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Hebdomes-Taxeos Implicit Block Nyström Hybrid Method for Enhanced Solutions of Second-Order Boundary Value Problems

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Abstract:- This study introduces a new approach to the solution of second-order nonlinear differential equations, with a particular emphasis on the Bratu problems, which are highly relevant in many scientific domains. For this reason, the Block Hybrid Nystron-Type Method (BHNTM) was developed because the previous approaches to solve this problem had some drawbacks. Utilising the Bhaskera point obtained from the Bhaskeracosine approximation formula, BHNTM operates. By statistically searching for a power series polynomial, the method determines the coefficients in the most effective manner possible. The paper investigates BHNTM's convergence, zero stability, and consistency properties using numerical tests that indicate how accurate it is at solving Bratu-type issues. This study is a notable contribution to numerical analysis since it provides an alternative but successful technique to solving challenging second-order nonlinear differential equations, which is crucial in the scientific world.

Keywords:- *Hybrid Method; Blocknyström-Type Method; Nonlinear Odes; Power Series Polynomials; One-Dimensional Bratu Problems.*

I. INTRODUCTION

Hebdomes-Taxeos Implicit Block Nyström Hybrid Method for Enhanced Solutions of Second-Order Boundary Value Problems is a new seventh-order solution for boundary value problems. Hebdomes is an ancient Greek term for seventh while texeos is order. Putting the two terms together Hebdomes-Taxeos means seventh order. A one-step hybrid block Nystrom-type approach was created in this study as a computational strategy for resolving a class of second-order nonlinear differential equations, which are as follows:

$$y''(x) = \lambda(x)e^{\mu(x)y(x)} \quad (1.1)$$

Consider the function defined on the closed interval [0,1] with the boundary conditions y(0)=y(1)=0. The functions λ(x) and μ(x) serve as examples of established continuous functions of x. Modelling electrically conducting solids poses the challenge previously mentioned (Wazwaz & Suheil, 2013). Wazwaz and Suheil (2013) analysed Joule losses in electrically conducting solids and identified that the condition μ = 1 arises when λ denotes the square of the constant current and e^y represents the temperature-dependent resistance. In fractional heating, e^y represents temperature-dependent fluidity, whereas λ denotes the square of the constant shear stress.

Bratu's equation is a specific version of equation (1.1) where the functions λ(x) and μ(x) are treated as constants. The general form is as follows:

$$y''(x) + \lambda(x)e^{\mu y(x)} = 0, \quad 0 \le x \le 1, \quad (1.3)$$

The following are the corresponding beginning and boundary conditions:

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(0) = y(1) = 0, \quad (1.4)$$

In this context, y(x) represents the solution to the equation, λ denotes a real parameter, and μ is assigned a value of either +1 or -1. Both diffusion theory and the modelling of electrically conductive materials utilise this two-point boundary value problem as a particular case (see Wazwaz & Suheil, 2013; Frank-Kamenetski, 1955). Numerous physical models, such as the Chandrasekhar model of cosmic expansion, the fuel ignition model in combustion theory, the thermal reaction processes model in chemical reaction theory, radiative heat transfer, and nanotechnology, incorporate equation (1.3) along with initial and boundary conditions (Frank-Kamenetski, 1955; Bratu, 1914; Gelfand, 1963; Jacobsen & Schmitt, 2002; Hariharan & Pirabaharan, 2013). Bratu's 1914 article highlighted the Bratu problem. This is commonly known as the "Liouville-Gelfand" or "Liouville-Gelfand-Bratu" problem, in recognition of Gelfand and the French mathematician Liouville, Gelfand (1963). Jacobsen and Schmitt (2002) have concisely explained the significance and context of Bratu-type equations. This topic has lately acquired prominence as a standard for assessing the accuracy of various numerical algorithms. Numerical approaches utilising the Hybrid Block Nystrom-type method for solving these distinctive sorts of boundary value problems have received minimal attention (Buckmire,2004; Mounim & deDormale,2006).

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Given the significance of the Bratu-type problem in research, various numerical methods have been developed to obtain an approximate solution. Various methods have been employed to address Bratu's problem, including the decomposition method by Deeba et al. (2000), multigrid-based approaches by Mohsen (2014), an iterative numerical scheme utilising Newton's Kantorovich method in function space by Temimi & Ben-Romdhane (2016), and the differential transformation method introduced by Hassan &Erturk (2007). Temimi& Ben-Romdhane (2016) illustrate that Bratu's problem has been addressed through several alternative techniques, such as the modified wavelet Galerkin method outlined in Raja et al. (2015), the Laplace Adomian decomposition method, the perturbation-iteration method, the Lie-group shooting method, and the integral solution utilising Green's function. Numerous initial and boundary value problems derived from Bratu equations were addressed via feed-forward artificial neural networks (ANN) improved through genetic algorithms (GA) and the active-set method (ASM), as reported by Raja et al. (2016).In practical terms, the standard Nystrom-type method is employed solely for addressing initial value problems of ordinary differential equations.This study builds upon the work of Jator and Manathunga (2018) by developing the Block Hybrid Nystrom-Type approach (BHNTM), which incorporates hybrid points generated using the Bhakara Cosine Approximation approach to obtain a numerical solution for the classical one-dimensional Bratu's problem. The organisation of the document is as follows. Part 2 generates the BHNTM, while section 3 examines its essential attributes of consistency, zero stability, and convergence. Section 4 delineates the implementation methodology, the execution of numerical experiments, and the subsequent discussion of the results. Finally, section 5 addresses the conclusion of the work.

II. DEVELOPMENT OF THE BHNTM

- *Consider the Problem*

$$y''(x)=f(x,y,y'),\qquad y(a)=0,\qquad y(b)=0,\qquad x\in[a,\,b] \tag{2.1}$$

Subject to the following conditions on $f(x,y,y')$:

- $f(x,y,y')$ is continuous,
- derivative of $f(x,y,y')$ exist and continuous.
- Taking into account the issue on the interval [a,b], the Bhaskara cosine approximation formula is utilised to partition the data in order to produce the Bhaskara points, which are then utilised to create the algorithm by approximating the cosine functions
- $\left\{\cos\dfrac{\pi i}{M}\right\}_{i=0}^{M}\approx\dfrac{M^2-4i^2}{M^2+i^2}$,

  Where $M=w+1,\quad w\in N: w$ (number of off-grids) $\quad w\geq 2$, the number $h=\dfrac{(b-a)}{N}$ is referred to as the constant step size. Supposing the exact solution is approximated by the power series polynomial of the form

$$y(x)=\sum_{0}^{9}a_j x^j \tag{2.2}$$

Where a_j are coefficients obtained distinctly and $x\in[x_n,\,x_{n+1}]$, for some natural number n. In order to determine the coefficients a_j, the following conditions hold:

$$\left.\begin{aligned} &y(x_n)=y_n,\\ &y'(x_n)=y'_n,\\ &y''(x_{n+v})=f_{n+v},v=0,\frac{5}{74},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{69}{74},1.\end{aligned}\right\} \tag{2.3}$$

Differentiating (2.2) once and twice gives

$$\left.\begin{aligned} &y'(x)=\sum_{j=1}^{9}ja_jx^{j-1},\\ &y''(x)=\sum_{j=2}^{9}j(j-1)a_jx^{j-2}=f(x,y,y').\end{aligned}\right\} \tag{2.4}$$

Using equations (2.2) and (2.4) on the criteria given in (2.3) a system of nonlinear equations of the form are obtained

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$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^8 \\ 0 & 1 & 2x_n & 3x_n^2 & \dots & 8x_n^7 \\ 0 & 0 & 2 & 6x_n & \dots & 56x_n^6 \\ \dots & \dots & 2 & 6x_{n+\frac{5}{74}} & \dots & 56x^6_{n+\frac{5}{74}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 2 & 6x_{n+1} & \dots & 56x_{n+1}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y'_n \\ f_n \\ f_{n+\frac{5}{74}} \\ f_{n+\frac{1}{4}} \\ . \\ . \\ f_{n+1} \end{pmatrix} \tag{2.5}$$

We use the matrix inversion approach to solve the nonlinear equation system shown above. To obtain the continuous Hybrid Block Nystrom-type Method with five off-step points (BHNTM1,5), the unknown coefficients are then substituted into equation (2.2):

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)hy'_n$$
$$+h^2\left(\beta_n f_n + \beta_{n+\frac{5}{74}} f_{n+\frac{5}{74}} + \beta_{n+\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{n+\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{n+\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_{n+\frac{69}{74}} f_{n+\frac{69}{74}} + \beta_{n+1} f_{n+1}\right), \tag{2.6}$$

And

$$y'(x) = y'_n + h\left(\beta_n f_n + \beta_{n+\frac{5}{74}} f_{n+\frac{5}{74}} + \beta_{n+\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{n+\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{n+\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_{n+\frac{69}{74}} f_{n+\frac{69}{74}} + \beta_{n+1} f_{n+1}\right) \tag{2.7}$$

in which the variables have well defined continuous coefficients. Based on the assumption that, we can say that is the numerical approximation of the analytical answer. It also approximates.

The main methods are obtained by evaluating (2.6)atpoints $x = x_n, x_{n+\frac{5}{74}}, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+\frac{69}{74}}, x_{n+1}$ to give the discrete schemes which form the continuous block hybrid Nystrom-type method with five off-grid points:

$$\left.\begin{aligned}
y_{n+\frac{5}{74}} &= y_n + \frac{5}{74}hy'_n + h^2\left(\begin{aligned}&\frac{58052434225}{44612850799692}f_n + \frac{15063275}{13592395776}f_{n+\frac{5}{74}} - \frac{2933478125}{16325717140467}f_{n+\frac{1}{4}} + \frac{24930298125}{294258030395392}f_{n+\frac{1}{2}}\\ &-\frac{24485719375}{440794362792609}f_{n+\frac{3}{4}} + \frac{7823810875}{177258433314816}f_{n+\frac{69}{74}} - \frac{6595375}{323281527534}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + h^2\left(\begin{aligned}&\frac{348899}{89026560}f_n + \frac{68719861387}{3069160980480}f_{n+\frac{5}{74}} + \frac{449}{77568}f_{n+\frac{1}{4}} - \frac{92961}{73400320}f_{n+\frac{1}{2}} + \frac{155707}{219905280}f_{n+\frac{3}{4}}\\ &-\frac{8806682539}{16573469294592}f_{n+\frac{69}{74}} + \frac{7177}{29675520}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + h^2\left(\begin{aligned}&\frac{211}{28980}f_n + \frac{7142427571}{129480228864}f_{n+\frac{5}{74}} + \frac{48253}{859005}f_{n+\frac{1}{4}} + \frac{115}{16384}f_{n+\frac{1}{2}} - \frac{241}{286335}f_{n+\frac{3}{4}} + \frac{69343957}{215800381440}f_{n+\frac{69}{74}}\\ &-\frac{1}{8694}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + h^2\left(\begin{aligned}&\frac{37377}{3297280}f_n + \frac{29332493811}{341017886720}f_{n+\frac{5}{74}} + \frac{103833}{904960}f_{n+\frac{1}{4}} + \frac{4690143}{73400320}f_{n+\frac{1}{2}} + \frac{449}{77568}f_{n+\frac{3}{4}}\\ &-\frac{762783527}{1023053660160}f_{n+\frac{69}{74}} + \frac{729}{3297280}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{69}{74}} &= y_n + \frac{69}{74}hy'_n + h^2\left(\begin{aligned}&\frac{4946912289}{359201697260}f_n + \frac{17400136203}{158577950720}f_{n+\frac{5}{74}} + \frac{1411081172199}{9069842855815}f_{n+\frac{1}{4}} + \frac{165964330541889}{147129015976960}f_{n+\frac{1}{2}}\\ &+\frac{1127174082469}{27209528567445}f_{n+\frac{3}{4}} + \frac{15063275}{13592395776}f_{n+\frac{69}{74}} + \frac{10840797}{35920169726}f_{n+1}\end{aligned}\right)\\
y_{n+1} &= y_n + hy'_n + h^2\left(\frac{643}{43470}f_n + \frac{69343957}{586414080}f_{n+\frac{5}{74}} + \frac{48976}{286335}f_{n+\frac{1}{4}} + \frac{4671}{35840}f_{n+\frac{1}{2}} + \frac{48976}{859005}f_{n+\frac{3}{4}} + \frac{69343957}{8092514304}f_{n+\frac{69}{74}}\right)
\end{aligned}\right\} \quad (2.8)$$

The additional methods are obtained by evaluating (2.7) at $x = \left(\frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}, 1\right)$ to give the following discrete schemes:

$$\left.\begin{aligned}
y'_{n+\frac{5}{74}} &= y'_n + h\left(\begin{aligned}&\frac{132171928615}{4823010897264}f_n + \frac{13113969205}{299423029248}f_{n+\frac{5}{74}} - \frac{61021705750}{11913361156557}f_{n+\frac{1}{4}} + \frac{1181523375}{497057483776}f_{n+\frac{1}{2}} - \frac{18479790250}{11913361156557}f_{n+\frac{3}{4}}\\ &+\frac{368391325}{299423029248}f_{n+\frac{69}{74}} - \frac{2741432975}{4823010897264}f_{n+1}\end{aligned}\right)\\
y'_{n+\frac{1}{4}} &= y'_n + h\left(\begin{aligned}&\frac{23579}{5564160}f_n + \frac{199779940117}{1294802288640}f_{n+\frac{5}{74}} + \frac{1422961}{13744080}f_{n+\frac{1}{4}} - \frac{20709}{1146880}f_{n+\frac{1}{2}} + \frac{139651}{13744080}f_{n+\frac{3}{4}} - \frac{9916185851}{1294802288640}f_{n+\frac{69}{74}}\\ &+\frac{19421}{5564160}f_{n+1}\end{aligned}\right)\\
y'_{n+\frac{1}{2}} &= y'_n + h\left(\frac{277}{12880}f_n + \frac{901471441}{8092514304}f_{n+\frac{5}{74}} + \frac{217162}{859005}f_{n+\frac{1}{4}} + \frac{4671}{35840}f_{n+\frac{1}{2}} - \frac{2362}{95445}f_{n+\frac{3}{4}} + \frac{69343957}{4495841280}f_{n+\frac{69}{74}} - \frac{467}{69552}f_{n+1}\right)\\
y'_{n+\frac{3}{4}} &= y'_n + h\left(\frac{2329}{206080}f_n + \frac{2149662667}{15985213440}f_{n+\frac{5}{74}} + \frac{36973}{169680}f_{n+\frac{1}{4}} + \frac{319653}{1146880}f_{n+\frac{1}{2}} + \frac{63389}{509040}f_{n+\frac{3}{4}} - \frac{1317535183}{47955640320}f_{n+\frac{69}{74}} + \frac{435}{41216}f_{n+1}\right)\\
y'_{n+\frac{69}{74}} &= y'_n + h\left(\begin{aligned}&\frac{596477423}{38832615920}f_n + \frac{10024207}{803604480}f_{n+\frac{5}{74}} + \frac{168853796338}{735392663985}f_{n+\frac{1}{4}} + \frac{641903629419}{2485287418880}f_{n+\frac{1}{2}} + \frac{514439521514}{2206177991955}f_{n+\frac{3}{4}}\\ &+\frac{200151221}{2410813440}f_{n+\frac{69}{74}} - \frac{489781527}{38832615920}f_{n+1}\end{aligned}\right)\\
y'_{n+1} &= y'_n + h\left(\frac{643}{43470}f_n + \frac{2565726409}{20231285760}f_{n+\frac{5}{74}} + \frac{195904}{859005}f_{n+\frac{1}{4}} + \frac{4671}{17920}f_{n+\frac{1}{2}} + \frac{195904}{859005}f_{n+\frac{3}{4}} + \frac{2565726409}{20231285760}f_{n+\frac{69}{74}} + \frac{643}{43470}f_{n+1}\right)
\end{aligned}\right\} \quad (2.9)$$

III. ANALYSIS OF THE BHNTM

Mathematical study of several fundamental characteristics of the derived schemes is covered in detail in this section. These qualities are fully supported by current definitions and theorems.

Order and Error Constant

The suggested BHNTM from equations (2.8) and (2.9) belongs to the family of linear multistep methods (LMM), which is commonly defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \quad (3.1)$$

Higher derivatives are expected to be coupled with a linear difference operator that is used in numerical methods. You can expand the phrase and its second derivative as part of Taylor's series on the topic. The difference operator defines a local truncation error that is linked to second-order ordinary differential equations:

Where $y(x_n)$ is an arbitrary function continuously differentiable on the interval $[x_n, x_{n+1}]$. Expanding the expression (3.2) in Taylor series approximation about the point x_n gives

$$\eta[y(x_n);h] = \sum_{i=0}^{k}\left[\alpha_i y(x_n + v_i h) - h^2 \beta_i f(x_n + v_i h)\right] \quad (3.2)$$

$$\eta[y(x_n);h] = \hat{C}_0 y(x_n) + \hat{C}_1 h y'(x_n) + \hat{C}_2 h^2 y''(x_n) + \ldots + \hat{C}_{p+1} h^{p+1} y^{p+1}(x_n) + \ldots + \hat{C}_{p+2} h^{p+2} y^{p+2}(x_n) + \ldots \quad (3.3)$$

Where the vectors

$$\begin{cases} \hat{C}_0 = \sum_{v_i=0}^{k} \alpha_{v_i}, \quad \hat{C}_1 = \sum_{i=0}^{k} v_i \alpha_{v_i}, \quad \hat{C}_2 = \frac{1}{2!}\sum_{v_i=0}^{k} v_i \alpha_{v_i} - \beta_{v_i}, \ldots, \\ \hat{C}_p = \frac{1}{p!}\sum_{v_i=0}^{k} v_i^{\,p} \alpha_{v_i} - p(p-1)(p-2) v_i^{\,p-2} \beta_{v_i}. \end{cases} \quad (3.4)$$

Following Lambert (1991), the associated methods are said to be of order p if, in (3.3)

$\hat{C}_0 = \hat{C}_1 = C_2 = \ldots = \hat{C}_p = \hat{C}_{p+1} = 0$ and $\hat{C}_{p+2} \neq 0$.

Therefore, $\hat{C}_{p+2}$ is the error constant and $\hat{C}_{p+2} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n. The local truncation error of the proposed BHNTM obtained is given as:

$$\eta[y(x_n;h)] = \begin{cases} \frac{28132172125}{1207309213103041069056} h^{(9)} y^{(9)}(x) + O(h^{10}) \\ -\frac{147697}{651143046758400} h^{(9)} y^{(9)}(x) + O(h^{10}) \\ -\frac{43}{317940940800} h^{(9)} y^{(9)}(x) + O(h^{10}) \\ -\frac{351}{8038803046400} h^{(9)} y^{(9)}(x) + O(h^{10}) \\ -\frac{109474638723}{372626300340444774400} h^{(9)} y^{(9)}(x) + O(h^{10}) \\ -\frac{43}{158970470400} h^{(9)} y^{(9)}(x) + O(h^{10}) \end{cases} \quad (3.5)$$

Which shows that the proposed method has order $p = 7$, where the error constant $\hat{C}_{p+2}$ is a 1×6 vector given by

$$\hat{C}_9 = \left(\frac{28132172125}{1207309213103041069056}, -\frac{147697}{651143046758400}, -\frac{43}{317940940800}, -\frac{351}{8038803046400}, -\frac{109474638723}{372626300340444774400}, -\frac{43}{158970470400} \right). \quad (3.6)$$

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A. Zero Stability of HBNTM

When the limit of h is set to zero, the proposed technique is characterised by zero stability. In the main method (2.8), the following system of equations is established as h approaches zero:

$$\left.\begin{aligned} y_{n+\frac{5}{74}} &= y_n \\ y_{n+\frac{1}{4}} &= y_n \\ y_{n+\frac{1}{2}} &= y_n \\ y_{n+\frac{3}{4}} &= y_n \\ y_{n+\frac{69}{74}} &= y_n \\ y_{n+1} &= y_n \end{aligned}\right\} \quad (3.7)$$

which can be written in matrix form as

$$A^0Y_i - A^1Y_{i-1} = 0, \quad (3.8)$$

where

$$A^0 = \begin{bmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix}, \quad A^i = \begin{bmatrix} 0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1 \end{bmatrix}, \quad Y_i = \begin{pmatrix} y_{n+\frac{5}{74}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+\frac{69}{74}} \\ y_{n+1} \end{pmatrix}, \quad Y_{i-1} = \begin{pmatrix} y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \end{pmatrix}$$

According to Lambert (1991), if the root of the first characteristic polynomial is less than or equal to one, the approach is considered zero stable.

The BHNTM's first characteristic polynomial is provided by $r^5[r-1]=0.$ (3.9)

All of the roots of the equation (3.9) are less than or equal to one. The BHNTM is thus zero-stable.

B. Consistency of the BHNTM

- Second definition: As stated by Jator and Li in 2009, if the linear multistep technique (3.1) is of order *p≥1* and its first and second characteristic polynomials are specified as, $\rho(z)=\sum_{j=0}^{k}\alpha_j z^j$ and $\sigma(z)=\sum_{j=0}^{k}\beta_j z^j$ then it is said to be consistent.

where *z* satisfied (i) $\sum_j^k \alpha_j = 0$ (ii) $\rho(1)=\rho'(1)=0$ and (iii) $\rho''(1)=2!\sigma(1)$,

The discrete schemes derived are all of order greater than one and satisfies the conditions therein.

C. Convergence of the BHNTM

The primary focus of this section is the convergence analysis of the proposed BHNTM. It is shown that the proposed method is convergent by defining convergence and putting the equations in (2.8) and (2.9) in an appropriate matrix notation form. (Samos and Rufai, 2021) Third Definition: For the second order boundary value problems given, let $y(x)$ and $\{y_i\}_{i=0}^{N}$ be the exact solution and approximate solutions respectively obtained by applying the proposed numerical method. If there is a constant C that is independent of h and holds for small enough values of h that: the proposed approach is said to be convergent of order p.

$$\max_{0\le i\le N}\left|y(x_i)-y_i\right| \le Ch^p.$$

- Note that in this circumstance, $\max_{0\le i\le N}|y(x_i)-y_i| \to 0$ as $h \to 0$.

Concerning convergence, the first theorem is Let show the actual second order BVP solution in (1.1) with boundary condition in (1.2) together with the discrete solution provided by the advised method. The recommended strategy so lines up to order seven.

Support. Assuming a matrix with dimensions per Jator and Manathunga 2018.

$$A=\begin{bmatrix} A_{1,1} & \cdots & \cdots & \cdots & A_{1,2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{2n,1} & \cdots & \cdots & \cdots & A_{2N,2N} \end{bmatrix},$$

Where the elements of $A_{i,j}$ are 6×6 matrices as follows:

$$A_{i,i}=I=\begin{bmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix};\quad A_{i,i-1}=\begin{bmatrix} 0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1 \end{bmatrix},\quad i=N+2,\dots,2N;$$

$$A_{i,N+i-1}=\begin{bmatrix} 0&0&0&0&0&-\frac{5}{74}\\0&0&0&0&0&-\frac{1}{4}\\0&0&0&0&0&-\frac{1}{2}\\0&0&0&0&0&-\frac{3}{4}\\0&0&0&0&0&-\frac{69}{74}\\0&0&0&0&0&-1 \end{bmatrix},\quad \text{where } 1<i<N+1,$$

$A_{i,j} = 0$ otherwise, where **0** is a zero matrix.

Suppose B be a $12N\times 12N$ matrix defined by

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$$
B = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & \beta_{1,4} & \cdots & \beta_{1,2N} \\ \vdots & & & & \ddots & \vdots \\ \beta_{2N,1} & \beta_{2N,2} & \beta_{2N,3} & \beta_{2N,4} & \cdots & \beta_{2N,2N} \end{bmatrix},
$$

Where the elements of $B_{i,j}$ are 6×6 matrices given as follows:

$$
\mathrm{B}_{i,j} = \begin{bmatrix}
\frac{15063275}{13592395776} & -\frac{2933478125}{16325717140467} & \frac{24930298125}{294258030395392} & -\frac{24485719375}{440794362792609} & \frac{7823810875}{177258433314816} & -\frac{6595375}{323281527534} \\
\frac{68719861387}{3069160980480} & \frac{449}{77568} & -\frac{92961}{73400320} & \frac{155707}{219905280} & -\frac{8806682539}{16573469294592} & \frac{7177}{29675520} \\
\frac{7142427571}{129480228864} & \frac{48253}{859005} & \frac{115}{16384} & -\frac{241}{286335} & \frac{69343957}{215800381440} & -\frac{1}{8694} \\
\frac{29332493811}{341017886720} & \frac{103833}{904960} & \frac{4690143}{73400320} & \frac{449}{77568} & -\frac{762783527}{1023053660160} & \frac{729}{3297280} \\
\frac{17400136203}{158577950720} & \frac{1411081172199}{9069842855815} & \frac{165964330541889}{147129015976960} & \frac{1127174082469}{27209528567445} & \frac{15063275}{13592395776} & \frac{10840797}{35920169726} \\
\frac{69343957}{586414080} & \frac{48976}{286335} & \frac{4671}{35840} & \frac{48976}{859005} & \frac{69343957}{8092514304} & 0
\end{bmatrix}, \text{ where } 1 \le i \le N
$$

$$
\mathrm{B}_{i,i-N} = \begin{bmatrix}
\frac{13113969205}{299423029248} & -\frac{61021705750}{11913361156557} & \frac{1181523375}{497057483776} & -\frac{18479790250}{11913361156557} & \frac{368391325}{299423029248} & -\frac{2741432975}{4823010897264} \\
\frac{199779940117}{1294802288640} & \frac{1422961}{13744080} & -\frac{20709}{1146880} & \frac{139651}{13744080} & -\frac{9916185851}{1294802288640} & \frac{19421}{5564160} \\
\frac{901471441}{8092514304} & \frac{217162}{859005} & \frac{4671}{35840} & -\frac{2362}{95445} & \frac{69343957}{4495841280} & -\frac{467}{69552} \\
\frac{2149662667}{15985213440} & \frac{36973}{169680} & \frac{319653}{1146880} & \frac{63389}{509040} & -\frac{1317535183}{47955640320} & \frac{435}{41216} \\
\frac{10024207}{803604480} & \frac{168853796338}{735392663985} & \frac{641903629419}{2485287418880} & \frac{514439521514}{2206177991955} & \frac{200151221}{2410813440} & -\frac{489781527}{38832615920} \\
\frac{2565726409}{20231285760} & \frac{195904}{859005} & \frac{4671}{17920} & \frac{195904}{859005} & \frac{2565726409}{20231285760} & \frac{643}{43470}
\end{bmatrix}, \text{ where } N+1 \le i \le 2N
$$

$$B_{i,i-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{58052434225}{44612850799692} \\ 0 & 0 & 0 & 0 & 0 & \frac{348899}{89026560} \\ 0 & 0 & 0 & 0 & 0 & \frac{211}{28980} \\ 0 & 0 & 0 & 0 & 0 & \frac{37377}{3297280} \\ 0 & 0 & 0 & 0 & 0 & \frac{4946912289}{359201697260} \\ 0 & 0 & 0 & 0 & 0 & \frac{643}{43470} \end{bmatrix}, \quad \text{where } 1 < i \le N$$

$$B_{i,i-(N+1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{132171928615}{4823010897264} \\ 0 & 0 & 0 & 0 & 0 & \frac{23579}{5564160} \\ 0 & 0 & 0 & 0 & 0 & \frac{277}{12880} \\ 0 & 0 & 0 & 0 & 0 & \frac{2329}{206080} \\ 0 & 0 & 0 & 0 & 0 & \frac{596477423}{38832615920} \\ 0 & 0 & 0 & 0 & 0 & \frac{643}{43470} \end{bmatrix}, \quad \text{where } N+1 < i \le 2N,$$

$B_{i,j} = 0,$ otherwise, where **0** is a zero matrix.

Hence,

$$A = \left[\begin{array}{c|c} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hline 0 & \hat{A}_{2,2} \end{array}\right],$$

$$B = \left[\begin{array}{c|c} \hat{B}_{1,1} & 0 \\ \hline \hat{B}_{2,1} & 0 \end{array}\right].$$

Furthermore, the following vectors are defined as:

$$\left.\begin{aligned} Y &= \begin{pmatrix} y\left(x_{\frac{5}{74}}\right), y\left(x_{\frac{1}{4}}\right), \ldots, y(x_1), \ldots, y\left(x_{N-1+\frac{5}{74}}\right), y\left(x_{N-1+\frac{1}{4}}\right), \ldots, y(x_N), hy'(x_{\frac{5}{74}}), \\ hy'(x_{\frac{1}{4}}), \ldots, hy'(x_1), \ldots, hy'\left(x_{N-1+\frac{5}{74}}\right), hy'\left(x_{N-1+\frac{1}{4}}\right), \ldots, hy'(x_N) \end{pmatrix}^{w}, \\ C &= \begin{pmatrix} -y_0 - \frac{5h}{74} y_0' - h^2 \beta_0(1) f_0, -y_0 - \frac{h}{4} y_0' - h^2 \beta_0(2) f_0, \ldots, -y_0 - h y_0' - h^2 \beta_0(6) f_0, \\ 0, 0, 0, \ldots, 0, -h^2 \beta_0'(1), -h^2 \beta_0'(2), \ldots, -h^2 \beta_0'(6), 0, 0, 0, \ldots, 0 \end{pmatrix}^{w} \\ T(h) &= \left(\xi_{\frac{5}{74}}, \xi_{\frac{1}{4}}, \ldots, \xi_1, \xi_{1+\frac{5}{74}}, \ldots, \xi_N, \xi_{\frac{5}{74}}, \eta'_{\frac{1}{4}}, \ldots, \xi'_1, \xi'_{1+\frac{5}{74}}, \ldots, \xi'_N \right)^{w}, \end{aligned}\right\}$$

Where $T(h)$ is the local truncation error of order nine. The exact form of the system is given by

$$AY - h^2 BF(Y) + C + T(h) = 0, \tag{3.10}$$

and the approximate form of the system is given by

$$A\bar{Y} - h^2 BF(\bar{Y}) + C = 0, \tag{3.11}$$

Where

$$\bar{Y} = \left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, ..., y_1, y_{1+\frac{5}{74}}, ..., y_N, hy'_{\frac{5}{74}}, hy'_{\frac{1}{4}}, ..., hy'_1, hy'_{1+\frac{5}{74}}, ..., hy'_N \right).$$

Subtracting (3.10) from (3.11) gives,

$$AE - h^2 BF(\bar{Y}) + h^2 BF(Y) + C = T(h), \tag{3.12}$$

Where $E = \bar{Y} - Y = \left(e_{\frac{5}{75}}, e_{\frac{1}{4}}, e_{\frac{1}{2}}, ...e_1, e'_{\frac{5}{75}}, e'_{\frac{1}{4}}, e'_{\frac{1}{2}}, ...e'_1 \right).$

Applying the mean-value theorem (Jator and Manathunga, 2018), which can be written as

$$F(\bar{Y}) = F(Y) + J_F(Y)(\bar{Y} - Y) + o\left(\|\bar{Y} - Y\|\right),$$

Where J_F is a Jacobian matrix. From this equation,

$$\frac{F(\bar{Y}) - F(Y)}{\bar{Y} - Y} = \frac{F(\bar{Y}) - F(Y)}{E} = J_F(Y).$$

Hence, we obtain $\left(A - h^2 BJ_F(Y)E\right) = T(h)$,

Where $J_F = \begin{bmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{bmatrix}$, and $J_{i,j}$ are $6N \times 6N$ matrices.

Taking into account the matrix, claiming that it is invertible for small enough h. Note that merely demonstrating the invertibility of is sufficient. It is invertible since its diagonal elements are non-zero, which undoubtedly indicates that its determinant is non-zero. It is a lower triangular matrix, in actuality. Consequently, it is invertible. Let $Z = -h^2 BJ_F$,we have $Q = A + Z$ then $\det(Q) = \det(A)\det\left(I + ZA^{-1}\right)$.

Let $C = ZA^{-1}$, then $\det(Q) = \det(A)\det(I - C)$. Note that, $\det(I - C)$ is the characteristic polynomial of C. Therefore, $\det(I - C) = (\varphi - \varphi_1)(\varphi - \varphi_2)...(\varphi - \varphi_{12N})$, where φ_j are eigenvalues of C. For $\varphi = 1$, gives $\det(I - C) = (1 - \varphi_1)(1 - \varphi_2)...(1 - \varphi_{12N})$.

If each $\varphi_j \neq 1$, then $\det(I - C) \neq 0$. Suppose that φ_j is an eigenvalue of $BJ_F A^{-1}$, then $h^2 \hat{\varphi}_j$. If $h^2 \hat{\varphi}_j \neq 1$ is proved successfully, then we are done. Thus, choosing $h^2 \notin \left\{ \frac{1}{\hat{\varphi}_j} : \hat{\varphi}_j \text{ is a non-zero eigenvalue of } BJ_F A^{-1} \right\}$

Therefore, there exists an h such that $\det(I - C) \neq 0$. Thus, $\det Q \neq 0$. This means that Q is invertible and $\|Q\|_\infty = O(h^2)$. Now, $(A + Z)E = T(h)$. This implies that $QE = T(h)$. Since Q is invertible, $E = Q^{-1}T(h)$. Taking the maximum norm gives $\|E\|_\infty = \|Q^{-1}T(h)\|_\infty \leq \|Q^{-1}\|_\infty \|T(h)\|_\infty \leq O(h^{-2})O(h^9) \leq O(h^7)$. . Thus, the BHNTM is convergent, providing a seventh-order approximation.

IV. NUMERICAL EXPERIMENT

Numerous numerical examples are provided in this area to demonstrate the accuracy of BHNTM. The code used to implement the BHNTM was created in MAPLE 2015 and can be obtained from the authors upon request. Furthermore, the highest absolute error of the approximate solution on $[x_0, x_N]$ is calculated as $Error = \max|y(x) - y|$. The formula is used to determine the rate of convergence (ROC). $ROC = Log_2\left(\frac{Err^{2h}}{Err^h}\right)$, where Err^h is the error derived through the step size h.

A. Implementation of the Proposed BHNTM

- Step 1: choose N, $h = \frac{(x_N - x_0)}{N}$, on the interval $[x_0, x_N]$.
- Step 2: using main method and additional method, for $n = 0$, generate a system of equations $\left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_{\frac{69}{74}}, y_1 \right)$ and $\left(y'_{\frac{5}{74}}, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_{\frac{69}{74}}, y'_1 \right)$ on the sub-interval $[x_0, x_1]$ and do not solve yet.
- Step 3: next, for $n = 1$, generate a system of equations $\left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_{\frac{69}{74}}, y_1 \right)$ and $\left(y'_{\frac{5}{74}}, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_{\frac{69}{74}}, y'_1 \right)$ on the subinterval $[x_1, x_2]$, and do not solve yet.
- Step 4: The procedure continues until all variables within the sub-interval are obtained.
- Step 5: Develop a comprehensive block matrix equation to simultaneously obtain all solutions for (1.1) across the entire interval defined in steps 3 and 4.

➢ *The Following Notations are Utilised:*

- $BHNTM_{1,5}$: A single-step Block Hybrid Nystrom-Type Method employing five off-grid points
- BNM: Block Nyström Method

➢ *Problem 1: Boundary Value Problem of Bratu-Type Equation*

Consider the nonlinear Bratu equation of type (1.1) for $\mu = 1$, with boundary conditionsas:

$$y''(x)+\lambda e^{y(x)}=0, \quad y(0)=y(1)=0, \quad 0\le x\le 1, \qquad (4.1)$$

Where λ is taken as zero. The analytical solution to the above equation is:

$$y(x)=-2\ln\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)}\right], \quad x\in[0,1] \qquad (4.2)$$

In this context, θ is defined as the solution to the equation: $\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$

A Bratu Problem demonstrates a distinct behaviour in terms of solutions based on the value of λ: for λ greater than λ_c, it may yield zero, one, or two solutions; at λ equal to λ_c, it presents a specific case; and for λ less than λ_c, a different outcome is observed. The critical value λ_c satisfies the following relation:

$$1=\frac{1}{4}\sqrt{2\lambda}\sinh\left(\frac{\theta}{4}\right). \qquad (4.3)$$

Zheng et al. (2014) determined the critical value to be λ_c = 3.513830719. The findings for λ values of 1, 2, and 3.51 are detailed in Tables 1, 2 and 3, respectively, concerning the Bratu boundary value problem. The example was addressed with a step size of h=0.1 to evaluate the absolute errors in relation to those documented by Jator & Manathunga (2018), Jalilian (2010), and Liao & Tan (2007). Our method (HBNTM) shows a distinct advantage when evaluated against most of the approaches reported in the literature. The addition of off-grid points within the system leads to a notable alteration in the accuracy of the proposed method.

Table 1: Assessment of Absolute Error for Problem 1,when $\lambda = 1$

X	McGough (1998)	Liao & Tan (2007)	Jalilian (2010)	Caglar et al. (2010)	Jator & Manathunga (2018)	$HBNTM_{1,5}$
0.1	1.98×10^{-6}	2.68×10^{-3}	5.77×10^{-10}	2.98×10^{-6}	1.03×10^{-15}	4.81×10^{-17}
0.2	3.94×10^{-6}	2.02×10^{-3}	2.47×10^{-10}	5.46×10^{-6}	1.68×10^{-15}	1.02×10^{-16}
0.3	5.85×10^{-6}	1.52×10^{-4}	4.56×10^{-11}	7.33×10^{-6}	3.04×10^{-15}	1.19×10^{-17}
0.4	7.70×10^{-6}	2.20×10^{-3}	9.64×10^{-11}	8.50×10^{-6}	3.77×10^{-15}	1.86×10^{-17}
0.5	9.47×10^{-6}	3.01×10^{-3}	1.46×10^{-10}	8.89×10^{-6}	3.83×10^{-15}	1.77×10^{-18}
0.6	1.11×10^{-5}	2.20×10^{-3}	9.64×10^{-11}	8.50×10^{-6}	3.77×10^{-15}	2.74×10^{-17}
0.7	1.26×10^{-5}	1.52×10^{-4}	4.56×10^{-11}	7.33×10^{-6}	3.03×10^{-15}	8.07×10^{-18}
0.8	1.35×10^{-5}	2.02×10^{-3}	2.47×10^{-10}	5.46×10^{-6}	1.67×10^{-15}	1.97×10^{-18}
0.9	1.20×10^{-5}	2.68×10^{-3}	5.77×10^{-10}	2.98×10^{-6}	1.01×10^{-15}	4.81×10^{-17}

Table 2: Assessment of Absolute Error for Problem 1 when $\boldsymbol{\lambda = 2}$

x	McGough (1998)	Liao & Tan (2007)	Jalilian (2010)	Caglar*et al.* (2010)	Jator & Manathunga (2018)	$BHNTM_{1,5}$
0.1	2.13×10^{-3}	1.52×10^{-2}	9.71×10^{-9}	1.72×10^{-5}	1.91×10^{-14}	1.01×10^{-15}
0.2	4.21×10^{-3}	1.47×10^{-2}	1.41×10^{-8}	3.26×10^{-5}	6.00×10^{-14}	3.20×10^{-15}
0.3	6.19×10^{-3}	5.89×10^{-3}	1.98×10^{-8}	4.49×10^{-5}	1.17×10^{-13}	6.17×10^{-15}
0.4	8.00×10^{-3}	3.25×10^{-3}	2.42×10^{-8}	5.28×10^{-5}	1.67×10^{-13}	8.84×10^{-15}
0.5	9.60×10^{-3}	6.98×10^{-3}	2.60×10^{-8}	5.56×10^{-5}	1.88×10^{-13}	9.92×10^{-15}
0.6	1.09×10^{-3}	3.25×10^{-3}	2.42×10^{-8}	5.28×10^{-5}	1.67×10^{-13}	8.84×10^{-15}
0.7	1.19×10^{-2}	5.89×10^{-3}	1.98×10^{-8}	4.49×10^{-5}	1.16×10^{-13}	6.17×10^{-15}
0.8	1.24×10^{-2}	1.47×10^{-2}	1.41×10^{-8}	3.26×10^{-5}	5.99×10^{-14}	3.20×10^{-15}
0.9	1.09×10^{-3}	1.52×10^{-2}	9.71×10^{-9}	1.72×10^{-5}	1.90×10^{-14}	1.01×10^{-15}

Table 3: Comparison of Absolute Error for Problem 1 when $\lambda = 3.51$

x	**Caglaret al. (2010)**	**Jalilian (2010)**	**Nasabet al. (2013)**	**Jator and Manathunga (2018)**	$BHNTM_{1,5}$
0.1	3.84×10^{-2}	6.61×10^{-6}	2.34×10^{-10}	2.75×10^{-12}	1.42×10^{-14}
0.2	7.48×10^{-2}	5.83×10^{-6}	3.20×10^{-10}	1.37×10^{-12}	7.07×10^{-14}
0.3	1.06×10^{-1}	6.19×10^{-6}	7.88×10^{-10}	6.91×10^{-12}	4.41×10^{-14}
0.4	1.27×10^{-1}	6.89×10^{-6}	1.11×10^{-9}	3.13×10^{-11}	2.87×10^{-14}
0.5	1.35×10^{-1}	7.31×10^{-6}	1.22×10^{-9}	4.92×10^{-11}	8.44×10^{-14}
0.6	1.27×10^{-1}	6.89×10^{-6}	1.11×10^{-9}	3.13×10^{-11}	2.66×10^{-14}
0.7	1.06×10^{-1}	6.19×10^{-6}	7.88×10^{-10}	6.91×10^{-12}	1.55×10^{-14}
0.8	7.48×10^{-2}	5.83×10^{-6}	3.20×10^{-10}	1.37×10^{-12}	5.70×10^{-14}
0.9	3.84×10^{-2}	6.61×10^{-6}	2.34×10^{-10}	2.75×10^{-12}	4.30×10^{-16}

- *Problem 2: IVPs of Bratu-type equations*

Consider the initial value problem of Bratu-type

$$y''(x) - 2e^{y(x)} = 0, \quad y(0) = y'(0) = 0, \quad 0 \le x \le 1. \tag{4.4}$$

With Exact Solution

Table 4: Maximum Absolute Error and Rate of Convergence for Problem 2

N	**Error in BNM (Jator, 2018)**	ROC	**Error in BHNT**	ROC
2	4.97×10^{-6}		3.13×10^{-7}	
4	4.05×10^{-8}	6.94	2.31×10^{-9}	7.08
8	2.15×10^{-10}	7.56	1.17×10^{-11}	7.63
16	9.26×10^{-13}	7.86	4.95×10^{-14}	7.88
32	1.33×10^{-15}	9.44	1.98×10^{-16}	7.96

Source: Jator and Manathunga (2018)

Table 4 presents the maximum absolute errors achieved at different values of N, comparing the proposed BHNTM with the BNM as described by Jator and Manathunga (2018). This highlights the advantages of the BHNTM. The ROCs demonstrate that the BHNTM operates as a seventh-order method, aligning with the theoretical order of p=7.

Table 5: Comparison of Absolute Error for Problem 2

x	**Jalilian (2010) (GA)**	**Raja et al. (2016) (ASM)**	**Raja et al. (2016) (RKM)**	**Raja et al. (2016) (GA-ASM)**	**Jator & Manathunga (2018)**	**BHNTM$_{1,5}$**
0.0	6.13×10^{-4}	4.66×10^{-8}	0.00	6.88×10^{-8}	0.00	0.00
0.1	5.85×10^{-4}	1.05×10^{-8}	7.45×10^{-9}	5.71×10^{-8}	3.50×10^{-15}	1.92×10^{-16}
0.2	6.19×10^{-4}	5.22×10^{-8}	1.09×10^{-8}	8.43×10^{-8}	1.63×10^{-14}	8.65×10^{-16}
0.3	7.08×10^{-4}	2.16×10^{-7}	1.36×10^{-8}	7.18×10^{-8}	4.46×10^{-14}	2.37×10^{-15}
0.4	8.03×10^{-4}	9.09×10^{-8}	1.75×10^{-8}	4.33×10^{-8}	1.05×10^{-13}	5.58×10^{-15}
0.5	8.73×10^{-4}	1.80×10^{-7}	2.41×10^{-8}	1.30×10^{-7}	2.37×10^{-13}	1.26×10^{-14}
0.6	9.42×10^{-4}	8.97×10^{-8}	3.56×10^{-8}	2.59×10^{-7}	5.47×10^{-13}	2.91×10^{-14}
0.7	1.05×10^{-3}	1.68×10^{-7}	5.43×10^{-8}	5.89×10^{-8}	1.33×10^{-12}	7.10×10^{-14}
0.8	1.21×10^{-3}	3.31×10^{-8}	7.92×10^{-8}	3.13×10^{-7}	3.54×10^{-12}	1.89×10^{-13}
0.9	1.39×10^{-3}	5.53×10^{-8}	1.14×10^{-7}	1.43×10^{-7}	1.06×10^{-11}	5.70×10^{-13}
1.0	1.73×10^{-3}	6.22×10^{-7}	1.66×10^{-7}	1.51×10^{-6}	3.78×10^{-11}	2.03×10^{-12}

Table 5, at various values of x when N=10, presents the absolute errors derived from different approaches. The BHNTM surpasses the methodologies documented in the literature.

B. *Results and Discussion*

This study details the stepwise implementation of the BHNTM, employing the block method technique of a continuous linear multistep method to formulate a system of equations on sub-intervals. A block matrix equation is formulated to obtain simultaneous solutions for the entire interval, rather than solving the equations directly. Solutions to the Bratu-type equation are assured to be accurately approximated by this methodological technique.

The numerical examples presented in the study demonstrate the effectiveness and precision of the proposed Block Hybrid Nystrom-Type Method (BHNTM) in addressing the Bratu-type problem (4.1). The results are comprehensively outlined for each topic addressed.

This study examines a specific Bratu-type equation under boundary conditions where λ is equal to zero. Equation (4.2) presents the analytical solution for this equation, which can yield zero, one, or two solutions based on the conditions outlined in equation 4.3. The critical value of the parameter λ is examined in relation to the solutions of the equation.

Table 1 presents an evaluation of the absolute errors for Problem 1, specifically the Bratu-type equation, at λ=1. Comparisons are made between errors from various approaches, including those of McGough (1998), Liao & Tan (2007), Jalilian (2010), Caglaret al. (2010), and Jator & Manathunga (2018), and the results obtained. The Block Hybrid Nystrom-Type Method with five off-grid points ($BHNTM_{1,5}$) demonstrates significantly lower absolute errors compared to alternative methods, indicating its enhanced accuracy.

An analogous comparison of absolute errors for Problem 1 at λ=2 is presented in Table 2. The accuracy of the $BHNTM_{1,5}$ exceeds that of alternative techniques, illustrating its superiority and efficacy.

The analysis of absolute errors for Problem 1 is extended to incorporate the case where λ=3.51, as presented in Table 3. The $BHNTM_{1,5}$ consistently exhibits absolute errors that are lower than those identified by other methodologies, such as those presented by Caglaret al. (2010), Jalilian (2010), Nasabet al. (2013), and Jator & Manathunga (2018), thereby demonstrating exceptional accuracy. The results clearly indicated that the BHNTM exhibits high accuracy across a broad spectrum of λ values. The numerical examples consistently validate the precision and effectiveness of the BHNTM in solving Bratu-type equations. It consistently outperforms alternative methods across various contexts, demonstrating its effectiveness in addressing similar second-order nonlinear differential equations with boundary value problems. The results evaluating the approach support its high accuracy, demonstrated in various scientific and engineering applications where these equations are utilised. Analysis of the Rate of Convergence (ROC) suggests that the BHNTM operates analogously to a method of order seven (7). The observed order of convergence aligns with the theoretical order of the BHNTM approach, thereby validating the method's claimed high accuracy. Table 4 presents the largest absolute errors generated by the proposed Block Hybrid Nystrom-Type (BHNTM) and the Block Nystrom Method (BNM) as outlined by Jator and Manathunga (2018) across different values of N, with N representing the number of iterations. The results clearly indicate that the BHNTM demonstrates greater accuracy compared to the BNM.

Table 5 presents a comprehensive assessment of the absolute errors associated with Problem 2 across various parameters. The comparisons involve the techniques employed by Jalilian (2010), Raja et al. (2016), Jator & Manathunga (2018), $BHNTM_{1,5}$, and Raja et al. (2016), utilising the Genetic Algorithm (GA), Active Set Method (ASM), and RK4 Method (RKM).

According to the study, the $BHNTM_{1,5}$ is the most accurate of all the methods because it consistently outguesses the others for the various objective values. One of them, the $BHNTM_{1,5}$ demonstrates its exceptional precision in solving differential equations by achieving incredibly low absolute errors, occasionally with values nearly close to 0.

To put it another way, the results for Problem 2 support the BHNTM's primary assertion, which is that it solves Bratu-type equation initial value issues with great accuracy and efficiency. The approach has the potential to solve such nonlinear differential equations since it performs better than alternative numerical techniques. The theoretical order of convergence of the BHNTM is further supported by the Rate of Convergence analysis, making it a reliable numerical method for applications in science and engineering where accuracy is crucial.

V. CONCLUSION

Many researchers have been working on Bratu problems, which are well-known and come from different areas. In this work, we solve the problems by removing the strong non-linear terms that cause difficulties. The primary benefit of the suggested method is its ability to address issues without reverting them to their initial value equivalents, so alleviating computational load and conserving time. Enhancing the off-grid points further augments precision. The numerical findings from BHNTM demonstrate its superiority over the approaches documented in the literature regarding accuracy and consistency.

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