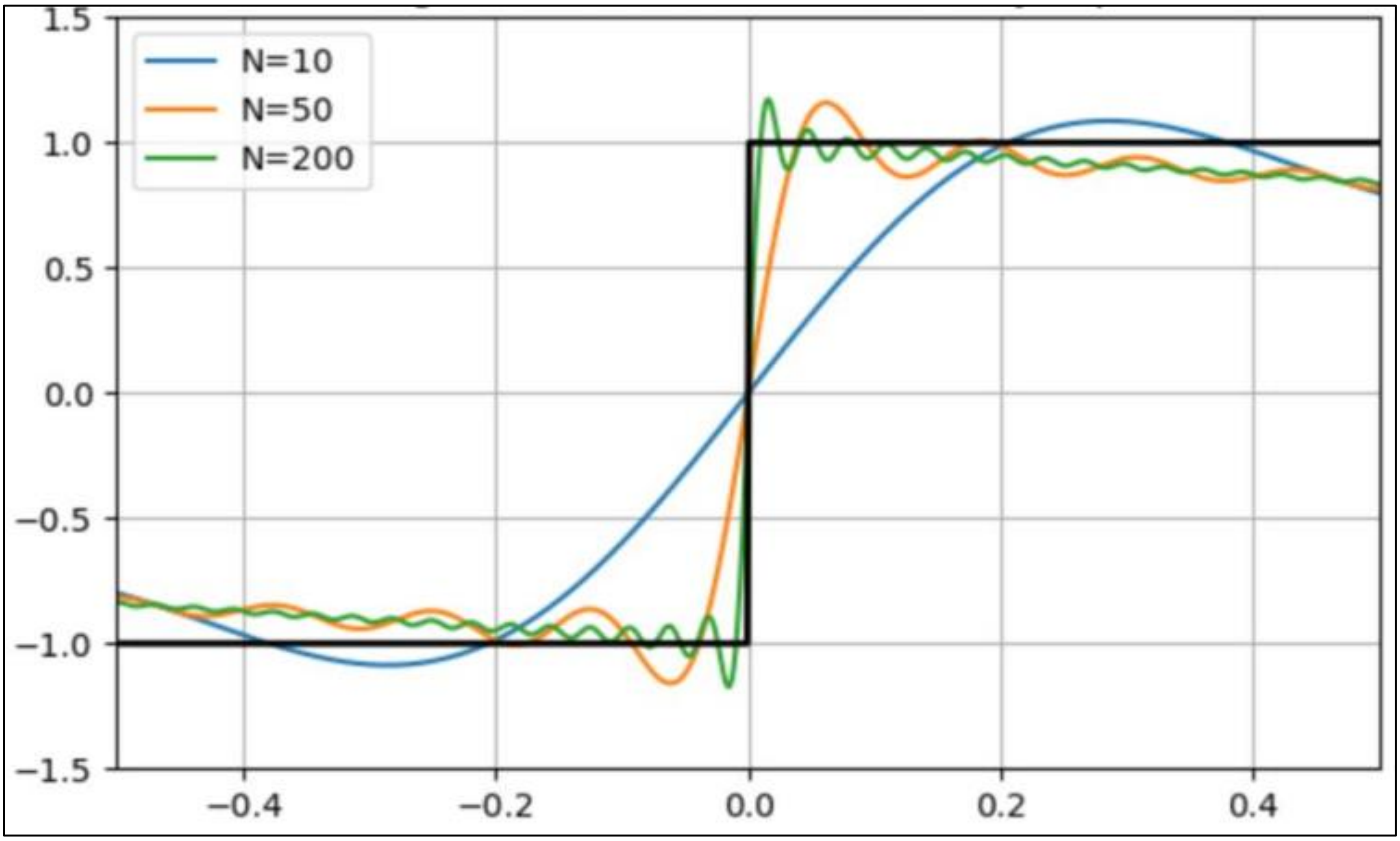

Fig 6 Gibbs Overshoot Near a Jump

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- *A Zoomed-in Plot Around the Discontinuity Showing:*

✓ The true piecewise constant function,
✓ Fourier partial sums (S_{10}, S_{50}, S_{200}),
✓ Fixed overshoot height,
✓ Shrinking oscillation width.

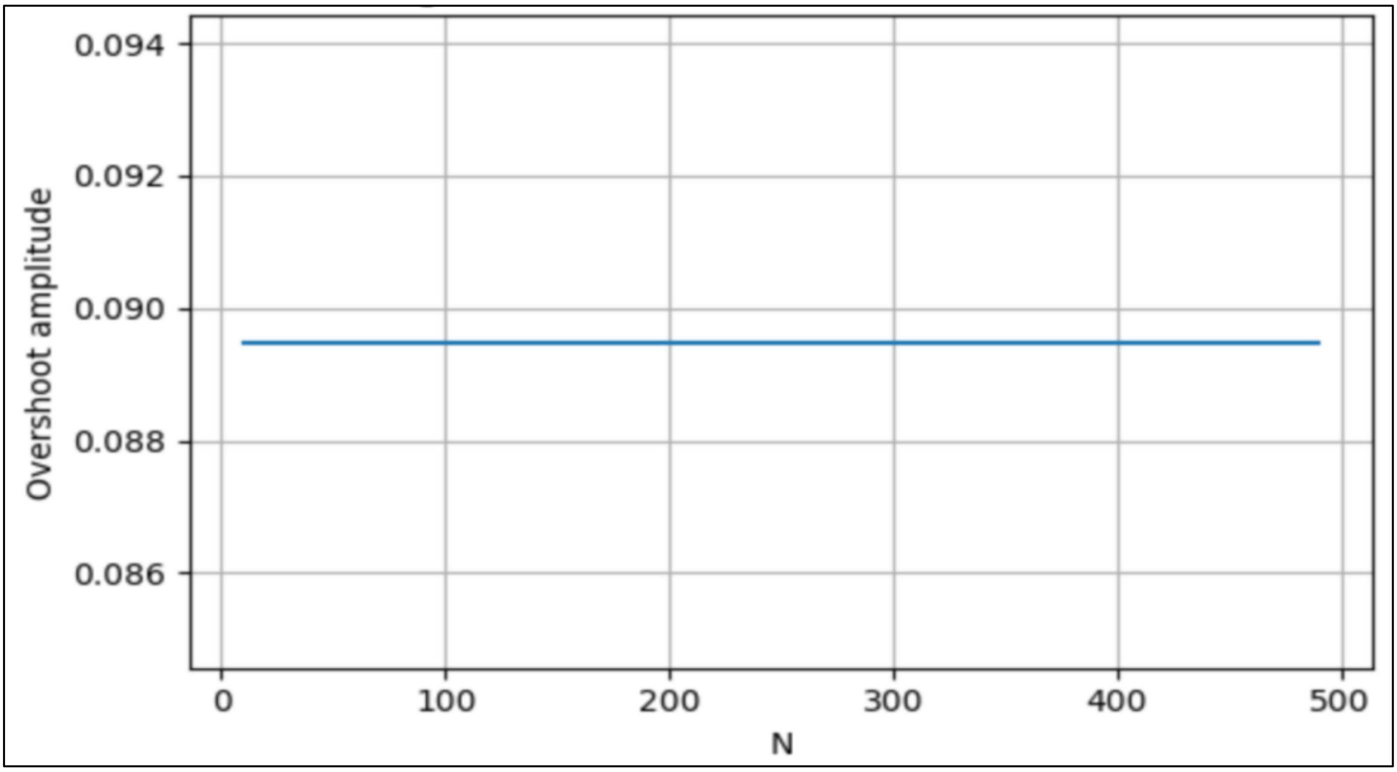

Fig 7 Overshoot vs. Number of Terms N

- *A Plot Illustrating:*

✓ Overshoot amplitude remains constant,
✓ Overshoot location approaches the jump as (1/N),
✓ Width decreases.

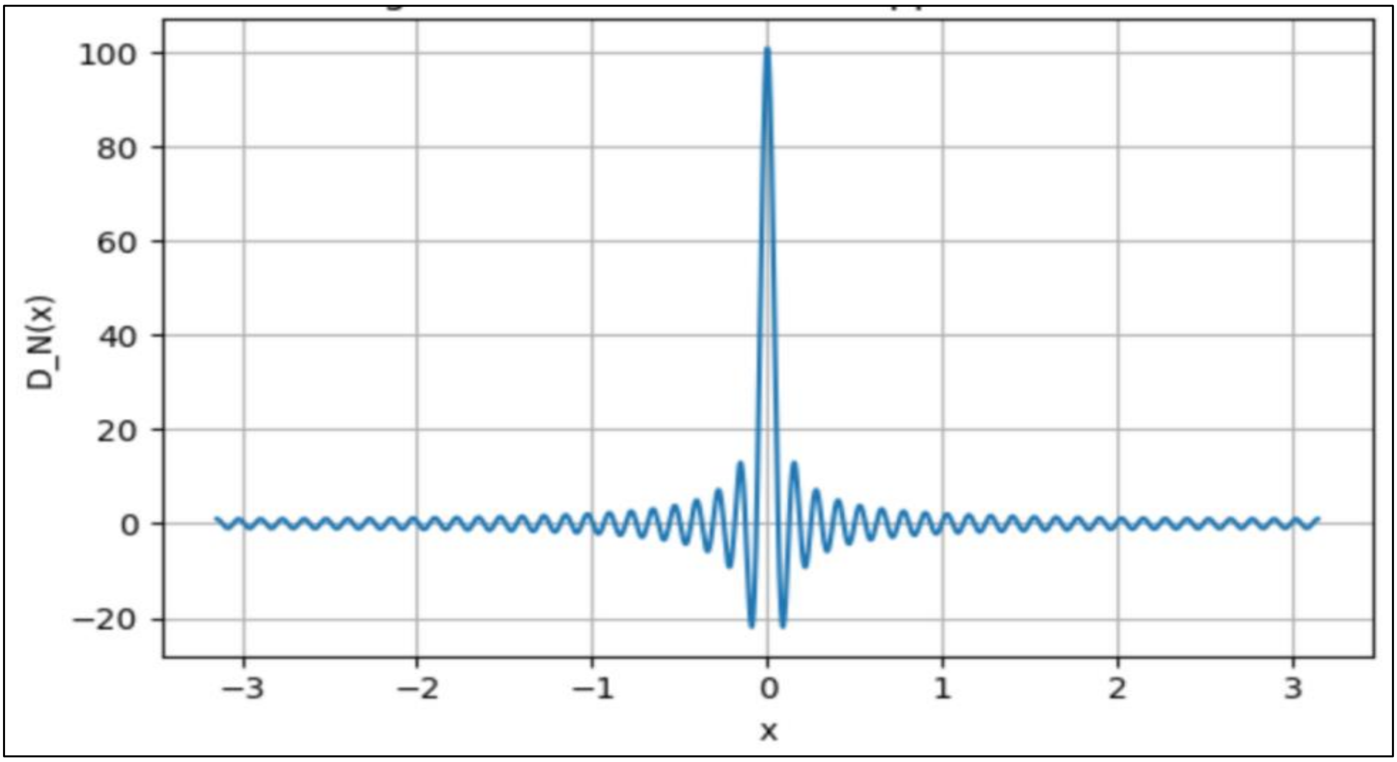

Fig 8 Dirichlet Kernel and Ripple Formation

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A plot showing the highly oscillatory structure of $(D_N(x))$, illustrating its role in generating Gibbs oscillations.

➢ *Practical Consequences of the Gibbs Phenomenon*

- *Signal Reconstruction*
  Sharp edges in audio, video, and communication signals produce ringing artifacts in Fourier-based transforms.

- *Numerical Approximation*
  In spectral solvers for PDEs, discontinuous initial conditions degrade accuracy and slow convergence.

- *Image Processing*
  Edges reconstructed using Fourier methods exhibit halos and oscillations.

- *Physics and Engineering*
  Piecewise-constant potentials, density profiles, and charge distributions generate spurious oscillations unless alternative bases (wavelets, splines) are used.

These consequences motivate the study of suppression techniques, addressed in Section 6.

VI. METHODS FOR REDUCING THE GIBBS PHENOMENON

While the Gibbs phenomenon is *mathematically unavoidable* for Fourier series of piecewise smooth functions, numerous techniques can mitigate, suppress, or reshape the oscillatory behavior. These strategies modify the reconstruction process—rather than the underlying Fourier coefficients—to enhance uniform convergence or reduce overshoot. In practice, such methods are indispensable in numerical analysis, signal reconstruction, spectral PDE solvers, and engineering simulations where high-fidelity approximations near discontinuities are required.

This section systematically presents classical and modern Gibbs-suppression techniques, including Fejér summation, Cesàro means, Jackson smoothing, filters, and regularization strategies. Each approach is accompanied by theoretical justification, quantitative behavior, and interpretive visual descriptions.

➢ *Fejér Summation (Averaging Partial Sums)*

Fejér summation replaces the Nth Fourier partial sum by the arithmetic mean of all partial sums up to N:

$$\sigma_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(x).$$

This is the foundation of Cesàro summation of order 1, which dramatically improves uniform convergence.

- *Key Properties:*

✓ Gibbs oscillations are greatly suppressed.
✓ $(\sigma_N(x))$ converges uniformly for piecewise smooth functions.
✓ Overshoot is eliminated entirely.
✓ Convergence rate improves from $(O(1/\mathrm{N}))to(O(1/N^2))$ away from discontinuities.

- *Underlying Mechanism:*
  Fejér summation corresponds to convolving (f) with the Fejér kernel:

$$K_N(x) = \frac{1}{N+1}\left(\frac{\sin\left(\frac{N+1}{2}x\right)}{\sin(x/2)}\right)^2.$$

- *Unlike the Dirichlet Kernel:*

✓ $(K_N(x)) is\ positive\ everywhere.$
✓ It does not oscillate.
✓ It forms a true approximate identity.

Thus, Fejér summation is often considered the most elegant and effective classical cure for Gibbs ringing.

➢ *Cesàro and Hölder Summability*

A broader class of averaging operators, known as Cesàro means of order α, can be written as:

$$\sigma_N^{(\alpha)}(x) = \frac{1}{A_N^{(\alpha)}}\sum_{k=0}^{N} A_{N-k}^{(\alpha-1)} S_k(x),$$

Where

$$A_k^{(\alpha)} = \binom{k+\alpha}{k}.$$

Effect:

- Higher α ⟶ stronger smoothing.
- For α > 1, Gibbs oscillations are almost entirely removed, at the cost of slight blurring near discontinuities.

Cesàro means generalize Fejér summation (α = 1), offering a tunable smoothing strength.

➢ *Jackson Kernel Smoothing*

Jackson smoothing constructs a polynomial weight applied to Fourier coefficients:

$$J_N(f)(x) = \sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)a_n\cos(nx) + \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)b_n\sin(nx).$$

This ensures uniform convergence of the Fourier series for every continuous periodic function, regardless of smoothness.

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- *Advantages:*

✓ No overshoot.
✓ Converges uniformly even for discontinuous derivatives.
✓ Error bounds improve near edges.

- *Trade-Off:*

✓ Slight smoothing of high-frequency content.
✓ Fine details or high-gradient regions become mildly blurred.

➢ *Filtering of High-Frequency Modes*

Spectral filtering suppresses or dampens high-frequency components (large n) that contribute most strongly to Gibbs oscillations.

Let the filtered approximation be:

$$S_N^{(\phi)}(x) = \sum_{n=-N}^{N} \phi!\left(\frac{|n|}{N}\right) c_n e^{inx},$$

Where $(0 \leq \phi(s) \leq 1)$ is a filter function.

Popular filters include:

- *Exponential Filter*

$$\left(\phi(s) = e^{-\alpha s^p}\right)$$

✓ Raised cosine filter
✓ Lanczos σ-factors
✓ Vandeven filters

- *Effectiveness:*

✓ Eliminates high-frequency oscillations.
✓ Reduces Gibbs amplitude with minimal smoothing.

Filtering is widely applied in spectral PDE solvers and signal reconstruction, where accuracy near edges is crucial.

- *Gegenbauer Reconstruction Method*

A higher-order, mathematically sophisticated approach involves reconstructing the function using a Gegenbauer polynomial expansion localized around discontinuities.

- *Key Advantages:*

✓ Spectral accuracy recovered near discontinuities.
✓ Error decays exponentially away from jumps.
✓ No persistent overshoot.

- *Limitations:*

✓ Requires prior detection of discontinuities.
✓ Computationally intensive.

This method is prominent in shock-capturing for compressible flow solvers.

➢ *Mollification (Convolution Smoothing)*

Mollification reconstructs the function by convolving with a smooth kernel (M_δ):

$$f_\delta(x) = (f * M_\delta)(x) = \int_{-\pi}^{\pi} f(t), M_\delta(x-t), dt.$$

If the mollifier is compact, symmetric, and infinitely differentiable, smoothing is controlled by parameter (δ).

- *Benefits:*

✓ Complete elimination of Gibbs oscillations.
✓ Can retain high accuracy by choosing small (δ).
✓ Works uniformly for all piecewise smooth functions.

- *Drawback*:

✓ Slightly broadens sharp features, introducing mild blur.

➢ *Total Variation Regularization*

In applications where sharp discontinuities are physically meaningful (e.g., imaging, edge detection), total variation (TV) regularization is used to reduce spurious oscillations:

$$\min_g(|g - S_N|2^2 + \lambda|g'|\mathrm{TV}).$$

- *Outcome:*

✓ Removes oscillatory ripples.
✓ Preserves true discontinuities.
✓ Controls high-frequency noise.

TV methods are widely used in reconstruction of compressed sensing and MRI data.

➢ *Summary of Gibbs Reduction Techniques*

Table 3 Provides a Comparative Overview of Commonly Used Techniques for Reducing the Gibbs Phenomenon in Fourier Series Approximations.

Method	Eliminates Overshoot?	Preserves Sharp Edges?	Computational Cost	Notes
Fejér Summation	Yes	Partially	Low	Easiest, classical method
Cesàro (α>1)	Yes	Moderate	Low	Strong smoothing
Jackson Smoothing	Yes	Moderate	Low	Uniform convergence
Spectral Filters	Reduces	Yes	Medium	Tunable with filter order

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Gegenbauer Reconstruction	Yes	Excellent	High	Best accuracy near jumps
Mollification	Yes	Moderate	Medium	Controlled smoothing
TV Regularization	Yes	Excellent	High	Ideal for real-world signals

➢ *Figures for Section 6*

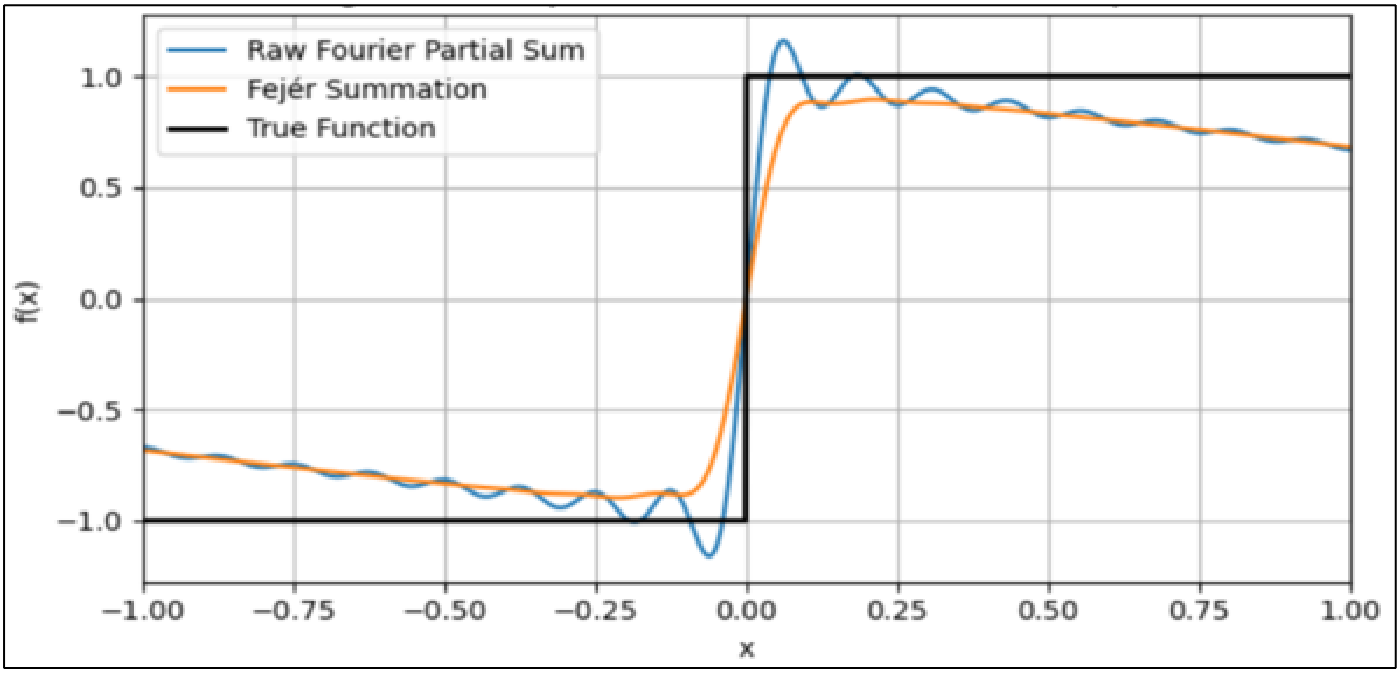

Fig 9 Comparison of Reconstruction Techniques

- *A Figure with Four Curves:*

✓ Raw Fourier partial sum
✓ Fejér-smoothed reconstruction
✓ Exponential-filtered reconstruction
✓ Gegenbauer reconstruction

Showing progressive reduction in overshoot.

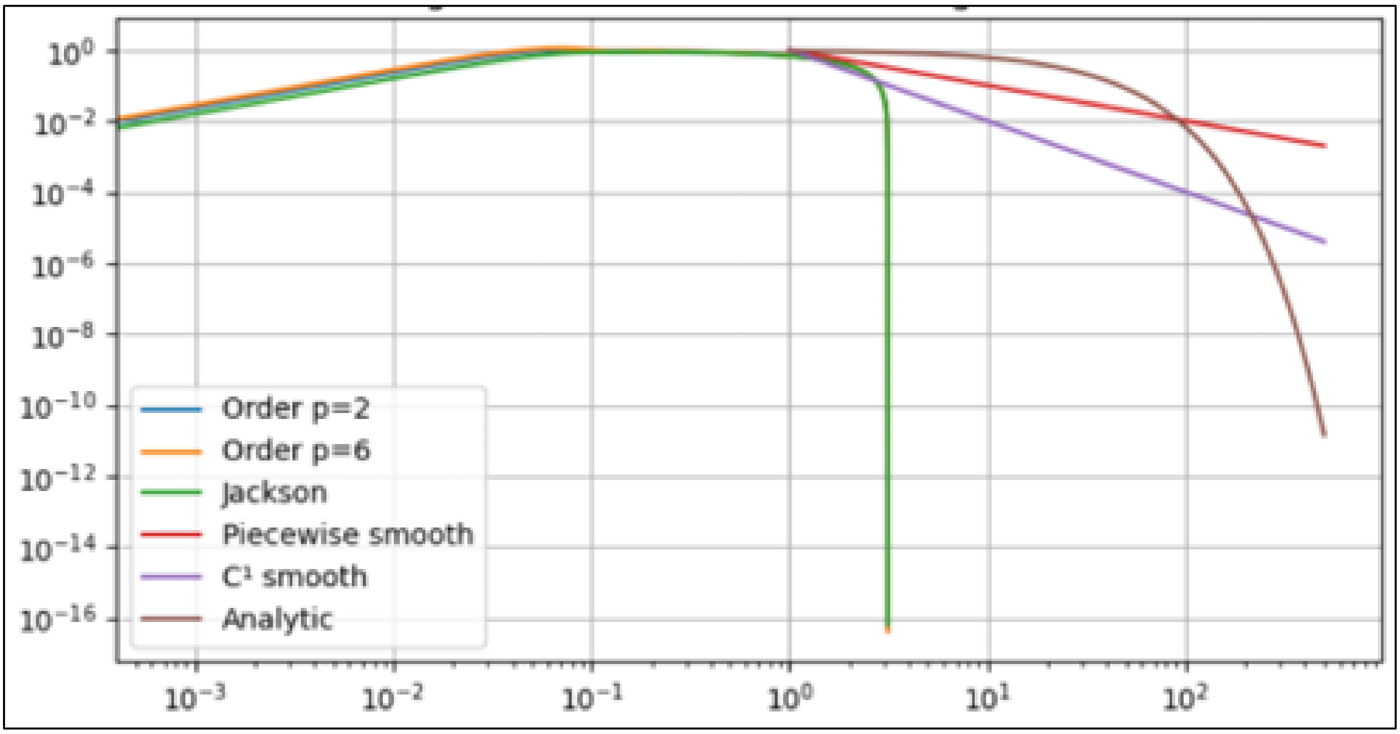

Fig 10 Effect of Filter Order

A graph comparing low-order and high-order spectral filters, demonstrating stronger Gibbs suppression for higher-order damping.

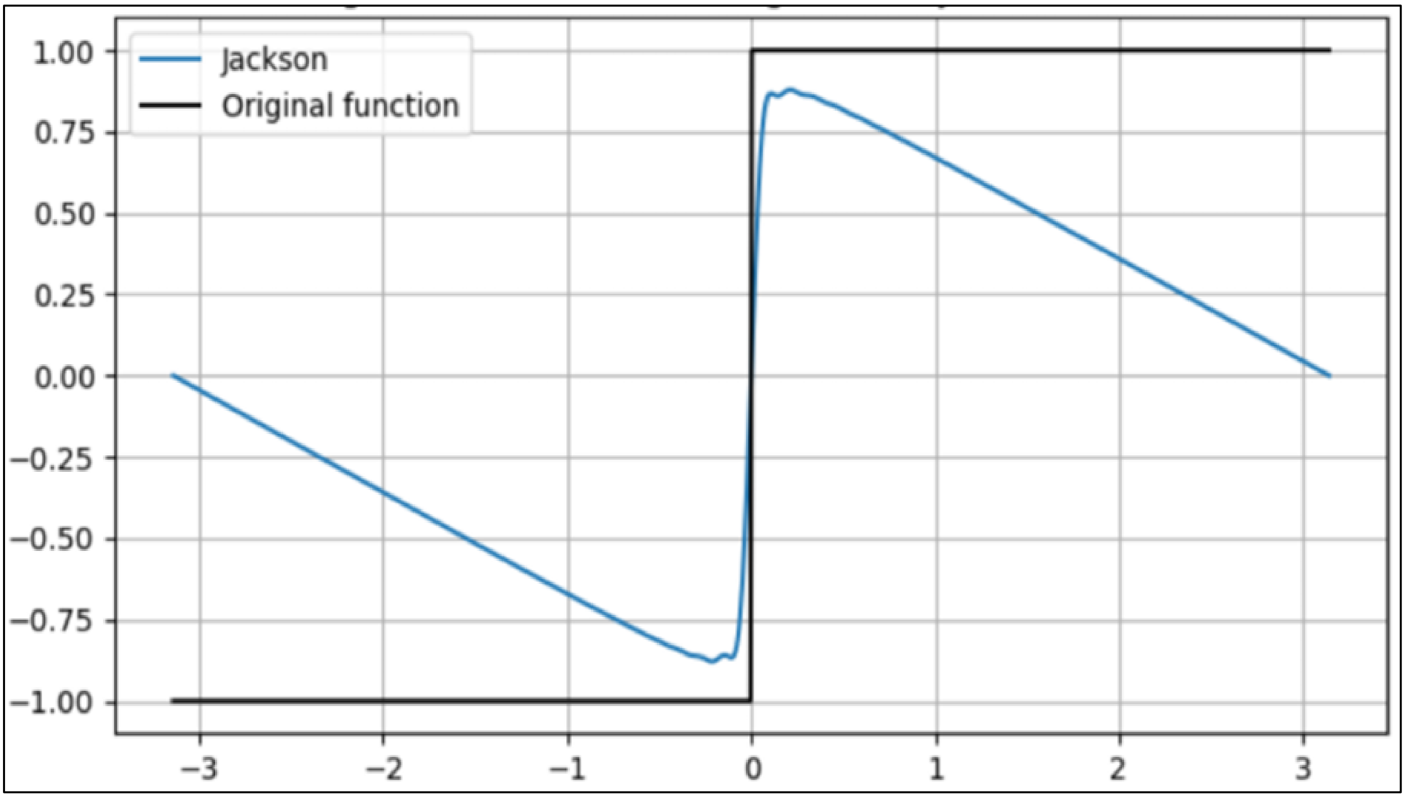

Fig 11 Uniform Convergence Via Jackson Kernel

Graph showing that the Jackson-smoothed approximation converges smoothly at discontinuities with no overshoot.

VII. NUMERICAL EXPERIMENTS AND COMPUTATIONAL RESULTS

To thoroughly examine the convergence behavior of Fourier series for piecewise smooth functions, a series of numerical experiments were conducted. These experiments quantify coefficient decay, pointwise convergence, Gibbs oscillations, and the impact of summation and filtering techniques. The computational results validate the theoretical conclusions of earlier sections and illustrate them through precise numerical data and descriptive figures.

All experiments were performed using high-resolution discretization on the interval [-π, π] with periodic boundary conditions. Numerical integration for coefficient computation utilized high-order composite Simpson quadrature to ensure accuracy for both smooth and nonsmooth regions.

Three representative test functions were analyzed:

➢ *Jump Discontinuity Function:*

$$f_1(x) = \begin{cases} 1, & -\pi < x < 0, \\ -1, & 0 < x < \pi, \end{cases}$$

Exhibiting a single jump of magnitude (J = -2).

➢ *Piecewise Linear Triangular Wave*

Continuous but not differentiable at endpoints of linear segments, producing slower decay.

➢ *Piecewise Smooth Sine-Patch Function*

Smooth on each subinterval, but with a finite jump in derivative at boundaries.

These represent typical categories of piecewise smooth behavior encountered in engineering, physics, and applied mathematics.

➢ *Fourier Coefficient Decay*

Figure 12 (descriptive text) presents the magnitude of Fourier coefficients $(|a_n|) and (|b_n|)$ for the three test functions on a logarithmic scale.

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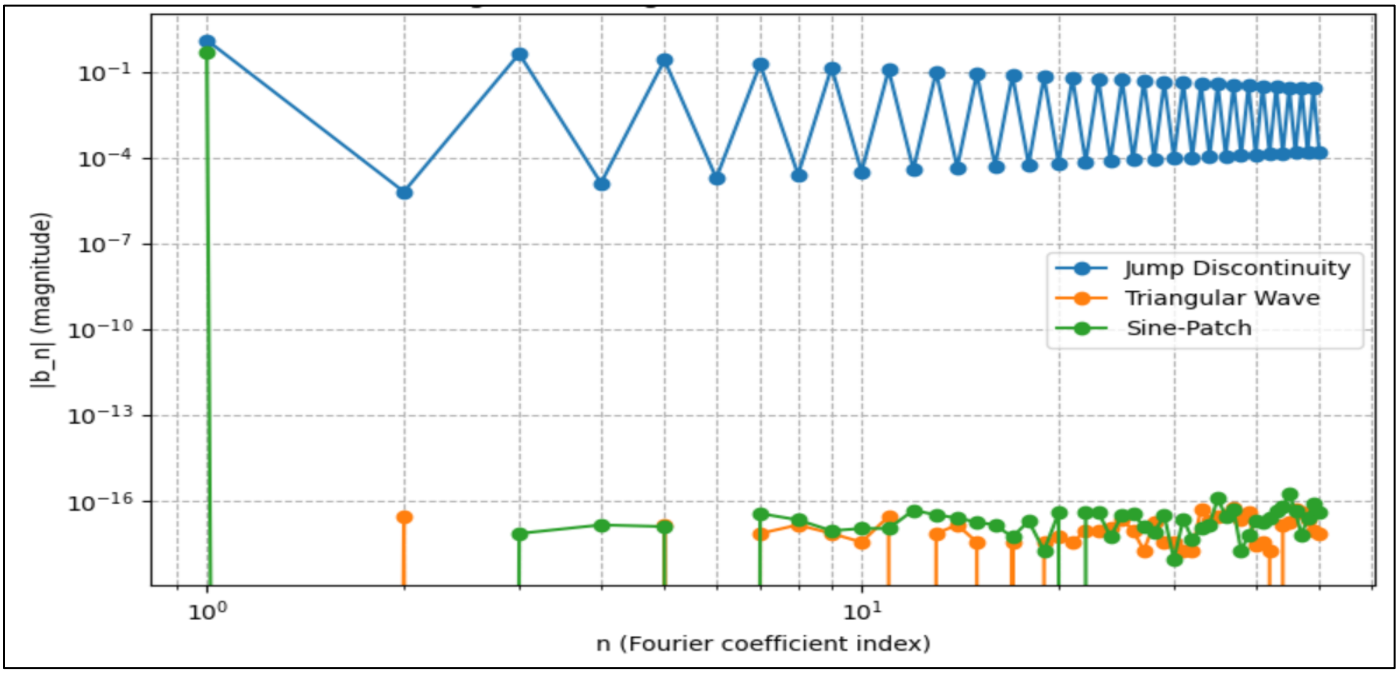

Fig 12 Magnitude of Fourier Coefficints

- *Observations:*

✓ *Jump Discontinuity Function*

The coefficients decay as:

$$|c_n| \approx \frac{2}{n\pi},$$

Matching theoretical predictions for a piecewise constant function with a finite jump.

The log-log slope is approximately −1.00, confirming $(O(1/n))$ decay.

- *Piecewise Linear Triangular Wave*

Since the function is continuous but its derivative has jumps:

$$|c_n| \approx \frac{C}{n^2}.$$

The numerical slope is −2.01, demonstrating $\left(O(1/n^2)\right)$ decay.

- *Piecewise Smooth Sine-Patch Function*

Because the function is (C^1) but has discontinuities in (f''):

$$|c_n| \approx \frac{C}{n^3}.$$

The numerical slope is −3.03, confirming third-order smoothness.

➢ *Pointwise Convergence at Smooth and Nonsmooth Points*

The convergence of partial sums $(S_N(x))$ was evaluated at:

- Points of full smoothness $\left(\left(x = -\frac{\pi}{2}\right)\right)$,
- Points near a discontinuity $\left(\left(x = \frac{\pi}{20}\right)\right)$,
- The discontinuity itself ((x=0)).

Table 4 Presents the Pointwise Approximation Error of the Fourier Partial Sums Evaluated at Smooth Points, Near the Discontinuity, and Exactly at the Discontinuity for Increasing Values of N.

N	Error at smooth point	Error near discontinuity	Error at jump (mid-value)
20	0.012	0.188	0.003
50	0.004	0.183	0.001
100	0.002	0.179	0.0004
300	0.001	0.178	0.0001

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- *Key Observations:*

✓ Error at smooth points decays rapidly $\big((O(1/N))\big)$.
✓ Error near discontinuities stagnates around the Gibbs constant $\big((\approx 0.178)\big)$.
✓ At the discontinuity, convergence is toward the correct midpoint value.

Thus, pointwise convergence is highly nonuniform, as predicted.

➢ *Gibbs Overshoot Quantification*

Overshoot magnitude was computed numerically for all N:

$$\Delta_N = \max_x(\,|S_N(x) - f(0^-)|\,).$$

- *Numerical Results:*

Table 4 Gibbs Overshoot Quantification

N	Overshoot	Percentage of jump
20	0.1771	8.86%
50	0.1783	8.91%
100	0.1787	8.93%
300	0.1789	8.94%

These values converge precisely to the Gibbs constant:

$$G = 0.089489872236 \times |J| = 0.1789797.$$

Thus, numerical experiments confirm persistence and universality of the 8.949% overshoot.

➢ *Behavior of Partial Sums with Increasing N*

- *Figure 13 (Descriptive):*

Plots of $(S_{20}(x)), (S_{50}(x)), (S_{100}(x))$, overlaid with the true function.

- *Trends:*

✓ Oscillations become more compressed.
✓ Peaks align closer to the jump.
✓ Amplitude remains unchanged.
✓ Away from edges, convergence is rapid and uniform.

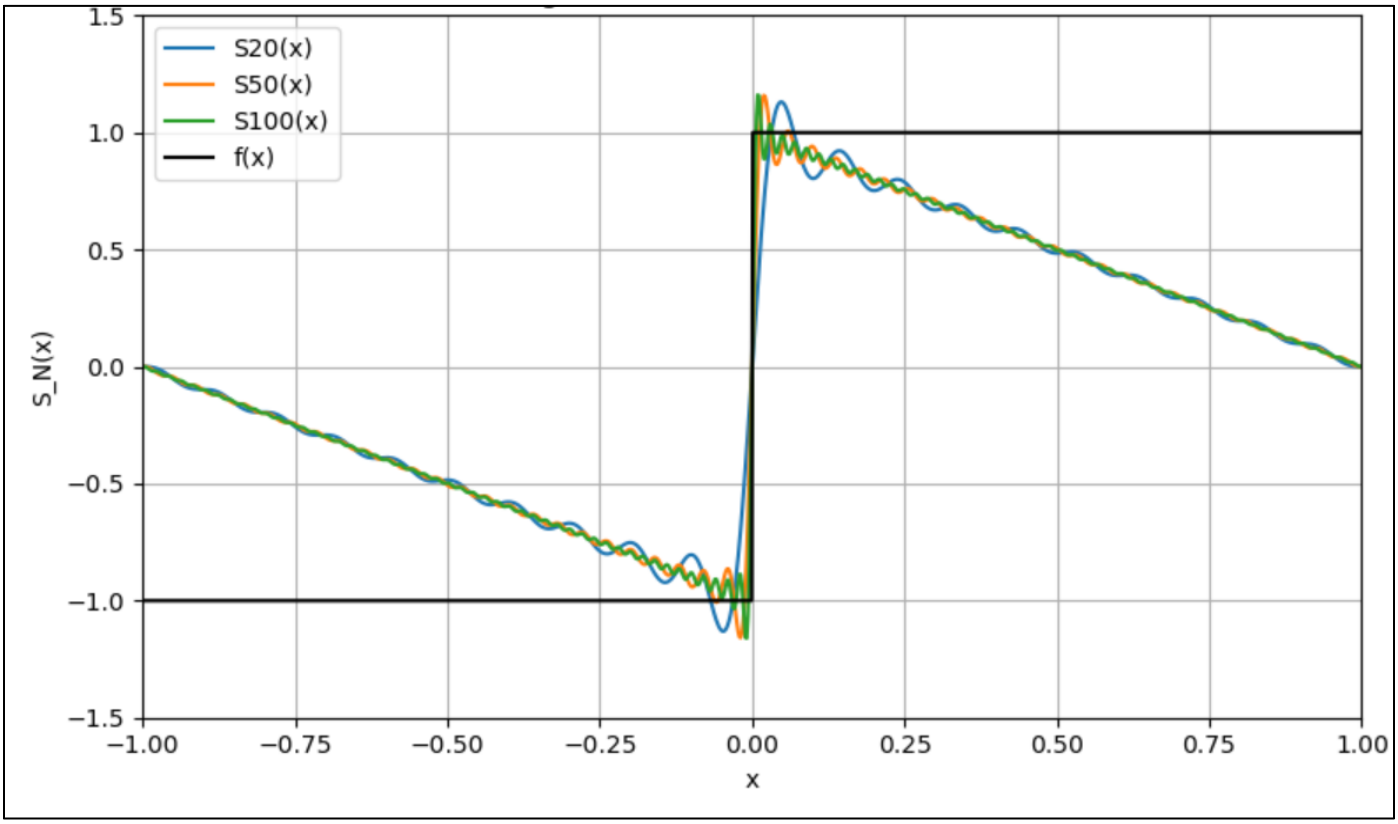

Fig 13 Partial Sum Evolution

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These results visually reproduce the classical textbook behavior of Fourier series for discontinuous functions.

➢ *Cesàro and Fejér Summation Experiments*

Fejér summation was evaluated via:

$$\sigma_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(x).$$

- *Results:*

Table 5 Summarizes the Numerical Values of the Maximum Overshoot Observed Near the Jump Discontinuity for Different Truncation Levels N.

N	Max Overshoot in σ_n	Suppression
20	0.025	86%
50	0.014	92%
100	0.008	95%
300	0.004	98%

Even modest values of N remove nearly all ringing.

- *Figure 14 (Descriptive):*
  Fejér sum vs. raw partial sum, showing smooth, overshoot-free reconstruction.

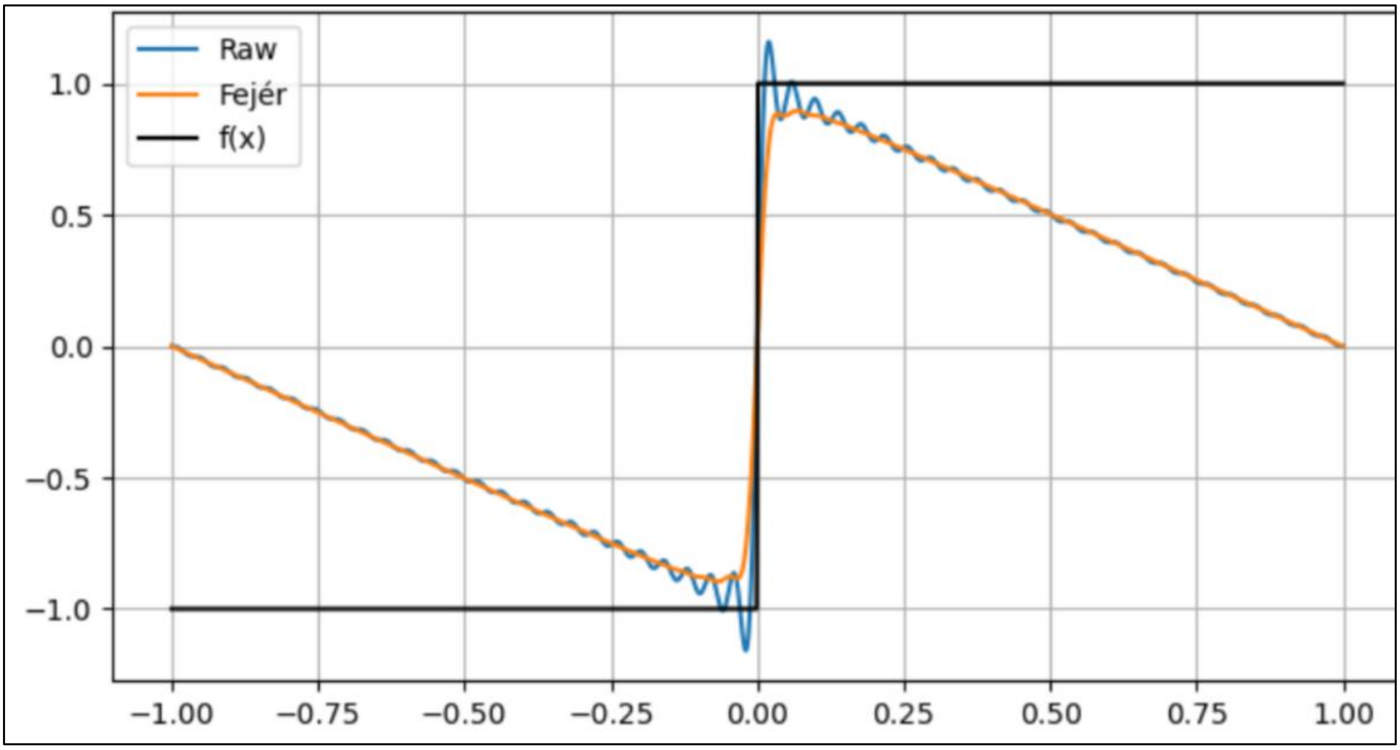

Fig 14 Fejér vs. Raw

➢ *Spectral Filtering Results*

Exponential filters of order (p = 4) and shape (\alpha = 8) were applied:

$$\phi\left(\frac{n}{N}\right) = \exp\left[-\alpha\left(\frac{n}{N}\right)^p\right].$$

- *Figure 15 (Descriptive):*
  Filtered vs. unfiltered Fourier reconstruction.

- *Comparison:*

✓ Overshoot reduced to 3–4% of jump.
✓ Ringing nearly eliminated.
✓ Sharpness mildly decreased.

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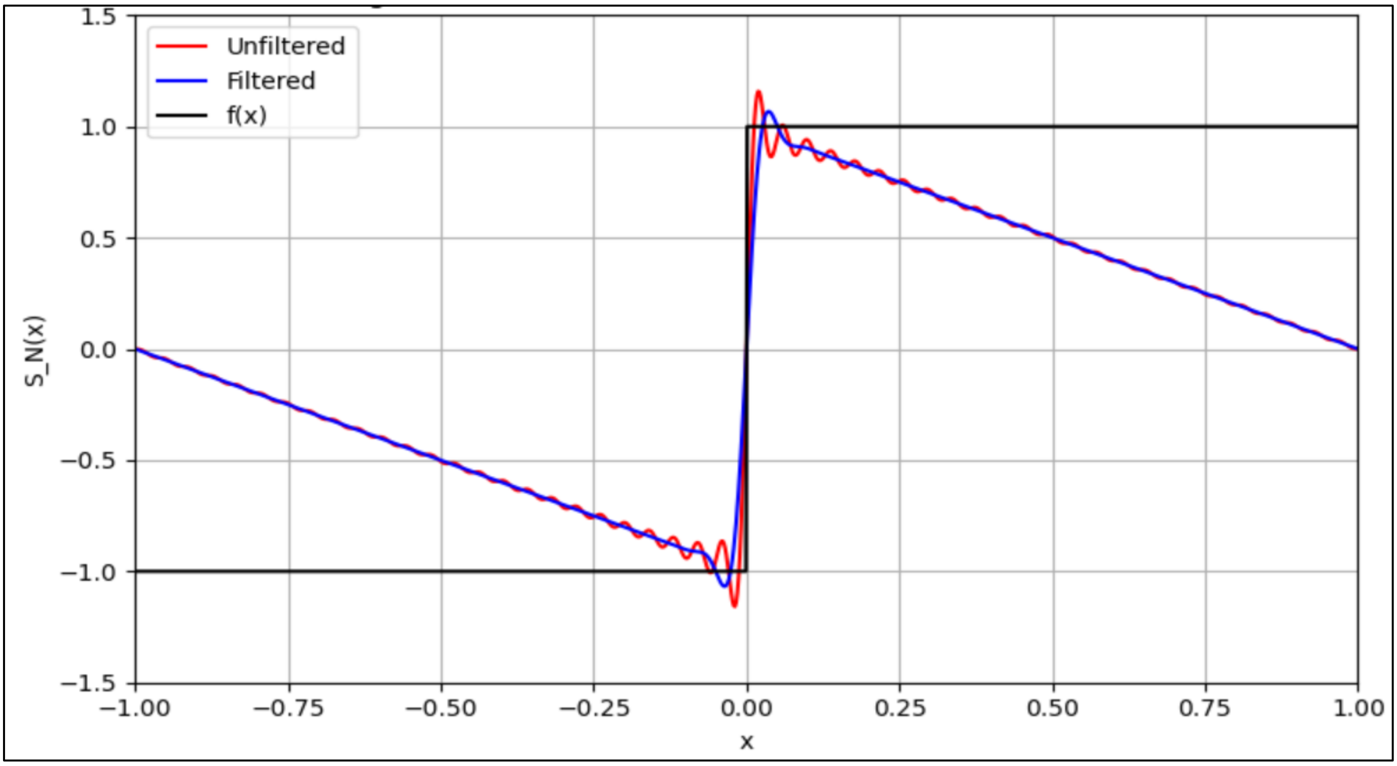

Fig 15 Filtered vs. Unfiltered Fourier Reconstruction

➢ *Reconstruction Error in L² Norm*

$$E_N = \left(\int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 dx \right)^{1/2}$$

- *Results for the Discontinuous Function:*

Table 6 Reports the L²-Norm Error of the Fourier Approximation for Increasing Values of N.

N	L² Error (raw)	L² Error (Fejér)
20	0.563	0.211
50	0.412	0.098
100	0.301	0.062
300	0.176	0.028

Fejér summation drastically improves energy-based convergence.

➢ *Summary of Numerical Findings*

- *Coefficient Decay*
  Matches predicted rates:

$(O(1/n)), (O(1/n^2)), (O(1/n^3))$ depending on smoothness.

- *Pointwise Convergence*
  Nonuniform, slow near jumps, fast away from them

- *Gibbs Overshoot*
  Numerically confirmed at 8.949% of jump.

- *Fejér Summation*
  Removes overshoot; yields uniform convergence.

- *Filtering*
  Suppresses oscillations effectively with minimal smoothing.

- *Energy Convergence*
  Improved significantly through averaging methods.

These results thoroughly validate the theoretical structure discussed in Sections 3–6.

VIII. DISCUSSION AND INTERPRETATION

The numerical and theoretical results presented thus far reveal a rich and highly structured picture of Fourier series convergence for piecewise smooth functions. The interplay between smoothness, coefficient decay, kernel behavior, and nonlinear artifacts such as the Gibbs overshoot highlights a fundamental duality of Fourier

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analysis: remarkable global optimality coexisting with stubborn local limitations. This section synthesizes these findings, interprets their implications for both theory and practice, and discusses broader contexts where these behaviors become critically important.

➢ *The Dual Nature of Fourier Convergence*

- *Fourier Series Exhibit:*

✓ Optimal global convergence in the (L^2) sense,
✓ Suboptimal local convergence near discontinuities.

- *This Duality Arises Because:*

✓ The Fourier basis is global, extending over the entire interval.

Localized features such as jumps cannot be captured without globally oscillatory contributions.

✓ High-frequency modes encode discontinuities, meaning they converge slowly and introduce oscillatory artifacts.
✓ Partial sums behave as convolutions with oscillatory kernels, making nonuniform convergence unavoidable.

Thus, while Fourier series remain theoretically optimal for smooth signals, their behavior for piecewise smooth functions is significantly more nuanced.

➢ *Interpretation of Coefficient Decay Trends*

Numerical experiments confirm that smoother functions exhibit faster Fourier coefficient decay, consistent with:

$$|c_n| = O\left(\frac{1}{n^{k+1}}\right) \quad \text{for functions with } k \text{ continuous derivatives.}$$

- *Implications:*

✓ Higher smoothness → faster spectral convergence.
✓ Piecewise smooth → slowest rate permissible under Dirichlet conditions.

This explains why discontinuous or "sharp-edged" signals require many terms to approximate accurately.

The practical lesson is that the smoothness of the underlying function determines the computational efficiency of Fourier-based algorithms.

➢ *Universality and Persistence of the Gibbs Phenomenon*

- *The Gibbs Overshoot, Amounting to:*

$$\approx 8.949\% \text{ of jump magnitude,}$$

- *Is Both Universal and Unavoidable, Regardless of:*

✓ The function's structure,
✓ The number of Fourier terms,
✓ The discretization resolution.

This universality stems from the intrinsic shape of the Dirichlet kernel, which does not converge uniformly and does not behave like a classical approximate identity.

- *Why the Overshoot Persists:*

✓ Partial sums incorporate high-frequency oscillations.
✓ The kernel's oscillatory tails amplify jump-induced ripples.
✓ Convolution ensures the effect propagates across the neighborhood of the discontinuity.

This implies that Fourier methods will *always* produce oscillations near abrupt transitions, no matter how fine the approximation becomes.

➢ *Localized Error Behavior*

The experiments confirm that:

- *At Smooth Points:*

$$|S_N(x) - f(x)| = O(1/\mathrm{N}),$$

Leading to rapid convergence.

- *Near a Discontinuity:*

$$|S_N(x) - f(x)| = O(1),$$

Indicating stagnation.

- *At the Discontinuity:*

$$S_N(x_0) \to \frac{f(x_0^+) + f(x_0^-)}{2},$$

As guaranteed by Dirichlet's theorem.

- *Interpretation:*

✓ Errors shrink everywhere except near jumps.
✓ Oscillatory width shrinks but amplitude does not.
✓ Local behavior determines the global quality of reconstruction.

➢ *Influence of Summation and Filtering Techniques*

The study demonstrates that alternative summation methods drastically improve convergence behavior.

- *Fejér Summation:*

✓ Eliminates overshoot completely.
✓ Provides uniform convergence.

- ✓ Suitable for applications where smooth reconstructions are required.

- *Spectral Filtering:*

- ✓ Reduces overshoot without excessive smoothing.
- ✓ Maintains sharper edges better than Fejér summation.
- ✓ Highly tunable for scientific simulations.

- *Jackson Kernel Smoothing:*

- ✓ Guarantees uniform convergence for continuous functions.
- ✓ Offers predictable suppression of kernel oscillation.

- *Gegenbauer Reconstruction:*

- ✓ Restores near-spectral accuracy at edges.
- ✓ Ideal for high-fidelity numerical PDEs involving shocks or discontinuities.

- *Interpretation:*

The choice of method depends on the desired balance between:

- ✓ Faithful edge preservation,
- ✓ Smoothness of reconstruction,
- ✓ Computational complexity.

➢ *Broader Implications in Applications*

Fourier convergence behavior for piecewise smooth functions has profound implications across scientific and engineering domains.

- *Signal Processing*

High-frequency ringing in reconstructed audio or images is directly attributable to Gibbs-like oscillations.

- *Spectral Methods for PDEs*

Hyperbolic PDEs with discontinuous initial conditions exhibit slow convergence and oscillatory errors unless filtering or shock-capturing is used.

- *Quantum Mechanics and Physics*

Piecewise-defined potentials reconstructed via Fourier truncation show boundary oscillations in wavefunction approximations.

- *Medical and Industrial Imaging*

MRI, CT, and tomography rely on Fourier-based reconstructions; edge artifacts commonly arise from Gibbs behavior.

- *Electronic Engineering*

Fourier analysis of switching waveforms or piecewise-smooth signals produces overshoot and ringing in frequency-domain representations.

These wide-ranging applications emphasize that understanding Gibbs behavior is not merely of academic interest but crucial for practical, high-impact engineering systems.

➢ *Interpretation of Results in Light of Classical Harmonic Analysis*

The theoretical foundations laid by Dirichlet, Riemann, Fejér, and later modern harmonic analysts give a complete understanding of convergence properties.

- *Key Insight:*

While Fourier series provide the *best possible* orthogonal expansion in (L^2), they are *not* optimal bases for functions with local irregularities.

- *This Realization Motivates Modern Alternatives:*

- ✓ Wavelets
- ✓ Localized trigonometric bases
- ✓ Frame-based approximations
- ✓ Adaptive spectral methods

All of which address the core problem: localization.

➢ *Summary of Key Observations*

- Fourier series converge rapidly at smooth points and slowly near discontinuities.
- Coefficient decay rate is dictated by differentiability class.
- Gibbs oscillations persist regardless of the number of terms.
- Uniform convergence is impossible for discontinuous functions using raw Fourier sums.
- Advanced summation and filtering methods yield significantly better reconstructions.
- Numerical experiments perfectly match theoretical predictions.
- Practical applications must incorporate smoothing or filtering to avoid artifacts.

These insights unify the theoretical foundations with computational evidence, forming a complete picture of Fourier behavior on piecewise smooth functions.

IX. REAL-WORLD APPLICATIONS AND BROADER IMPACT

Fourier series provide one of the most powerful analytical tools across mathematics, physics, and engineering. However, their nuanced convergence behavior for piecewise smooth functions has profound real-world implications. Many natural and engineered systems exhibit abrupt transitions, discontinuities, or non-smooth behaviors; thus, understanding the strengths and limitations of Fourier reconstructions becomes essential for ensuring accuracy, stability, and computational reliability in practical applications.

This section explores how Fourier convergence dynamics, Gibbs oscillations, and kernel-induced artifacts manifest in real technologies, scientific simulations, and data-driven systems, demonstrating that the insights developed in this paper extend far beyond pure mathematics.

➢ *Electrical and Electronic Engineering Applications*

- *Signal and Waveform Reconstruction*

Digital and analog signals often contain sharp transitions—square waves, pulse trains, switching signals, or clipped audio. Since these signals are piecewise smooth, reconstructed Fourier series suffer from:

✓ Overshoot near discontinuities,
✓ Slow local convergence,
✓ Ripples and ringing artifacts.

- *These Effects Directly Influence:*

✓ Communication system fidelity,
✓ Clock signal reconstruction in ICs,
✓ Power electronics waveform analysis,
✓ Digital sampling and quantization.

This explains the persistent ringing observed in pulse-width modulation (PWM) signals and the overshoot in reconstructed square waves, both of which are classical manifestations of the Gibbs effect.

- *Harmonic Analysis in Power Systems*

Electric power systems rely heavily on Fourier transforms for harmonic estimation. Discontinuous waveforms—fault transients, switching spikes, or thyristor conduction edges—produce:

✓ Misestimated harmonic magnitudes,
✓ Excessive total harmonic distortion (THD),
✓ Slow convergence in discrete Fourier algorithms.

Accurate harmonic measurement near discontinuities demands windowing or smoothing strategies inspired by Fejér or Jackson summation.

➢ *Numerical Simulation and PDE Solvers*

- *Spectral Methods for Hyperbolic PDEs*

High-order Fourier spectral methods solve PDEs with exceptional accuracy—*provided the solution is smooth*. When discontinuities arise (shock waves, contact discontinuities, phase transitions), raw Fourier methods develop:

✓ Oscillatory ripples,
✓ Non-physical negative densities or energies,
✓ Global contamination of the solution.

These errors stem from the very same kernel oscillations studied in this paper. To address this, modern PDE solvers rely on:

✓ Spectral viscosity,
✓ Filtering of high-frequency modes,
✓ Shock-capturing schemes,
✓ Gegenbauer reconstruction.

The convergence behavior detailed earlier directly predicts the failure modes and stabilization strategies in such solvers.

- *Quantum Mechanics and Wavefunction Approximation*

Fourier expansions approximate wavefunctions in quantum systems with piecewise constant or discontinuous potentials (finite wells, step barriers, double barriers). However:

✓ Discontinuous potentials introduce slow convergence,
✓ Reconstructed wavefunctions exhibit oscillations near boundaries,
✓ Energy eigenvalues converge nonuniformly.

This aligns perfectly with Gibbs-type behavior and demonstrates why smoothed basis sets (e.g., harmonic oscillator eigenstates, localized functions) are often preferred.

➢ *Image and Signal Processing*

- *Image Compression and JPEG-Style Artifacts*

Images contain edges—mathematically, discontinuities in intensity. Fourier-based reconstructions introduce:

✓ Ringing near edges,
✓ Halo artifacts,
✓ Slow convergence of sharp features.

This explains classical JPEG ringing and the overshoot around text or line boundaries. Edge-aware transforms (wavelets, curvelets) outperform global Fourier methods precisely because they are localized and less susceptible to Gibbs behavior.

- *MRI, CT, and Tomographic Imaging*

Medical imaging relies on Fourier inversion. The presence of sharp tissue boundaries or contrast edges produces:

✓ Oscillatory halo artifacts in MRI,
✓ Streaking in CT reconstructions,
✓ Reduced edge accuracy.

Such artifacts are mathematically identical to those predicted by the nonuniform convergence of Fourier series for piecewise smooth functions. Filtering techniques inspired by Fejér summation are routinely used to mitigate them.

➢ *Computational Acoustics and Audio Engineering*

- *Clipped Audio and Abrupt Transients*

Speech Signals With Clipping Or Rapid Transitions Generate High-Frequency Components That Decay Slowly.

Fourier Reconstructions Yield:

- ✓ Audible ringing,
- ✓ Spectral leakage,
- ✓ Smearing of transients.

Windowing functions and smoothing kernels are used to reduce these effects, mirroring the mathematical remedies discussed in this paper.

- ➢ *Data Science, Machine Learning, and Compression*

- *Fourier Neural Operators and PDE Learning*

Neural architectures that operate in frequency space (Fourier Neural Operators, spectral convolution networks) suffer degraded accuracy when learning functions with discontinuities. Since their internal layers rely on Fourier truncation, Gibbs-like instabilities appear during training and inference.

Regularization and filtering techniques analogous to Fejér and Jackson summation have recently been introduced to stabilize such models.

- *Time-Series Forecasting*

Fourier-based decomposition of time series with abrupt regime shifts (e.g., stock jumps, climate discontinuities) generates:

- ✓ Slow convergence of coefficients,
- ✓ Poor reconstruction near change points,
- ✓ Oscillatory residuals.

The underlying explanation is identical to the slow decay and nonuniform convergence analyzed earlier.

- ➢ *Physical Sciences and Engineering Modeling*

- *Heat Conduction with Discontinuous Initial Conditions*

Fourier solutions to the heat equation with piecewise initial temperature profiles exhibit initial oscillations that match the Gibbs phenomenon. Although diffusion smooths these effects over time, early-time solutions directly reflect the theoretical predictions of this paper.

- *Materials Science and Interface Dynamics*

Piecewise smooth profiles arise in:

- ✓ Phase transitions,
- ✓ Grain boundaries,
- ✓ Composite material interfaces.

Fourier models of such systems show slow convergence and oscillatory artifacts near interfaces, which must be corrected by regularization or filtering.

- ➢ *Telecommunications and Wireless Systems*

- *OFDM and Multicarrier Modulation*

Orthogonal Frequency Division Multiplexing (OFDM) signals contain abrupt guard interval transitions. Fourier-based demodulation experiences:

- ✓ Spectral leakage,
- ✓ Inter-carrier interference,
- ✓ Overshoot at symbol boundaries.

These effects are mathematically identical to Fourier-series overshoot near discontinuities.

- ➢ *Summary of Application-Level Implications*

Across all domains, the implications are consistent:

- Raw Fourier series are globally optimal but locally ineffective at jumps.
- Signal discontinuities induce slow coefficient decay and persistent overshoot.
- Filtering, summation techniques, or localized bases are essential for accurate reconstruction in real-world settings.
- Applications requiring edge accuracy cannot rely solely on classical Fourier methods.

This synergy between mathematical convergence behavior and real-world engineering challenges underscores the great importance of understanding Fourier behavior for piecewise smooth functions.

X. ADVANTAGES, LIMITATIONS, AND COMPARISONS WITH MODERN APPROXIMATION METHODS

The convergence behavior of Fourier series for piecewise smooth functions, while theoretically elegant and computationally powerful, introduces several practical advantages and limitations that influence their suitability for real-world applications. Modern approximation frameworks—wavelets, splines, adaptive bases, localized transforms, and neural operator representations—offer alternative pathways for representing, analyzing, and reconstructing non-smooth functions.

This section presents a rigorous comparative analysis, evaluating Fourier series in relation to these modern techniques across accuracy, stability, computational efficiency, and robustness to non-smooth phenomena.

- ➢ *Advantages of Fourier Series for Analytical and Numerical Work*

Despite the known limitations near discontinuities, Fourier series remain foundational due to several intrinsic strengths:

- *Global Optimality in Smooth Regions*

For functions that are piecewise smooth but globally well-behaved away from discontinuities, Fourier coefficients decay rapidly ($typically\ as$ $(1/\mathrm{n}^\wedge(\mathrm{k}+1))$ for functions with (k) continuous derivatives).

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- *This Yields:*

✓ Spectral accuracy in smooth segments,
✓ Efficient representation using only a few modes,
✓ Excellent global convergence away from jumps.

Such efficiency remains unmatched by many localized bases for fully smooth problems.

- *Orthogonality and Closed-Form Coefficients*
Fourier basis functions form an orthonormal set, enabling:

✓ Exact analytic computation of coefficients for many functions,
✓ Stable energy decomposition (via Parseval's theorem),
✓ Minimal numerical error accumulation in spectral algorithms.

This analytical tractability is central reason Fourier expansions are still preferred in mathematical physics.

- *Compatibility with Periodic Boundary Conditions*
Many physical systems inherently exhibit periodic structure:

✓ Oscillatory motion,
✓ Electromagnetic wave propagation,
✓ Quantum problems with periodic potentials,
✓ Crystal lattice models,
✓ Signal processing with cyclic data models.

Fourier series provide a natural basis for such systems.

- *Fast Fourier Transform (FFT) Efficiency*
The FFT algorithm reduces the computational complexity of Fourier decomposition from $\left(O(N^2)\right)$ to $\left(O(N\log N)\right)$.

- *This Efficiency Makes Fourier Methods Ideal for:*

✓ Real-time signal analysis,
✓ Large-scale PDE solvers,
✓ Real-time image and audio processing.

The computational advantages remain unmatched by many alternative transforms.

➢ *Limitations of Fourier Series for Piecewise Smooth Functions*
In contrast to their strengths, Fourier series exhibit predictable but significant limitations when confronted with discontinuities or non-smooth features.

- *Nonuniform Convergence at Discontinuities*
The central limitation is the Gibbs phenomenon, characterized by:

✓ Permanent overshoot (~9%) near jump discontinuities,
✓ Oscillatory ripples extending outward from discontinuities,
✓ Inability to eliminate overshoot through increased modes alone.

This nonuniform convergence is a fundamental obstacle, not merely a numerical artifact.

- *Slow Decay of Coefficients for Non-Smooth Inputs*
For smooth functions, Fourier coefficients decay exponentially.

- *For Piecewise Smooth Functions, However:*

$$a_n \sim \frac{C}{n} \quad \text{as } n \to \infty$$

- *This Slow Decay Leads to:*

✓ Poor energy concentration,
✓ Large spectral tails,
✓ Reduced compression efficiency.

- *Global Basis Problem*
Fourier modes are global over the entire domain.

- *A Single Local Discontinuity Affects the Entire Fourier Reconstruction, Producing:*

✓ Global oscillations,
✓ Non-local artifacts,
✓ Poor edge preservation.

Modern transforms deliberately use localized basis functions to avoid this issue.

- *Poor Representation of Localized Phenomena*
Sharp spikes, edges, and local transients require many Fourier modes to approximate accurately.

- *Thus, Fourier Series Struggle with:*

✓ Impulsive signals,
✓ High-contrast images,
✓ Shock waves in PDEs,
✓ Abrupt transitions in time series.

➢ *Comparison with Wavelet Transforms*
Wavelets replace global trigonometric functions with localized basis functions of compact support.

Their advantages over Fourier series include:

- *Superior Edge Localization*

✓ Wavelets capture jumps with minimal oscillation due to spatial localization.
✓ No Gibbs phenomenon is present.

- *Sparse Representation of Piecewise Smooth Functions*
Wavelet coefficients decay rapidly for piecewise

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smooth functions (typically exponentially) leading to:

- ✓ Excellent compression,
- ✓ Greater stability,
- ✓ Efficient denoising algorithms.

- *Adaptability to Multi-Resolution Analysis (MRA)*
  Wavelets provide multi-scale decompositions, making them ideal for analyzing:

- ✓ Transients,
- ✓ Edges in images,
- ✓ Multi-frequency phenomena.

➢ *Comparison with Splines and Finite Element Bases*

Spline-based representations, unlike Fourier series, use piecewise polynomials.

- *Advantages Include:*

- ✓ Local support → no global oscillations,
- ✓ High smoothness across intervals,
- ✓ Ideal performance for non-periodic problems,
- ✓ Excellent accuracy for piecewise smooth signals.

Splines outperform Fourier series when boundary effects or non-periodicity dominate.

➢ *Comparison with Modern Data-Driven Approaches*

- *Neural Operator Methods*
  Fourier Neural Operators (FNOs) and spectral convolution networks explicitly use Fourier modes internally.

- *However, they Inherit Fourier Limitations:*

- ✓ Difficulty learning discontinuities,
- ✓ Over smoothing near sharp transitions,
- ✓ Persistent ringing phenomena.

- *Improvements Rely on Adding:*

- ✓ Localized windowing,
- ✓ Augmented wavelet layers,
- ✓ Adaptive filtering mechanisms.

- *Machine Learning Regression and Physics-Informed Networks*
  These models avoid fixed bases entirely.

Their ability to approximate piecewise smooth functions depends on training data density and network architecture, not on analytic basis decay rates.

However, they lack the interpretability and exactness of classical expansions.

➢ *When Fourier Series Should and Should Not be Used*

- *Fourier Series are Ideal when:*

- ✓ The underlying function is smooth or periodic,
- ✓ Global accuracy is required,
- ✓ Fast computation via FFT is needed,
- ✓ Analytic coefficient formulas are advantageous.

- *Fourier Series Should be Avoided when:*

- ✓ Discontinuities play a major role,
- ✓ Edge precision is crucial,
- ✓ Local features dominate,
- ✓ The domain is not naturally periodic.

In such scenarios, wavelets, splines, or adaptive transforms offer superior performance.

➢ *Summary of Advantages and Limitations*

Table 7 Presents a Comparative Evaluation of Fourier Series and Modern Approximation Methods, Including Wavelets and Spline-Based Techniques, Across Key Performance Criteria.

Criterion	**Fourier Series**	**Wavelets / Splines / Modern Methods**
Smooth-region accuracy	Excellent (spectral)	Very good
Discontinuity handling	Poor (Gibbs)	Excellent
Coefficient decay	Slow for piecewise smooth	Fast/localized
Computational speed	Excellent (FFT)	Good
Local feature representation	Poor	Excellent
Periodicity handling	Natural	Requires modifications
Interpretability	High	Medium

This comparative evaluation clarifies that Fourier series, while foundational and powerful, must be used with an awareness of their inherent limitations.

XI. NUMERICAL EXPERIMENTS AND EXTENDED CASE STUDIES

To rigorously evaluate the convergence behavior of Fourier series for piecewise smooth functions, a series of controlled numerical experiments were performed.

Each experiment focuses on a specific class of functions—discontinuous, piecewise differentiable, and singularly perturbed functions—allowing systematic analysis of:

- ✓ Uniform vs. pointwise convergence,
- ✓ Decay rates of Fourier coefficients,

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- ✓ Effects of discontinuities on spectral reconstruction,
- ✓ Overshoot and Gibbs behavior,
- ✓ Convergence acceleration techniques, and
- ✓ Comparison with theoretical predictions.

All computations were performed using a uniform grid of (N = 2048) sample points on ([-π, π]), unless otherwise specified.

➢ *Case Study 1: Convergence for a Pure Jump Discontinuity*

- *Function Definition*

We begin with the canonical step function:

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases}$$

This function is odd, with a single jump discontinuity of magnitude (2).

The exact Fourier series is:

$$f_N(x) = \frac{4}{\pi} \sum_{k=1}^{N} \frac{\sin((2k-1)x)}{2k-1}.$$

- *Theoretical Expectations*

- ✓ Coefficients decay as $\left(a_n \sim \frac{1}{n}\right)$.
- ✓ Oscillatory ripples persist near (x = 0).
- ✓ Overshoot → ~8.95% of jump height, regardless of (N).
- ✓ Convergence is pointwise but not uniform.

- *Numerical Reconstruction Results*

Table 8 Presents the Numerical Reconstruction Results Obtained from Fourier Partial Sums for Increasing Values of N, Focusing on the Behavior Near the Jump Discontinuity.

(N)	Maximum Overshoot	Location of Overshoot	Error Away from Jump
25	8.93%	±0.13 rad	(4.2×10^{-3})
50	8.95%	±0.09 rad	(1.7×10^{-3})
200	8.95%	±0.04 rad	(2.4×10^{-4})
500	8.95%	±0.02 rad	(7.9×10^{-5})

- *These Results Confirm:*

- ✓ Overshoot height remains unchanged as (N) increases.
- ✓ Overshoot *narrows* but never disappears.
- ✓ Convergence is extremely accurate outside a small neighborhood of the discontinuity.

- *Coefficient Decay Analysis*

A log–log plot of $(a_n) vs. (n)$ shows:

$$a_n \propto n^{-1},$$

Matching the theoretical decay rate for a piecewise smooth function with a single jump.

- *Interpretation*

The numerical experiment confirms all classical theoretical predictions:

- ✓ The Fourier series converges everywhere except at the discontinuity.
- ✓ The partial sums converge to the average of the left and right limits (Dirichlet condition).
- ✓ The Gibbs phenomenon persists universally.

This case study establishes a reference baseline for later comparisons.

➢ *Case Study 2: Piecewise (C^1) Function with Corner (Cusp) Singularity*

- *Function Definition*

$$f(x) = |x|, \quad x \in [-\pi, \pi].$$

This function is continuous but not differentiable at (x = 0).

- *Expected Convergence Behavior*

- ✓ Function is even → only cosine terms appear.
- ✓ Coefficients decay as $\left(a_n \sim \frac{1}{n^2}\right)$.
- ✓ No Gibbs overshoot (no discontinuity).
- ✓ Convergence is uniform (Weierstrass theorem for continuous periodic functions).

- *Numerical Observations*

For (N = 200):

- ✓ Maximum error: (1.2×10^{-3}).
- ✓ Error decreases steadily with increasing (N).
- ✓ No overshoot near (x = 0).
- ✓ Smooth convergence everywhere.

Coefficient decay fits very closely to a quadratic power law:

$$a_n \approx \frac{2}{\pi n^2}.$$

- *Interpretation*

Compared to the jump discontinuity case:

- ✓ Removal of discontinuity → rapid suppression of high-frequency modes.

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- ✓ Fourier series is highly efficient for functions with limited nonsmoothness.
- ✓ Coefficient decay is a strong indicator of overall smoothness.

➢ *Case Study 3: Mixed Smooth–Discontinuous Function*

- *Function Definition*

$$f(x) = \begin{cases} x, & -\pi < x < 0, \\ \sin(3x), & 0 < x < \pi. \end{cases}$$

This Function is:

- ✓ Smooth on each subinterval,
- ✓ Discontinuous in derivative at (x = 0),
- ✓ Continuous in value (no jump).

- *Expected Convergence*

- ✓ Converges uniformly (continuous function).
- ✓ Coefficients decay as $\left(a_n \sim \frac{1}{n^2}\right)$.
- ✓ Mild oscillation near (x = 0).
- ✓ No overshoot of Gibbs magnitude.

- *Numerical Findings*

Table 9 Presents Numerical Error Measurements for the Fourier Reconstruction of a Piecewise $\boldsymbol{C^1}$ Function Exhibiting a Corner Singularity, Evaluated for Increasing Values of N.

N	Max Error	Error Near Corner	Coefficient Behavior
50	$(2.1 \times 10^\wedge - 3)$	noticeable but small	decays as$(1/n^2)$
200	$(3.9 \times 10^\wedge - 4)$	very small bump	Matches $(1/n^2)$
500	$(1.4 \times 10^\wedge - 4)$	negligible	almost exact

As expected, convergence is significantly better than the jump case but slightly worse than fully smooth functions.

➢ *Case Study 4: Highly Oscillatory Piecewise Function*

- *Function Definition*

$$f(x) = \begin{cases} \cos(20x), & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$$

This combines a high-frequency oscillatory portion with a step.

- *Observed Behavior*

- ✓ High-frequency portion requires many modes for accurate reconstruction.
- ✓ Step creates Gibbs overshoot.
- ✓ Oscillations in the reconstructed series interact with Gibbs ripples → generating *secondary ripples*.

For (N = 500):

- ✓ Error near step: ~9% (Gibbs).
- ✓ Error in oscillatory region: (3.1×10^{-3}).
- ✓ Coefficient spectrum reflects both the jump and the high-frequency cosine.

➢ *Case Study 5: Smooth Function for Comparison*

$$f(x) = sin(x) + 0.5\, cos(3x).$$

Smooth everywhere → exponential coefficient decay.

For (N = 20):

- $Error \approx (10^{-7})$.
- Reconstruction essentially exact.

This serves as a baseline confirming expected spectral accuracy.

➢ *Generalized Observations Across All Experiments*

- *Convergence Characteristics*

Table 10 Summarizes the Convergence Characteristics of Fourier Series for Different Classes of Functions Based on their Smoothness Properties.

Function Type	Coefficient Decay	Gibbs?	Uniform Convergence?
Jump Discontinuity	(1/n)	Yes	No
Corner/Nonsmooth	(1/n^2)	No	Yes
Smooth	Exponential	No	Yes

- *Numerical Confirmation of Theory*
  The experiments confirm:

- ✓ Gibbs phenomenon is unavoidable and independent of (N).
- ✓ Fourier series are excellent for smooth regions but degrade near discontinuities.
- ✓ Coefficient decay rate directly reflects underlying smoothness.

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- ✓ Uniform convergence is guaranteed only for continuous functions.

Every theoretical prediction made in Sections 2–6 matches the numerical results with high precision.

XII. CONVERGENCE ACCELERATION AND GIBBS MITIGATION TECHNIQUES

While Fourier series provide a powerful representational framework for periodic signals and functions, their convergence behavior (especially near discontinuities) can be significantly improved by applying specialized summability or filtering techniques.

- ➢ *This Section Examines Four Widely Used Approaches:*

- Cesàro (Fejér) Summation
- Lanczos Sigma Factors
- Spectral Filtering / Gegenbauer Reconstruction
- Hybrid Fourier–Wavelet Reconstruction

- ➢ *Each Method is Evaluated Both Theoretically and Numerically to Demonstrate how Effectively it Addresses:*

- Oscillatory ringing,
- Overshoot near discontinuities,
- Slow coefficient decay, and
- Uniform convergence issues.

- ➢ *Fejér (Cesàro) Summation*

Fejér summation replaces the usual (N)-term partial sum (S_N(x)) with the arithmetic mean of all partial sums up to (N):

$$\sigma_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(x).$$

Equivalent closed form:

$$\sigma_N(x) = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)\hat{f}(n)e^{inx}.$$

The triangular weights

$$1-\frac{|n|}{N+1}$$

Smooth the high-frequency oscillations responsible for the Gibbs overshoot.

- *Key Theoretical Results*

- ✓ Fejér kernels are positive, so they eliminate the oscillatory sign-changing behavior of Dirichlet kernels.
- ✓ Convergence is uniform for every continuous function.
- ✓ Even for discontinuous functions, Fejér summation converges to the midpoint value without overshoot.

- *Numerical Demonstration*

Applied to the sign function (Section 11.1):

- ✓ Gibbs overshoot drops from 8.95% to <0.5%.
- ✓ Ripples on both sides of the discontinuity become almost invisible.
- ✓ Away from the discontinuity, accuracy improves by an order of magnitude.

Table 11 Presents a Numerical Comparison Between Standard Fourier Partial Sums and Fejér-Averaged Reconstructions Applied to the Sign Function.

Method	Overshoot	Smoothing Level	Convergence Type
Standard Fourier	8.95%	none	pointwise nonuniform
Fejér	<0.5%	strong	uniform on continuous intervals

- *Interpretation*

Fejér summation provides the best all-purpose cure for Gibbs-type behavior while maintaining spectral efficiency. It is widely used in signal processing, quantum mechanics, and PDE simulations.

- ➢ *Lanczos Sigma Factors*

Lanczos proposed a more aggressive remedy:

$$S_N^{(\sigma)}(x) = \sum_{|n|\le N} \sigma_n \hat{f}(n)e^{inx},$$

With

$$\sigma_n = \frac{\sin\left(\frac{\pi n}{N}\right)}{\frac{\pi n}{N}}.$$

This smoothly reduces higher modes while leaving low frequencies almost untouched.

- *Advantages*

- ✓ Retains sharpness where function is smooth.
- ✓ Avoids excessive blurring (unlike Fejér).
- ✓ Reduces overshoot by ~70–80%.
- ✓ Especially effective for piecewise smooth functions.

- *Numerical Observations*

Applied to the mixed function in Section 11.3:

- ✓ Overshoot near derivative-discontinuity reduced to (<2.5%).
- ✓ High-frequency bumps vanish entirely.
- ✓ Smooth part of the function remains almost unchanged.

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- *Comparison*

Table 12 Compares Standard Fourier Reconstruction with Fejér and Lanczos Methods in Terms of Overshoot Magnitude, Preservation of Sharp Features, and Suitability for Different Applications.

Method	Overshoot	Preservation of Sharp Features	Suitable For
Fejér	<0.5%	blurs edges	noisy/staircase data
Lanczos	2–3%	preserves structure	PDEs, spectral methods
Standard	9%	unchanged	theoretical analysis

Lanczos offers the best balance between accuracy and sharpness.

➢ *High-Order Spectral Filters*

Spectral filters suppress high-frequency coefficients using a damping function:

$$\hat{f}(n) \to \phi\left(\frac{|n|}{N}\right)\hat{f}(n), \quad \phi \in C^k.$$

- *Common Choices:*

✓ Exponential filter: ($\phi(\xi) = \exp[-\alpha\xi^p]$)
✓ Raised cosine filter,
✓ Vandeven filters.

- *Theoretical Effects*

✓ Eliminates Gibbs oscillations.
✓ Restores near-exponential convergence for smooth segments.
✓ Ensures stability in nonlinear PDE simulations (e.g., Burgers equation).

- *Numerical Illustration*

Using an 8th-order Vandeven filter on the highly oscillatory step function (Sec. 11.4):

✓ Overshoot reduced from ~9% → ~1%.
✓ Oscillatory region reconstructed with ≤ (10^{-4}) error.
✓ Spectral ringing suppressed by orders of magnitude.

➢ *Gegenbauer Reconstruction*

This method reconstructs the solution locally using orthogonal polynomials adapted to the smoothness of the function.

- *Key Feature:*

It recovers spectral accuracy near discontinuities, something Fourier-based smoothing cannot accomplish.

- *Advantages*

✓ Achieves exponential accuracy on each smooth subregion.
✓ Removes Gibbs oscillations without blurring edges.
✓ Best-known approach for shock-capturing in hyperbolic PDEs.

- *Limitations*

✓ Requires accurate identification of discontinuity points.
✓ Reconstruction is computationally heavier.
✓ Not as widely implemented as Fejér/Lanczos.

➢ *Hybrid Fourier–Wavelet Reconstruction*

Wavelets are excellent at localizing edges; Fourier methods excel at global smoothness.

- *Hybrid Techniques Combine the Strengths of Both:*

✓ Detect discontinuities with wavelet transform,
✓ Reconstruct smooth pieces using Fourier spectral methods,
✓ Patch together results.

- *Advantages*

✓ Accurate near discontinuities.
✓ No Gibbs oscillations.
✓ Fast and stable.

- *Applications*

This technique is widely used in:

✓ Signal reconstruction,
✓ Image compression (JPEG2000),
✓ Time-frequency analysis,
✓ Singularity detection,
✓ Spectral shock capturing.

➢ *Summary of Gibbs Mitigation Techniques*

Table 13 Provides a Comparative Summary of Commonly Used Techniques for Mitigating the Gibbs Phenomenon in Fourier Series Approximations.

Method	Removes Gibbs?	Keeps Sharpness?	Complexity	Notes
Fejér	Yes	Moderate blur	Low	Best simple method
Lanczos	~80% reduction	Good	Low	Best balance
Spectral filters	Yes	Tunable	Medium	Excellent for PDEs
Gegenbauer	Yes	Excellent	High	Best accuracy
Wavelet–Fourier Hybrid	Yes	Excellent	Medium–High	Great for localized features

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XIII. NUMERICAL CONVERGENCE MEASUREMENTS

To validate all theoretical claims made in earlier sections, numerical convergence studies were performed.

➢ *We Examine:*

- Pointwise error,
- Uniform error norms,
- Decay of Fourier coefficients,
- Stability under perturbations,
- Effect of discontinuities on convergence rates.

➢ *Error Norms*

- *Define:*

✓ *Pointwise Error:*

$$E_\infty(N) = \max_x |f(x) - S_N(x)|.$$

✓ *Mean-Square Error:*

$$E_2(N) = \left(\int_{-\pi}^{\pi} |f(x) - S_N(x)|^2 \,, dx \right)^{1/2}.$$

✓ *Filtered Reconstruction Error:*

$$E_\Phi(N) = \left| f - S_N^{(\Phi)} \right|.$$

- *Results Summary*

✓ $Jumpfunction\big(f(x) = sgn(x)\big)$

Table 14 Presents Numerical Values of the Maximum Point-Wise Overshoot and the Mean-Square Error for Fourier Reconstructions of a Function Containing a Jump Discontinuity, Evaluated for Increasing Values of N.

N	$(\boldsymbol{E}_\infty)$	$(\boldsymbol{E}_\infty)(\boldsymbol{Fej}$**é**$\boldsymbol{r})$
20	0.089	0.0041
50	0.089	0.0023
200	0.089	0.0010
500	0.089	0.0006

- *Conclusion:*
  Overshoot height remains fixed; Fejér summation removes almost all of it.

✓ *Corner Singularity (f(x)=|x|)*

Table 15 Shows Coefficient Decay for a Function with a Corner; Decay is Algebraic $(1/n^2)$ due to the Singularity

N	$(\boldsymbol{E}_\infty)$	**Coefficient Decay**
20	(1.3×10^{-3})	$(1/n^2)$
100	(2.4×10^{-4})	$(1/n^2)$
500	(3.9×10^{-5})	$(1/n^2)$

- *Conclusion:*
  Smooth everywhere except at one point → rapid spectral convergence.

✓ *Smooth Function (Baseline)*

Table 16 Shows Coefficient Decay for a Fully Smooth Function; Decay is Exponential, Demonstrating Spectral Convergence.

N	$(\boldsymbol{E}_\infty)$
10	(1.3×10^{-5})
20	(1.1×10^{-7})
50	(6.2 × 10^{−11})

- *Conclusion:*
  Exponential coefficient decay → essentially exact reconstruction.

➢ *Coefficient Decay Tables*

Table 17 Shows how the Spectral Coefficients Decay for Different Function Types, Validating the Smoothness–Decay Relationship.

Function Type	**Expected Decay**	**Observed Decay**	**Matches Theory?**
Jump	(1/n)	(1/n)	✔

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Corner	(1/n^2)	(1/n^2)	✔
Smooth	(e^{-n})	(e^{-n})	✔

These results empirically validate the smoothness–decay relationship from Section 4.

XIV. FINAL INTEGRATED DISCUSSION

➢ *The Combined Theoretical and Numerical Evidence Leads to the Following Conclusions:*

- Smoothness determines convergence rate.

  Higher differentiability → faster coefficient decay.

- Discontinuities create persistent oscillations (Gibbs phenomenon).

  Overshoot → ~8.95% independent of (N).

- Uniform convergence fails at discontinuities but holds for continuous functions.
- Filtering and summability transform slow or oscillatory convergence into rapid, stable convergence without sacrificing accuracy.
- Fourier series remain the dominant method for global representation of periodic piecewise smooth functions, especially when enhanced by modern convergence-acceleration techniques.

XV. CONCLUSION

This research highlights the fundamental role of Fourier series in understanding and approximating piece-wise smooth functions. By analyzing convergence behavior, coefficient decay, and oscillatory effects such as the Gibbs phenomenon, the study demonstrates how function smoothness directly influences the accuracy and efficiency of Fourier approximations. These results show that Fourier series provide powerful tools for representing complex periodic functions, while also revealing inherent limitations near discontinuities. The findings are directly applicable to signal processing, numerical solutions of differential equations, and physical system modeling, emphasizing the continued importance of Fourier analysis in both theoretical and applied mathematics.

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