Calculation of Eigenvalues of a PT Symmetric Complex Cubic Oscillator through Morse-Feshbach Non-Linear Perturbation Series

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Abstract:- We study the convergence of the Morse-Feshbach nonlinear perturbation series (MFNPS) series to find out the energy levels of a PT symmetric complex cubic anharmonic oscillator. Perturbation series on energy has been calculated up to 15^{th} order for the ground state and the first excited state. All orders of the MFNPS are found to be real and positive for this non-Hermitian but PT - symmetric Hamiltonian. The convergent energy spectra nicely match with the results of calculation of matrix diagonalization method. Some discussions on wave functions have been made using the nonlinear series.

Keywords:- *Morse-Feshbach; Non-Linear Perturbation Series; Anharmonic Os Cillator; Energy Levels.*

I. INTRODUCTION

Perturbation theory is a powerful method for obtaining energy levels. It has been first studied extensively by Bender and Wu[1] on anharmonic oscillator. Later on Halliday and Suryani[2] suggested feasible way of dealing with perturbation theory in large order. Further Weniger[3], Janke and Klernet[4], Rath[5,6] have focused on larger order perturbation series on anharmonic oscillator. However the convergence criteria by Halliday and Suryani can hardly be applied to complex cubic oscillator having Hamiltonian of the type[7, 8, 10, 11]

$$H = p^2 + x^2 + i\lambda x^3 \tag{1}$$

In fact the Hamiltonian of this type is non-Hermitian having PT-symmetric nature satisfying the condition [7, 8, 10, 11]

$$[H, \mathrm{PT}]=0 \tag{2}$$

Here P stands for parity reflection operator, whose effect is to change the sign of position and momentum operator and T stands for time reversal operator, which changes the sign of momentum operator and complex number i as it is an antiunitary operator.

The operator behavior of P is

$$\mathrm{P}\, x\, \mathrm{P}^{-1} = -x \qquad \mathrm{P}\, p\, \mathrm{P}^{-1} = -p \qquad \mathrm{P}\, i\, \mathrm{P}^{-1} = i \tag{3}$$

The operator behavior of T is

$$\mathrm{T}\, x\, T^{-1} = x \qquad \mathrm{T}\, p\, \mathrm{T}^{-1} = -p \qquad \mathrm{T}\, i\, \mathrm{T}^{-1} = -i \tag{4}$$

It is noted that the Hamiltonian H is neither invariant under parity P nor under time reversal T, but invariant under PT. Hence the reality and positive value of the spectrum of H is due to PT symmetry [7]. Spectral analysis on this oscillator, has basically focused using non-perturbative analysis by many authors. Some authors [14] believe that, wave functions of this model oscillator are non-orthogonal in nature without any explicit calculations. Considering the previous literature, we focus our attention on this oscillator. In fact, Feranchuk et.al [8, 9] have successfully applied perturbation series but for constant series. Apart from constant perturbation series, we applied for non linear perturbation series [12], in which both LHS and RHS are unknown functions of energy eigenvalue E as

$$E = f(E) \tag{5}$$

Such type of series is well known Morse-Feshbach non-linear perturbation series [12].

II. MORSE-FESHBACH NONLINEAR PERTURBATION SERIES (MFNPS)

Before applying to complex cubic anharmonic oscillator Hamiltonian, we write the MFNPS in a simplified language. we split the Hamiltonian H as follows. Here we consider

$$H_0 = p^2 + x^2 \tag{6}$$

As unperturbed Hamiltonian having known eigenenergy $E_n^{(0)}$ and eigenstate $|n\rangle$

$$H_0|n\rangle = E_n^{(0)}|n\rangle \tag{7}$$

Where $E_n^{(0)} = 2n+1$. Here we consider the perturbation term H_p as

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$$H_p = \lambda \tilde{H} = i\lambda x^3 \tag{8}$$

According to MFNPS, the explicit form of the series for n^{th} eigenstate energy E_n for the PT-symmetric Hamiltonian H can be written as

$$\begin{aligned} E_n &= E_n^{(0)} + \lambda\langle n|\tilde{H}|n\rangle \\ &+ \lambda^2 \sum_{m\neq n} \frac{\langle n|\tilde{H}|m\rangle\langle m|\tilde{H}|n\rangle}{(E_n - E_m^{(0)})} \\ &+ \lambda^3 \sum_{m,k\neq n} \frac{\langle n|\tilde{H}|m\rangle\langle m|\tilde{H}|k\rangle\langle k|\tilde{H}|n\rangle}{(E_n - E_m^{(0)})(E_n - E_k^{(0)})} \\ &+ \lambda^4 \sum_{m,k,p\neq n} \frac{\langle n|\tilde{H}|m\rangle\langle m|\tilde{H}|k\rangle\langle k|\tilde{H}|p\rangle\langle p|\tilde{H}|n\rangle}{(E_n - E_m^{(0)})(E_n - E_k^{(0)})(E_n - E_p^{(0)})} \\ &+ \cdots + \lambda^K \sum_{m,\cdots z\neq n} \frac{\langle n|\tilde{H}|m\rangle\cdots\langle z|\tilde{H}|n\rangle}{(E_n - E_m^{(0)})\cdots(E_n - E_z^{(0)})} \end{aligned} \tag{9}$$

Here K is the order of the perturbation in MFNP series.

III. CALCULATION OF EIGENVALUES OF A PT SYMMETRIC COMPLEX CUBIC OSCILLATOR

Adopting second quantization formalism, we use the transformation

$$x = \frac{a + a^\dagger}{\sqrt{2}} \tag{10}$$

$$p = -i\frac{a - a^\dagger}{\sqrt{2}} \tag{11}$$

Here $a^\dagger$ is the creation operator and a is the annihilation operation satisfying the commutation relation

$$[a, a^\dagger] = 1 \tag{12}$$

Using unperturbed eigenstate $|n\rangle$ for harmonic oscillator, we calculate the non zero expectation values of the perturbation Hamiltonian $\tilde{H}$ and are given as

$$\langle n|\tilde{H}|n+1\rangle = \langle n+1|\tilde{H}|n\rangle = 3i\frac{(n+1)^{3/2}}{\sqrt{8}} \tag{13}$$

$$\langle n|\tilde{H}|n-1\rangle = \langle n-1|\tilde{H}|n\rangle = 3i\frac{(n)^{3/2}}{\sqrt{8}} \tag{14}$$

$$\langle n+3|\tilde{H}|n\rangle = \langle n|\tilde{H}|n+3\rangle = i\frac{\sqrt{(n+1)(n+2)(n+3)}}{\sqrt{8}} \tag{15}$$

$$\langle n-3|\tilde{H}|n\rangle = \langle n|\tilde{H}|n-3\rangle = i\frac{\sqrt{n(n-1)(n-2)}}{\sqrt{8}} \tag{16}$$

Now we use MFNPS in Eq. 9 to calculate the energy levels for the ground state and first excited state energy of the complex cubic anharmonic oscillator using numerical programming.

- *The Results of Eigenvalues are Illustrated in the Table 1*

Table 1 Numerical Energy levels for Ground State and first Excited State of Complex Cubic Oscillator for λ = 0.1 Using MFNPS.

K^{th} order of Perturbation	Ground state energy E_0 up to K^{th} order	First excited state energy E_1 up to K^{th} order
0	1	3
1	1	3
2	1.006 689 5	3.035 588 3
3	1.006 689 5	3.045 588 3
4	1.006 691 5	3.041 972 9
5	1.006 691 5	3.041 972 9
6	1.006 703 2	3.042 414 0
7	1.006 703 2	3.042 414 8
8	1.006 702 3	3.042 349 0
9	1.006 702 3	3.042 349 0
10	1.006 702 3	3.042 359 7
11	1.006 702 3	3.042 359 7
12	1.006 702 3	3.042 357 8
13	1.006 702 3	3.042 357 8
14	1.006 702 3	3.042 357 8
15	1.006 702 3	3.042 357 8

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IV. MATRIX DIAGONALIZATION METHOD

In order to compare the convergent results of the MFNPS, we use matrix diagonalization method (MDM) on solving the eigenvalue relation as

$$H|\psi\rangle = E|\psi\rangle \tag{17}$$

Where

$$|\psi\rangle = \sum A_m |m\rangle$$

Here we solve a five-term recurrence relation satisfied by A_m as

$$P_m A_{m-3} + Q_m A_{m-1} + R_m A_m + S_m A_{m+1} + T_m A_{m+3} = 0 \tag{18}$$

Where

$$P_m = \langle m|H|m-3\rangle = \lambda i \frac{\sqrt{(m+1)(m+2)(m+3)}}{\sqrt{8}} \tag{19}$$

$$Q_m = \langle m|H|m-1\rangle = 3\lambda i \frac{(m+1)^{3/2}}{\sqrt{8}} \tag{20}$$

$$R_m = \langle m|H|m\rangle = 2m+1 \tag{21}$$

$$S_m = \langle m|H|m+1\rangle = 3\lambda i \frac{(m)^{3/2}}{\sqrt{8}} \tag{22}$$

$$T_m = \langle m|H|m+3\rangle = \lambda i \frac{\sqrt{m(m-1)(m-2)}}{\sqrt{8}} \tag{23}$$

The eigenvalues for different matrix sizes calculated from the MDM are tabulated in Table 2.

Table 2 Numerical Energy Levels for Ground State and first Excited State of Complex Cubic Oscillator for λ = 0.1 Using MDM.

n	Matrix size (500 × 500)	Matrix size (750 × 7500)
0	1.006 702 3	1.006 702 3
1	3.042 357 8	3.042 358 1

V. RESULTS AND DISCUSSION

A comparison reflects that convergent energy from MFNPS in Table 1 and eigenenergy from MDM in Table 2 are the same up to 5 decimals. So, we consider the standard ground state energy calculated as E_0=1.00678 and the first excited state energy as E_1=3.04235 for the complex cubic anharmonic oscillator. Convergent eigen values for the ground state energy E_0 and first excited state energy of the PT-symmetric cubic anharmonic oscillator using MFNPS is illustrated in Fig. 1. Using the above calculated values, we can now analyze the wave functions as follows

$$
\begin{aligned}
|\psi_n(K)\rangle &= |\varphi_n\rangle + \lambda \sum_{p\neq n} |\varphi_p\rangle \frac{\langle \varphi_p|\tilde{H}|\varphi_n\rangle}{(E_n - E_p^{(0)})} \\
&+ \lambda^2 \sum_{p,q\neq n} |\varphi_p\rangle \frac{\langle \varphi_p|\tilde{H}|\varphi_q\rangle\langle \varphi_q|\tilde{H}|\varphi_n\rangle}{(E_n - E_p^{(0)})(E_n - E_q^{(0)})} \\
&+ \lambda^3 \sum_{p,q,r\neq n} |\varphi_p\rangle \frac{\langle \varphi_p|\tilde{H}|\varphi_q\rangle\langle \varphi_q|\tilde{H}|\varphi_r\rangle\langle \varphi_r|\tilde{H}|\varphi_n\rangle}{(E_n - E_p^{(0)})(E_n - E_q^{(0)})(E_n - E_r^{(0)})} + \cdots + \\
&\lambda^K \sum_{p,q,r,z\neq n} |\varphi_p\rangle \frac{\langle \varphi_p|\tilde{H}|\varphi_q\rangle\langle \varphi_q|\tilde{H}|\varphi_r\rangle\langle \varphi_r|\tilde{H}|\varphi_s\rangle\cdots\langle \varphi_z|\tilde{H}|\varphi_n\rangle}{(E_n - E_p^{(0)})(E_n - E_q^{(0)})\cdots(E_n - E_z^{(0)})}
\end{aligned} \tag{24}
$$

Here, $|\varphi_p\rangle$ is the unperturbed state and the series is infinite, contains contribution of odd states. The corresponding relation can be defined as

$$\left|\langle \varphi_{p=2,4,6\cdots}|\psi_0(K)\rangle\right| = 0 \tag{25}$$

Similarly for the first excited state

$$\left|\langle \varphi_{p=3,5,7\cdots}|\psi_1(K)\rangle\right| = 0 \tag{26}$$

However, it is easy to see that

$$\langle \psi_0(K)|\psi_1(K)\rangle \neq 0 \tag{27}$$

which proves the assumption of Siegl and Krejecirik [14] reported previously.

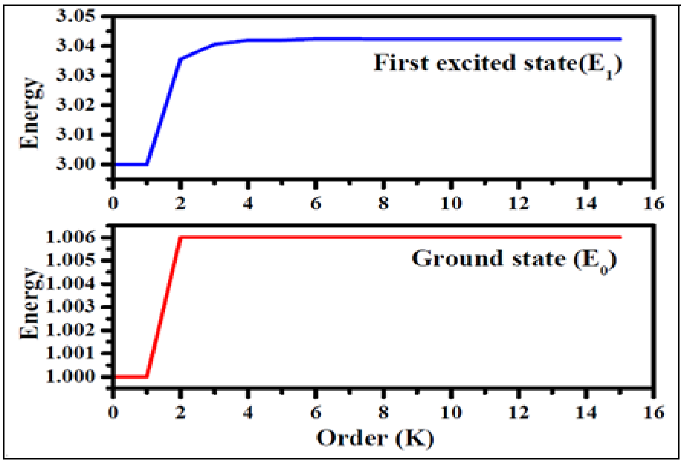

Fig 1 Convergent Eigen Values for the Ground State Energy E_0 and first Excited State Energy of the PT-Symmetric Cubic An Harmonic Oscillator Using MFNPS.

VI. CONCLUSION

The spectrum of PT -symmetric cubic anharmonic oscillator is confidently obtained using MFNP series. This nonlinear series can be easily used to check the reality of the energy spectra of new infinite classes of complex Hamiltonians, which are invariant under PT -symmetry. MFNP series can now be extended to analyze the nature of wave functions for non-Hermitian but PT -symmetric Hamiltonian.

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