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Computational & Functional Analysis of Special Functions with Arbitrary Parameters

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Abstract:- This work presents a comprehensive computational and functional analysis of special functions, specifically focusing on cases involving arbitrary integer parameters. Using integral transformations and identities, such as those from the Beta, Gamma, poly-gamma, and Zeta functions, we explore and derive solutions to various complex integral expressions. The problem sets address combinations of logarithmic, trigonometric, and exponential functions, including of the form $\ln(x)\tan(\frac{x}{b})$ and $\text{arcsinh}(\text{csch}(mx))$, where $b, m \in$ Z. Each solution is derived under generalized conditions, allowing for a range of integer parameter values. The study demonstrates the use of advanced mathematical techniques, including substitution, binomial expansions, and Fourier series, to simplify and compute the integrals. The results offer insights into the computational strategies required for complex special functions and serve as a reference for future explorations of such functions in both theoretical and applied mathematics.

I. INTRODUCTION

In this paper, we present a comprehensive study on the computational and functional analysis of special functions with arbitrary values. Special functions such as Beta, Gamma, and poly-gamma functions play a significant role in various mathematical and physical applications. Our work aims to derive and explore general solutions for complex integral equations involving these functions. By leveraging algebraic identities, binomial expansion, and trigonometric transformations, we investigate a series of special problems that encompass multiple levels of mathematical complexity. This study is particularly focused on expressions involving logarithmic and trigonometric functions, as well as exponential and hyperbolic terms, providing generalized solutions for integer-based parameters and constraints. Through systematic evaluation and the use of advanced mathematical techniques, we offer valuable insights into the behaviors and properties of these special functions. Our findings have potential applications in fields requiring precise mathematical computations, including theoretical physics and engineering.

II. MAIN RESULTS

This study investigates several problem sets involving integrals of complex functions, where parameters are arbitrary integers. The primary results for each problem set are as follows:

[1:1] Problem Set 1: Integration of Logarithmic and Trigonometric Functions

- *Objective*

To derive a general solution for integrals of the form:

$$\int_0^\infty \frac{ln^n(x)\tan^{-1}(\frac{x}{b})}{x^2+b^2}\,dx\,, \qquad where\ b,n \in \mathbb{Z}$$

- *Approach*

By using substitutions involving $u = \tan^{-1}(\frac{x}{b})$ and properties of logarithmic functions, we express the integral in terms of Beta and Gamma Functions.

- *Consider the Following:*

$$\int_0^\infty \frac{ln^n(x)\tan^{-1}\left(\frac{x}{b}\right)}{x^2+b^2}\,dx$$

Let:

$$u = \tan^{-1}(\frac{x}{b})$$

$$\therefore x = b\tan(u)$$

$$\therefore du = \frac{dx}{\frac{x^2}{b^2}+1}\frac{1}{b} = \frac{b\,dx}{x^2+b^2}$$

Therefore, by substituting:

$$\int_0^\infty \frac{ln^n(x)\tan^{-1}\left(\frac{x}{b}\right)}{x^2+b^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{u\,ln^n(b\tan(u))}{b}\,du$$

$$= \frac{1}{b}\int_0^{\frac{\pi}{2}} u(ln^n(b) + ln^n(\tan(u))\,du = \frac{1}{b}\int_0^{\frac{\pi}{2}} u(\ln(b) + \ln(\tan(u)))^n\,du$$

Applying the Binomial Expansion definition:

$$(s+t)^m = \sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k$$

We can apply the following algebraic identity:

$$\frac{1}{b}\int_0^{\frac{\pi}{2}} u(\ln(b) + \ln(\tan(u)))^n\,du = \frac{1}{b}\int_0^{\frac{\pi}{2}} u(\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)ln^k(\tan(u)))\,du = \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)\int_0^{\frac{\pi}{2}} uln^k(\tan(u))\,du$$

Using the trigonometric identity:

$$\tan(u) = \frac{\sin(u)}{\cos(u)}$$

We can substitute the following identity:

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)\int_0^{\frac{\pi}{2}} uln^k(\tan(u))\,du$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)\int_0^{\frac{\pi}{2}} uln^k\left(\frac{\sin(u)}{\cos(u)}\right)du$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)\int_0^{\frac{\pi}{2}} u(\ln(\sin(u)) - \ln(\cos(u)))^k\,du$$

Applying the Difference Binomial Expansion:

$$(s-t)^j = \sum_{m=0}^{j}\binom{j}{m}s^{j-m}t^m(-1)^m$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)\int_0^{\frac{\pi}{2}} u(\ln(\sin(u)) - \ln(\cos(u)))^k\,du$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k}ln^{n-k}(b)\int_0^{\frac{\pi}{2}} u\sum_{m=0}^{j}\binom{j}{m}ln^{j-m}(\sin(u))\,(-1)^m ln^m(\cos(u))\,du$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^m \int_0^{\frac{\pi}{2}} u(ln^{j-m}(\sin(u))\, ln^m(\cos(u))\, du$$

To simplify the algebraic expression, we can denote:

$$\int_0^{\frac{\pi}{2}} u(ln^{j-m}(\sin(u))\, ln^m(\cos(u))\, du = \delta$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^m \int_0^{\frac{\pi}{2}} u(ln^{j-m}(\sin(u))\, ln^m(\cos(u))\, du$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^m\ \delta$$

Since:

$$\delta = \int_0^{\frac{\pi}{2}} u(ln^{j-m}(\sin(u))\, ln^m(\cos(u))\, du$$

We can apply Kings Property to expand the following expression:

$$\int_0^{\frac{\pi}{2}} u(ln^{j-m}(\sin(u))\, ln^m(\cos(u))\, du = \int_0^{\frac{\pi}{2}} (\frac{\pi}{2}-u)(ln^{j-m}(\sin(\frac{\pi}{2}-u))\, ln^m(\cos(\frac{\pi}{2}-u))\, du$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} ln^{j-m}(\cos(u))\, ln^m(\sin(u))\, du - \delta$$

Therefore, by simplifying the following expression:

$$2\delta = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} ln^{j-m}(\cos(u))\, ln^m(\sin(u))\, du$$

$$4\delta = \pi\int_0^{\frac{\pi}{2}} ln^{j-m}(\cos(u))\, ln^m(\sin(u))\, du$$

To simplify the following expressions, we will change the parameters of the integral:

$$w = \sin(u)$$

$$\therefore \sqrt{1-w^2} = \cos(u)$$

$$\therefore \cos(u)\, du = dw$$

$$\therefore du = \frac{dw}{\sqrt{1-w^2}}$$

We can substitute the following:

$$4\delta = \pi\int_0^{1} \frac{ln^{j-m}(\sqrt{1-w^2})ln^m(w)}{\sqrt{1-w^2}} dw$$

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By simplifying with more parameter simplification:

$$w^2 = t$$

$$\therefore w = t^{1/2}$$

$$\therefore dw = \frac{1}{2}t^{-1/2}$$

We can compress the expression under new parameters:

$$4\delta = \pi \int_0^1 \frac{ln^{j-m}(\sqrt{1-t})ln^m(t^{1/2})}{\sqrt{1-t}} \frac{1}{2}t^{-1/2}\ dt$$

$$4\delta = \pi \int_0^1 \frac{\left(\frac{1}{2}\right)^{j-m} ln^{j-m}(1-t)ln^m(t)(\frac{1}{2})^m \frac{1}{2}t^{-1/2}}{\sqrt{1-t}}\ dt$$

$$4\delta = (\frac{1}{2})^{j+1}\pi \int_0^1 t^{-\frac{1}{2}}ln^{j-m}(1-t)ln^m(t)(1+t)^{-1/2}\,dt$$

$$4\delta = (\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \int_0^1 t^{-\frac{1}{2}}ln^{j-m}(1-t)ln^m(t)(1+t)^{-1/2}\,dt$$

$$4\delta = (\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \int_0^1 t^{a-\frac{1}{2}}ln^{j-m}(1-t)(1-t)^{-1/2}\,dt$$

By simplifying the following statement above:

$$1 - t = s$$

$$\therefore 1 - s = t$$

$$\therefore dt = -ds$$

We can expand the integration:

$$4\delta = -(\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \int_0^1 (1-s)^{a-\frac{1}{2}}ln^{j-m}(s)(s)^{-1/2}\,ds$$

$$4\delta = -(\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \left.\frac{d^{j-m}}{dz^{j-m}}\right|_{z=0} \int_0^1 (1-s)^{a-\frac{1}{2}}s^z s^{-1/2}\,ds$$

$$4\delta = -(\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \left.\frac{d^{j-m}}{dz^{j-m}}\right|_{z=0} \int_0^1 (1-s)^{a-\frac{1}{2}}s^{z-\frac{1}{2}}\,ds$$

By applying the Beta Function substitution to Zeta Function, we obtain the following:

$$4\delta = -(\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \left.\frac{d^{j-m}}{dz^{j-m}}\right|_{z=0} B(z+\frac{1}{2}, a+\frac{1}{2})$$

$$4\delta = -(\frac{1}{2})^{j+1}\pi \left.\frac{d^m}{da^m}\right|_{a=0} \left.\frac{d^{j-m}}{dz^{j-m}}\right|_{z=0} \frac{\Gamma(z+\frac{1}{2})\Gamma(a+\frac{1}{2})}{\Gamma(z+a+1)}$$

$$\delta = -\frac{\pi}{4}(\frac{1}{2})^{j+1} \frac{d^m}{da^m}\bigg|_{a=0} \frac{d^{j-m}}{dz^{j-m}}\bigg|_{z=0} \frac{\Gamma(z+\frac{1}{2})\Gamma(a+\frac{1}{2})}{\Gamma(z+a+1)}$$

By obtaining the original function, as well as its new denoted piece, we can finally expand:

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^m \ \delta$$

$$= \frac{1}{b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^m \ -\frac{\pi}{4}\left(\frac{1}{2}\right)^{j+1} \frac{d^m}{da^m}\bigg|_{a=0} \frac{d^{j-m}}{dz^{j-m}}\bigg|_{z=0} \frac{\Gamma\left(z+\frac{1}{2}\right)\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(z+a+1)}$$

$$= \frac{\pi}{4b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^{m+1} \ (\frac{1}{2})^{j+1} \frac{d^m}{da^m}\bigg|_{a=0} \frac{d^{j-m}}{dz^{j-m}}\bigg|_{z=0} \frac{\Gamma(z+\frac{1}{2})\Gamma(a+\frac{1}{2})}{\Gamma(z+a+1)}$$

Therefore, by expanding the integration in respect to its multiple summation and Zeta function, we obtained the general solution for [1] Problem Set 1:

$$\frac{\pi}{4b}\sum_{k=0}^{n}\binom{n}{k} ln^{n-k}(b) \sum_{m=0}^{j}\binom{j}{m}(-1)^{m+1} \ (\frac{1}{2})^{j+1} \frac{d^m}{da^m}\bigg|_{a=0} \frac{d^{j-m}}{dz^{j-m}}\bigg|_{z=0} \frac{\Gamma(z+\frac{1}{2})\Gamma(a+\frac{1}{2})}{\Gamma(z+a+1)}$$

III. RESULT

The derived solution incorporates a series expansion and further simplifies using the relationship between trigonometric functions, yielding a closed-form expression under the conditions $b, n \in \mathbb{Z}$.

[1:2] Analysis of Problem Set 1: Parameters $b, n \in \mathbb{Z}$

Let $n = 1$ and $b = 1$:

$$\int_0^{\infty} \frac{ln^n(x)\tan^{-1}(\frac{x}{b})}{x^2+b^2}\,dx \xrightarrow[(n,b)=(1,1)]{} \int_0^{\infty} \frac{ln^1(x)\tan^{-1}(\frac{x}{1})}{x^2+1^2}\,dx$$

Let:

$$w = \tan^{-1} x$$

$$\therefore \mathrm{x} = \tan(w)$$

$$\therefore dw = \frac{dx}{1+x^2}$$

$$\int_0^{\infty} \frac{ln^1(x)\tan^{-1}(\frac{x}{1})}{x^2+1^2}\,dx = \int_0^{\pi/2} \mathrm{w}\ln(\tan(w))\,dw$$

Using the identity:

$$\tan(w) = \frac{\sin(w)}{\cos(w)}$$

$$\int_0^{\pi/2} \mathrm{w}\ln(\tan(w))\,dw = \int_0^{\pi/2} \mathrm{w}\ln(\sin(w))\,dw - \int_0^{\pi/2} \mathrm{w}\ln(\cos(w))\,dw$$

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Let's denote the following:

$$\phi = \int_0^{\pi/2} \mathrm{w} \ln(\sin(w))\, dw$$

$$\psi = \int_0^{\pi/2} \mathrm{w} \ln(\cos(w))\, dw$$

Therefore:

$$\int_0^{\pi/2} \mathrm{w} \ln(\tan(w))\, dw = \int_0^{\pi/2} \mathrm{w} \ln(\sin(w))\, dw - \int_0^{\pi/2} \mathrm{w} \ln(\cos(w))\, dw = \phi - \psi$$

Solving for ϕ:

$$\phi = \int_0^{\pi/2} \mathrm{w} \ln(\sin(w))\, dw$$

Using Fourier Series:

$$\ln(sin(w)) = -\ln(2) - \sum_{n=1}^{\infty} \frac{\cos(2nw)}{n}$$

Substitute the following:

$$\phi = \int_0^{\frac{\pi}{2}} \mathrm{w} \ln(\sin(w))\, dw$$

$$= \int_0^{\pi/2} \mathrm{w} \left(-\ln(2) - \sum_{n=1}^{\infty} \frac{\cos(2nw)}{n} \right) dw = -\ln(2) \int_0^{\frac{\pi}{2}} w\, dw - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{2}} w \cos(2nw)\, dw$$

$$= -\ln(2) \frac{w^2}{2} |_0^{\frac{\pi}{2}} - \sum_{n=1}^{\infty} \frac{1}{n} \{ \frac{w}{2n} \sin(2nw) + \frac{1}{4n^2} \cos(2nw) \}_0^{\pi/2}$$

$$= -\frac{\pi^2}{8} \ln(2) - \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{4n^3} - \frac{1}{4n^3} \right\} = -\frac{\pi^2}{8} \ln(2) - \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

We know:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) = Zeta\ Function$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = 2^{-s} (2 - 2^s) \zeta(s)$$

$$= -\frac{\pi^2}{8} \ln(2) - \frac{1}{4} \left(2^{-3} (2 - 2^3) \right) \zeta(3) + \frac{1}{4} \zeta(3) = -\frac{\pi^2}{8} \ln(2) + \frac{7}{16} \zeta(3)$$

Hence,

$$\phi = -\frac{\pi^2}{8} \ln(2) + \frac{7}{16} \zeta(3)$$

Solving for ψ:

$$\psi = \int_0^{\pi/2} \mathrm{w} \ln(\cos(w))\, dw$$

Using Kings Property:

$$\psi = \int_0^{\pi/2} \mathrm{w} \ln(\cos(w))\, dw = \psi = \int_0^{\pi/2} (\frac{\pi}{2} - \mathrm{w}) \ln\left(\cos\left(\frac{\pi}{2} - w\right)\right) dw$$

$$= \int_0^{\pi/2} (\frac{\pi}{2} - \mathrm{w}) \ln(\sin(\mathrm{w}))\, dw$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \ln(\sin(\mathrm{w}))\, dw - \int_0^{\pi/2} \mathrm{w} \ln(\sin(w))\, dw$$

Denote:

$$\eta = \frac{\pi}{2}\int_0^{\pi/2} \ln(\sin(\mathrm{w}))\, dw$$

$$= \frac{\pi}{2}\int_0^{\pi/2} (-\ln(2) - \sum_{n=1}^{\infty} \frac{\cos(2nw)}{n})\, dw = \frac{\pi}{2}\left[-\ln(2)\int_0^{\frac{\pi}{2}} 1\, dw - \sum_{n=1}^{\infty} \frac{1}{n}\int_0^{\frac{\pi}{2}} \cos(2nw)\, dw\right]$$

$$= \frac{\pi}{2}\left[-\ln(2)\, w|_0^{\frac{\pi}{2}} - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\sin(2nw)}{2n}\right)|_0^{\frac{\pi}{2}}\right] = \frac{-\pi}{2}\left[-\ln(2)\frac{\pi}{2}\right] = \frac{-\pi^2}{4}\ln(2)$$

$$\psi = \frac{\pi}{2}\int_0^{\pi/2} \ln(\sin(w))\, dw - \int_0^{\pi/2} \mathrm{w} \ln(\sin(w))\, dw = \eta - \phi$$

$$\psi = \frac{-\pi^2}{4}\ln(2) - \left(-\frac{\pi^2}{8}\ln(2) + \frac{7}{16}\zeta(3)\right) = \frac{-\pi^2}{8}\ln(2) - \frac{7}{16}\zeta(3)$$

Therefore, the following condition where $b = 1$ and $n = 1$:

$$\int_0^{\pi/2} \mathrm{w} \ln(\tan(w))\, dw = \int_0^{\pi/2} \mathrm{w} \ln(\sin(w))\, dw - \int_0^{\pi/2} \mathrm{w} \ln(\cos(w))\, dw = \phi - \psi$$

$$= -\frac{\pi^2}{8}\ln(2) + \frac{7}{16}\zeta(3) - \left(\frac{-\pi^2}{8}\ln(2) - \frac{7}{16}\zeta(3)\right) = \frac{7}{8}\zeta(3) \approx 1.0517 \ldots$$

[1:3] Continuation for all n and b arbitrary values, where $n, b \in \mathbb{Z}$

The following data was compiled through Wolfram Alpha:

$$For\ n = 1, b = 1: \int_0^{\infty} \frac{ln(x)\tan^{-1}(x)}{x^2 + 1}\, dx \approx 1.0518$$

$$For\ n = 1, b = 2: \int_0^{\infty} \frac{ln(x)\tan^{-1}(\frac{x}{2})}{x^2 + 2^2}\, dx = 0.953468$$

$$For\ n = 1, b = 3: \int_0^{\infty} \frac{ln(x)\tan^{-1}(\frac{x}{3})}{x^2 + 3^2}\, dx = 0.802386\ldots$$

$$For\ n = 2, b = 1: \int_0^\infty \frac{ln^2(x)\tan^{-1}(x)}{x^2 + 1^2}\,dx = 3.04403$$

$$For\ n = 2, b = 2: \int_0^\infty \frac{ln^2(x)\tan^{-1}(\frac{x}{2})}{x^2 + 2^2}\,dx = 2.54744\ ...$$

The following method can be easily calculated for $n, b \in \mathbb{Z}$

[2:1] Problem Set 2: Analysis of Gamma Function Integrals

➢ *Objective*

To solve the integrals of the form:

$$\int_0^\infty ln^n(x)e^{-ax}\,dx, \quad where\ \ a, n \in \mathbb{Z}$$

➢ *Approach*

Using substitutions and identities associated with the Gamma function, this set evaluates the integral by expressing it in terms of poly-gamma and Zeta functions.

Consider:

$$\int_0^\infty ln^n(x)e^{-ax}\,dx$$

By doing the following:

$$t = e^{-ax}$$

$$\therefore \ln(t) = -ax$$

$$\therefore x = \frac{-1}{a}\ln(t)$$

$$\therefore dx = \frac{-1}{a}\frac{dt}{t}$$

Hence,

$$\int_0^\infty ln^n(x)e^{-ax}\,dx = -\int_0^\infty \frac{1}{a} ln^n\left(\frac{-1}{a}\ln(t)\right) dt = -\frac{1}{a}\int_0^\infty [\ln(t^{-1}) - \ln(a)]^n dt$$

Applying the Difference Binomial Expansion:

$$(x - y)^j = \sum_{m=0}^{j} \binom{j}{m} x^{j-m} y^m (-1)^m$$

$$= -\frac{1}{a}\int_0^\infty [\ln(t^{-1}) - \ln(a)]^n dt = \frac{-1}{a}\int_0^\infty [\sum_{k=0}^{n} \binom{n}{k} ln^{n-k}(\ln(t^{-1}))ln^k(a)(-1)^k]\,dt$$

$$= \frac{1}{a}\sum_{k=0}^{\infty} \binom{n}{k} (-1)^{k+1} ln^k(a) \int_0^\infty ln^{n-k}(\ln(t^{-1}))dt$$

Denote:

$$\rho = \frac{1}{a}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{k+1}ln^{k}(a)$$

$$= \frac{1}{a}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{k+1}ln^{k}(a)\int_{0}^{\infty} ln^{n-k}(\ln(t^{-1}))dt = \rho\int_{0}^{\infty} ln^{n-k}(\ln(t^{-1}))dt$$

Let:

$$w = \ln(t^{-1})$$

$$\therefore\ e^{w} = t^{-1}$$

$$\therefore t = e^{-w}$$

$$\therefore dt = \frac{-e^{-w}}{w}dw$$

$$\rho\int_{0}^{\infty} ln^{n-k}(\ln(t^{-1}))dt = \rho\int_{0}^{\infty} ln^{n-k}(\ln(e^{w}))\frac{-e^{-w}}{w}dw = \rho\int_{0}^{\infty} ln^{n-k}(w)\ \frac{-e^{-w}}{w}dw$$

$$\rho\ \frac{d}{d\psi^{n-k}}|_{\psi=0}\int_{0}^{\infty} w^{\psi-1}\ e^{-w}dw = \ \rho\ \frac{d}{d\psi^{n-k}}|_{\psi=0}\ \Gamma(\psi)$$

Therefore, by compressing the expression into a Gamma function we obtained the general solution for [2] Problem Set 2:

$$\frac{1}{a}\sum_{k=0}^{\infty}\binom{n}{k}(-1)^{k+1}ln^{k}(a)\frac{d}{d\psi^{n-k}}|_{\psi=0}\ \Gamma(\psi)$$

➢ *Result*

The results yield a general solution, expressed as a series involving $\Gamma(\psi)$ (Zeta function) and various combinations of poly-gamma functions. This solution is adaptable for different integer values of a and n.

[2:2] Analysis of Problem Set 2: Parameters $a, n \in \mathbb{Z}$

Conditions for values n and a:

$$\int_{0}^{\infty} ln^{n}(x)e^{-ax}\ dx = \frac{d}{d\psi^{n}}|_{\psi=0}\int_{0}^{\infty} x^{\psi}e^{-ax}\ dx$$

Let:

$$w = ax$$

$$\therefore x = \frac{w}{a}$$

$$\therefore dx = \frac{dw}{a}$$

$$\frac{d}{d\psi^{n}}|_{\psi=0}\int_{0}^{\infty} x^{\psi}e^{-ax}\ dx = \frac{d}{d\psi^{n}}|_{\psi=0}\int_{0}^{\infty}\frac{1}{a}(\frac{w}{a})^{\psi}e^{-w}\ dw = \frac{1}{a}\frac{d}{d\psi^{n}}|_{\psi=0}\int_{0}^{\infty}\frac{1}{a^{\psi}}\Gamma(b+1)$$

$$= \frac{1}{a}\frac{d}{d\psi^{n}}|_{\psi=0}\int_{0}^{\infty}\frac{1}{a^{\psi}}\Gamma(b+1)$$

$$= \frac{1}{a}\frac{d}{d\psi^n}|_{\psi=0}\frac{\psi!}{a^\psi}$$

We will generate solutions for $n, a \epsilon$ Z, using the following formula:

$$\frac{1}{a}\frac{d}{d\psi^n}|_{\psi=0}\frac{\psi!}{a^\psi}$$

Let's consider the following generated solutions for $n = 1,2,3$ and $a = 1$:

Consider $n = 1, a = 1$:

$$\frac{1}{a}\frac{d}{d\psi^n}|_{\psi=0}\frac{\psi!}{a^\psi}\xrightarrow{a=1,n=1}\frac{d}{d\psi^1}|_{\psi=0}\frac{\psi!}{a^\psi}$$

$$\frac{d}{d\psi^1}|_{\psi=0}\frac{\psi!}{a^\psi} = \frac{d}{d\psi^n}|_{\psi=0}[a^{-\psi}\psi!]$$

$$= [\psi(1) - Log(1)] = \psi(1) = -\gamma.$$

$$Where\ \gamma\ is\ the\ Euler - Mascheroni\ constant$$

Consider $n = 2, a = 1$:

$$\frac{1}{a}\frac{d}{d\psi^n}|_{\psi=0}\frac{\psi!}{a^\psi}\xrightarrow{a=1,n=2}\frac{d^2}{d\psi^2}|_{\psi=0}\frac{\psi!}{a^\psi}$$

$$= \frac{d^2}{d\psi^2}|_{\psi=0}[\Gamma(\psi+1)] = \Gamma(\psi+1)\left[\Psi^{(0)}(\psi+1)^2 + \Psi^{(1)}(\psi+1)\right]$$

$$= \Gamma(1)\left[\Psi^{(0)}(1)^2 + \Psi^{(1)}(1)\right] = \gamma^2 + \frac{\pi^2}{6}.$$

Consider $n = 3, a = 1$:

$$\frac{1}{a}\frac{d}{d\psi^n}|_{\psi=0}\frac{\psi!}{a^\psi}\xrightarrow{a=1,n=3}\frac{d^3}{d\psi^3}|_{\psi=0}\frac{\psi!}{a^\psi}$$

$$= \frac{d^3}{d\psi^3}|_{\psi=0}\frac{\psi!}{a^\psi}$$

$$= \Gamma(\psi+1)\left[\Psi^{(2)}(\psi+1) + 3\Psi^{(1)}(\psi+1)\Psi^{(0)}(\psi+1) + \left(\Psi^{(0)}(\psi+1)\right)^3\right]$$

$$= \Gamma(1)\left[\Psi^{(2)}(1) + 3\Psi^{(1)}(1)\Psi^{(0)}(1) + \left(\Psi^{(0)}(1)\right)^3\right] = -2\zeta(3) - \frac{\pi^2}{2}\gamma - \gamma^3.$$

[2:3] Continuation for all a and n arbitrary values, where $a, n \in \mathbb{Z}$

The following data was compiled through Wolfram Alpha:

$$For\ n = 1, a = 1: \int_0^\infty ln^1(x)e^{-1x}\,dx \approx -0.57721$$

$$For\ n = 2, a = 1: \int_0^\infty ln^2(x)e^{-1x}\,dx \approx 1.97811$$

$$For\ n = 3, a = 1: \int_0^\infty ln^3(x)e^{-1x}\,dx \approx -5.44487\ \ldots$$

$$For\ n = 1, a = 2: \int_0^\infty ln^1(x)e^{-2x}\,dx = \frac{1}{2}(-\gamma - \log(2)) \approx -0.635181$$

$$For\ n = 2, a = 2: \int_0^\infty ln^2(x)e^{-2x}\,dx = \frac{1}{12}(\pi^2 + 6(\gamma + \log(2))^2) \approx 1.62938$$

$$For\ n = 3, a = 2: \int_0^\infty ln^3(x)e^{-2x}\,dx = -\zeta(3) - \frac{1}{4}(\gamma + \log(2))(\pi^2 + 2(\gamma + \log(2))^2) \approx -5.36162$$

...

The following method can be easily calculated for $a, n \in \mathbb{Z}$

[3:1] Problem Set 3: Integrals of Logarithmic and Sinusoidal Functions

- *Objective*

To determine solutions for integrals like:

$$\int_0^\infty ln^n(ax)\sin(bx)\,e^{-cx}\,dx, \quad where\ a, b, c, n\ \in \mathbb{Z}$$

- *Approach*

By expanding $\sin(bx)$ in terms of complex exponentials, the integral transforms into an imaginary part of a complex function. Further simplification using Gamma function derivatives is applied:

- Consider:

$$\int_0^\infty ln^n(ax)\sin(bx)\,e^{-cx}\,dx$$

By expanding in terms of complex exponentials:

$$\sin(bx) = Im(e^{ibx})$$

$$\int_0^\infty ln^n(ax)\sin(bx)\,e^{-cx}\,dx = Im\{\int_0^\infty ln^n(ax)e^{ibx-cx}dx\} = Im\{\int_0^\infty ln^n(ax)e^{-x(c-ib)}dx\}$$

Using a known property of $ln^n(x)$:

$$\frac{d^n}{dz^n}|(ax)^z = ln^n(ax)$$

$$Im\{\int_0^\infty ln^n(ax)e^{-x(c-ib)}dx\} = Im\{\frac{d^n}{dz^n}|_{z=0}\,a^z\int_0^\infty x^z e^{-x(c-ib)}dx\}$$

Let:

$$y = x(c - ib)$$

$$\therefore x = \frac{y}{c - ib}$$

$$\therefore dx = \frac{dy}{c - ib}$$

$$= Im\{\frac{d^n}{dz^n}|_{z=0}\,a^z\int_0^\infty x^z e^{-x(c-ib)}dx\}$$

$$= Im\{\frac{d^n}{dz^n}|_{z=0}\ a^z \int_0^{\infty} \left(\frac{y}{c-ib}\right)^z e^{-y}\frac{dy}{c-ib}\}$$

$$= Im\{\frac{d^n}{dz^n}|_{z=0}\ \frac{a^z}{(c-ib)^{z+1}}\int_0^{\infty} y^z e^{-y}dy\}$$

Therefore, by expanding the sine function in terms of complex exponentials, as well as the Zeta function, we obtained the general solution for [3] Problem Set 3:

$$\frac{d^n}{dz^n}|_{z=0}\ \Gamma(z+1)Im\left\{\frac{a^z}{(c-ib)^{z+1}}\right\}$$

- *Results*

The solutions are represented as imaginary components of Gamma functions and poly-gamma derivatives, providing insight into the oscillatory nature of these functions when combined with logarithmic growth.

[3:2] Analysis of Problem Set 3: Parameters $\boldsymbol{a, b, c, n} \in \mathbb{Z}$

<u>Let's consider the following generated solutions for $n = 1$ and $a = b = c = 1$:</u>

Consider $n = 1$ and $a = b = c = 1$:

Using the general solution:

$$\frac{d^n}{dz^n}|_{z=0}\ \Gamma(z+1)Im\left\{\frac{a^z}{(c-ib)^{z+1}}\right\}$$

$$\frac{d^n}{dz^n}|_{z=0}\ \Gamma(z+1)Im\left\{\frac{a^z}{(c-ib)^{z+1}}\right\}\xrightarrow{a=b=c=n=1}\frac{d^1}{dz^1}|_{z=0}\ \Gamma(z+1)\ Im\left\{\frac{1}{(1-i)^{z+1}}\right\}$$

$$=\frac{d^1}{dz^1}|_{z=0}\ \Gamma(z+1)\ \omega, \qquad where\ \omega = Im\left\{\frac{1}{(1-i)^{z+1}}\right\}$$

Solving for ω:

$$\omega = Im\left\{\frac{1}{(1-i)^{z+1}}\right\} = Im\left\{\left(\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}i}\right)^{z+1}\right\}$$

$$= Im\left\{\left(\frac{\sqrt{2}}{2}\right)^{z+1}\left[\cos\left(\frac{\pi}{4}(z+1)\right) + isin\left(\frac{\pi}{4}(z+1)\right)\right]\right\}$$

$$= (\frac{\sqrt{2}}{2})^{z+1}\left[\sin\left(\frac{\pi}{4}(z+1)\right)\right] = 2^{\frac{-1}{2}(z+1)}\left[\sin\left(\frac{\pi}{4}(z+1)\right)\right]$$

Therefore,

$$\frac{d^1}{dz^1}|_{z=0}\ \Gamma(z+1)\ \omega = \frac{d^1}{dz^1}|_{z=0}\ \Gamma(z+1)\ 2^{\frac{-1}{2}(z+1)}\left[\sin\left(\frac{\pi}{4}(z+1)\right)\right]$$

$$= \Gamma(z+1)\{2^{\frac{-1}{2}(z+1)}\frac{\pi}{4}\cos\left(\frac{\pi}{4}(z+1)\right) - 2^{\frac{-1}{2}(z+3)}\ln(2)\sin\left(\frac{\pi}{4}(z+1)\right\} + \Gamma(z+1)\Psi^{(0)}(z+1)2^{\frac{-1}{2}(z+1)}\sin\left(\frac{\pi}{4}(z+1)\right)\xrightarrow{z=0}\ \frac{\pi}{8}-\frac{\ln(2)}{4}-\frac{\gamma}{2}$$

[3:3] Continuation for all a, b, c, and n arbitrary values, where $\boldsymbol{a, b, c, n} \in \mathbb{Z}$

➢ *The Following Data was Compiled through Wolfram Alpha:*

$$Let\ n = a = b = c = 1: \int_0^\infty ln^1(x)\sin(x)\, e^{-x}\, dx = \frac{\pi}{8} - \frac{\ln(2)}{4} - \frac{\gamma}{2} \approx -0.0691955$$
$$Let\ n = 2, a = b = c = 1: \int_0^\infty ln^2(x)\sin(x)\, e^{-x}\, dx \approx 0.215193$$
$$Let\ n = 3, a = b = c = 1: \int_0^\infty ln^3(x)\sin(x)\, e^{-x}\, dx \approx -0.319803$$
$$\ldots$$
$$Let\ n = 2, a = b = c = 2: \int_0^\infty ln^2(2x)\sin(2x)\, e^{-2x}\, dx = 0.207596$$
$$Let\ n = 3, a = b = c = 2: \int_0^\infty ln^3(2x)\sin(2x)\, e^{-2x}\, dx \approx -0.159901$$
$$Let\ n = 3, a = b = c = 3: \int_0^\infty ln^3(3x)\sin(3x)\, e^{-3x}\, dx \approx -0.106601$$
$$\ldots$$
$$Let\ n = 3, a = b = c = 4: \int_0^\infty ln^3(4x)\sin(4x)\, e^{-4x}\, dx = -\ 0.0799506$$

The following method can be easily calculated for $n, a, b, c \in \mathbb{Z}$

[4:1] Problem Set 4: Logarithmic Fractional Integrals

➢ *Objective*
To analyze the integrals of the form:

$$\int_0^1 \frac{ln^n(\frac{x}{1-x})}{1+x} dx, \quad where\ n \in \mathbb{Z}$$

➢ *Approach*
Using fractional transformation $y = \frac{x}{1-x}$, the integral is reformatted, allowing for simplification with logarithmic identities and the Liouville operator:

➢ *Consider*:

$$\int_0^1 \frac{ln^n(\frac{x}{1-x})}{1+x} dx$$

Using the fractional transformation:

$$y = \frac{x}{1-x}$$
$$\therefore x = \frac{y}{1+y}$$
$$\therefore dx = \frac{dy}{(y+1)^2}$$

Applying transformation:

$$\int_0^1 \frac{ln^n(\frac{x}{1-x})}{1+x} dx = \int_0^1 \frac{ln^n(y)}{1+\frac{y}{y+1}} \frac{dy}{(y+1)^2} = \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)} dy$$

$$\int_0^\infty \frac{ln^n(y)}{(2y+1)(y+1)}dy = \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)}dy + \int_1^\infty \frac{ln^n(y)}{(2y+1)(y+1)}dy$$

Let:

$$\int_1^\infty \frac{ln^n(y)}{(2y+1)(y+1)}dy \xrightarrow{y\to 1/y} \int_0^1 \frac{ln^n(\frac{1}{y})}{(1+y)(1+2y)}dy$$

$$= \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)}dy + \int_1^\infty \frac{ln^n(y)}{(2y+1)(y+1)}dy$$

$$= \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)}dy + \int_0^1 \frac{ln^n\left(\frac{1}{y}\right)}{(1+y)(1+2y)}dy$$

$$= \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)}dy + (-1)^n \int_0^1 \frac{ln^n(y)}{(y+1)(y+2)}dy$$

Denote the following:

$$\theta = \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)}dy$$

$$\varphi = (-1)^n \int_0^1 \frac{ln^n(y)}{(y+1)(y+2)}dy$$

Solving for θ:

$$\theta = \int_0^1 \frac{ln^n(y)}{(2y+1)(y+1)}dy = 2\int_0^1 \frac{ln^n(y)}{(2y+1)}dy - \int_0^1 \frac{ln^n(y)}{(y+1)}dy = \theta_a - \theta_b$$

Denote the following:

$$\theta_a = 2\int_0^1 \frac{ln^n(y)}{(2y+1)}dy$$

$$\theta_b = \int_0^1 \frac{ln^n(y)}{(y+1)}dy$$

Solving for θ_a:

$$\theta_a = 2\int_0^1 \frac{ln^n(y)}{(2y+1)}dy = 2\sum_{k=0}^{\infty}(-2)^k \int_0^1 ln^n(y)y^k dy$$

By changing the parameters:

$$-u = \ln(y)$$

$$\therefore e^{-u} = y$$

$$\therefore dy = -e^{-u}du$$

$$= 2\sum_{k=0}^{\infty}(-2)^k \int_0^1 ln^n(y)y^k dy$$

$$= 2\sum_{k=0}^{\infty}(-2)^k \int_0^{\infty}(e^{-u})^k(-u)^n(e^{-u}du) = 2\sum_{k=0}^{\infty}(-2)^k\,(-1)^n\int_0^{\infty}e^{-u(k+1)}(u)^n du$$

By changing parameters:

$$u(k+1) = w$$

$$\therefore u = \frac{w}{k+1}$$

$$\therefore du = \frac{dw}{k+1}$$

$$2\sum_{k=0}^{\infty}(-2)^k\,(-1)^n\int_0^{\infty}e^{-u(k+1)}(u)^n du = 2\sum_{k=0}^{\infty}(-2)^k\,(-1)^n\int_0^{\infty}e^{-w}\left(\frac{w}{k+1}\right)^n\frac{dw}{k+1}$$

$$= 2\sum_{k=0}^{\infty}\frac{(-2)^k(-1)^n}{(k+1)^{n+1}}\int_0^{\infty}e^{-w}(w)^n dw = 2\sum_{k=0}^{\infty}\frac{(-2)^k(-1)^n}{(k+1)^{n+1}}\Gamma(n+1)$$

$$= 2\sum_{k=0}^{\infty}\frac{(-2)^k(-1)^n}{(k+1)^{n+1}}\Gamma(n+1) = 2(-1)^n n!\sum_{k=0}^{\infty}\frac{(-2)^k}{(k+1)^{n+1}} = (-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{(-2)^k}{(k)^{n+1}}$$

Note the following for simplification:

$$\sum_{k=1}^{\infty}\frac{x^k}{k^2} = Li_2(x)$$

$$= (-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{(-2)^k}{(k)^{n+1}} = (-1)^{n+1}n!\,Li_{n+1}(-2)$$

Hence:

$$\theta_a = (-1)^{n+1}n!\,Li_{n+1}(-2)$$

Solving θ_b:

$$\theta_b = \int_0^1\frac{ln^n(y)}{(y+1)}dy = \sum_{k=0}^{\infty}(-1)^k\int_0^1 ln^n(y)y^k dy = \sum_{k=0}^{\infty}(-1)^k\left[\frac{(-1)^n n!}{(k+1)^{n+1}}\right]$$

$$= (-1)^n n!\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{n+1}} = (-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{(-1)^k}{(k)^{n+1}} = (-1)^{n+1}n!\,Li_{n+1}(-1)$$

Hence:

$$\theta_b = (-1)^{n+1}n!\,Li_{n+1}(-1)$$

Therefore:

$$\theta = \theta_a - \theta_b$$

$$= (-1)^{n+1}n!\,Li_{n+1}(-2) - \left((-1)^{n+1}n!\,Li_{n+1}(-1)\right)$$

$$= (-1)^{n+1}n!\,[Li_{n+1}(-2) - Li_{n+1}(-1)]$$

Solving for φ:

$$\varphi =$$

$$(-1)^n \int_0^1 \frac{ln^n(y)}{(y+1)(y+2)} dy$$

$$= (-1)^n \left[\int_0^1 \frac{ln^n(y)}{(y+1)} dy - \int_0^1 \frac{ln^n(y)}{(y+2)} dy \right] = (-1)^n \left[\theta_b - \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{-1}{2} \right)^k \int_0^1 ln^n(y) y^k dy \right]$$

$$= (-1)^n \left[\theta_b - \frac{1}{2} \sum_{k=0}^{\infty} (\frac{-1}{2})^k \left[\frac{(-1)^n n!}{(k+1)^{n+1}} \right] \right]$$

$$= (-1)^n \left[\theta_b - \frac{(-1)^n n!}{2} \sum_{k=0}^{\infty} \left[\frac{(\frac{-1}{2})^k}{(k+1)^{n+1}} \right] \right]$$

$$= (-1)^n \left[\theta_b - \{ -(-1)^n n! \sum_{k=0}^{\infty} \left[\frac{(\frac{-1}{2})^{k+1}}{(k+1)^{n+1}} \right] \} \right]$$

$$= (-1)^n \left[\theta_b - \{ (-1)^{n+1} n! \sum_{k=1}^{\infty} \left[\frac{(\frac{-1}{2})^k}{(k+1)^{n+1}} \right] \} \right]$$

$$= (-1)^n \left[\theta_b - \left\{ (-1)^{n+1} n!\, Li_{n+1} \left(-\frac{1}{2} \right) \right\} \right]$$

$$= (-1)^n \left[(-1)^{n+1} n!\, Li_{n+1}(-1) - \left\{ (-1)^{n+1} n!\, Li_{n+1} \left(-\frac{1}{2} \right) \right\} \right]$$

$$= (-1)^n (-1)^{n+1} n!\, (Li_{n+1}(-1) - Li_{n+1}(-\frac{1}{2}))$$

Therefore:

$$\varphi = (-1)^n (-1)^{n+1} n!\, (Li_{n+1}(-1) - Li_{n+1}(-\frac{1}{2}))$$

Since:

$$\int_0^1 \frac{ln^n(\frac{x}{1-x})}{1+x} dx = \theta + \varphi$$

$$= (-1)^{n+1} n!\, [Li_{n+1}(-2) - Li_{n+1}(-1)] + (-1)^n (-1)^{n+1} n!\, (Li_{n+1}(-1) - Li_{n+1}(-\frac{1}{2}))$$

Therefore, by applying the fractional transformation to the following integral, we obtained a general solution of combinations of special Liouville functions in [4] Problem Set 4:

$$n!\, [(-1)^{n+1}(Li_{n+1}(-2) - Li_{n+1}(-1)) + (-Li_{n+1}(-1) + Li_{n+1}(-1/2)]$$

➢ *Results*

The final solution is given in terms of special Liouville Functions ($Li_n(x)$), which describe the properties of the integral with respect to complex series expansions.

[4:2] Analysis of Problem Set 3: Parameters $n \in \mathbb{Z}$

Conditions for values n:

Let $n = 1$:

$$\int_0^1 \frac{ln^1(\frac{x}{1-x})}{1+x}dx = 1!\,[(-1)^2(Li_2(-2) - Li_2(-1)) + (-Li_2(-1) + Li_2(-1/2)] = Li_2(-2) + \frac{\pi^2}{6} + Li_2(-\frac{1}{2})$$

Let $n = 2$:

$$\int_0^1 \frac{ln^2(\frac{x}{1-x})}{1+x}dx = 2!\,[(-1)^3(Li_3(-2) - Li_3(-1)) + (-Li_3(-1) + Li_3(-1/2)] = -2Li_3(-2) + 2Li_3(-\frac{1}{2})$$

Let $n = 3$:

$$\int_0^1 \frac{ln^3(\frac{x}{1-x})}{1+x}dx = 3!\,[(-1)^4(Li_4(-2) - Li_4(-1)) + (-Li_4(-1) + Li_4(-1/2)] = 6Li_4(-2) - 12Li_4(-1) + 6Li_4(-\frac{1}{2})$$

The following method can be easily calculated for $n \in \mathrm{Z}$

[5:1] Problem Set 5: Logarithmic Integrals with Polynomial Growth

➢ *Objective*
To derive solutions for:

$$\int_0^1 \frac{x^k ln^n(x)}{x^w+1}dx, \quad where\; n, k, w \in \mathbb{Z}$$

➢ *Approach*
Using substitution and series expansion techniques, particularly focusing on the binomial expansion for logarithmic terms, the integral is expressed in terms of hypergeometric series:

- *Consider the Following:*

$$\int_0^1 \frac{x^k ln^n(x)}{x^w+1}dx$$

$$\int_0^1 \frac{x^k ln^n(x)}{x^w+1}dx = \sum_{j=0}^{\infty}(-1)^j \int_0^1 x^k x^{wj} ln^n(x)\,dx$$

Let:

$$\ln(x) = -y$$

$$\therefore x = e^{-y}$$

$$\therefore dx = -e^{-y}dy$$

$$\sum_{j=0}^{\infty}(-1)^j \int_0^1 x^k x^{wj} ln^n(x)\,dx = \sum_{j=0}^{\infty}(-1)^j \int_0^{\infty} (e^{-y})^k (e^{-y})^{wj} (-y)^n\,(-e^{-y})\,dy$$

$$= \sum_{j=0}^{\infty}(-1)^j \int_0^{\infty} e^{-yk-ywj-y}(-1)^n(y)^n\,dy = \sum_{j=0}^{\infty}(-1)^j(-1)^n \int_0^{\infty} y^n e^{-y(k+wj+1)}\,dy$$

Let:

$$y(k + wj + 1) = m$$

$$\therefore y = \frac{m}{k + wj + 1}$$

$$\therefore dy = \frac{dm}{k + wj + 1}$$

$$= \sum_{j=0}^{\infty} (-1)^j (-1)^n \int_0^{\infty} y^n e^{-y(k+wj+1)}\, dy$$

$$= \sum_{j=0}^{\infty} (-1)^j (-1)^n \int_0^{\infty} \left(\frac{m}{k + wj + 1}\right)^n e^{-m} \left(\frac{dm}{k + wj + 1}\right)$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j (-1)^n}{(k + wj + 1)^{n+1}} \int_0^{\infty} m^n e^{-m}\, dm = \sum_{j=0}^{\infty} \frac{(-1)^j (-1)^n}{(k + wj + 1)^{n+1}} \Gamma(n + 1)$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j (-1)^n}{(k + wj + 1)^{n+1}} n!$$

$$= (-1)^n n! \sum_{j=0}^{\infty} \frac{(-1)^j}{(k + wj + 1)^{n+1}} = (-1)^n n! \sum_{j=0}^{\infty} \frac{(-1)^j}{\left[w\left(j + \left(\frac{k+1}{w}\right)\right)\right]^{n+1}} = \frac{(-1)^n n!}{w^{n+1}} \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(j + \left(\frac{k+1}{w}\right)\right)^{n+1}}$$

We know:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^m} = \left[\frac{(-1)^m}{2^m \Gamma(m)}\right] \left[\Psi^{(m-1)}\left(\frac{a}{2}\right) - \Psi^{(m-1)}\left(\frac{a+1}{2}\right)\right]$$

Where:

$$m = n + 1$$

$$a = \frac{k+1}{w}$$

$$= \frac{(-1)^n n!}{w^{n+1}} \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(j + \left(\frac{k+1}{w}\right)\right)^{n+1}}$$

$$= \frac{(-1)^n n!}{w^{n+1}} \left[\frac{(-1)^{n+1}}{2^{n+1} \Gamma(n + 1)}\right] \left(\Psi^{(n)}\left(\frac{k+1}{2w}\right) - \Psi^{(n)}\left(\frac{k+w+1}{2w}\right)\right)$$

Therefore, the general expression involves hypergeometric functions combined with polynomial and logarithmic growth for [5] Problem Set 5:

$$\left[\frac{-1}{(2w)^{n+1}}\right] \left(\Psi^{(n)}\left(\frac{k+1}{2w}\right) - \Psi^{(n)}\left(\frac{k+w+1}{2w}\right)\right) \; for\ n, k, w \in \mathbb{Z}.$$

➢ *Result*

The resulting expression involves hypergeometric functions that account for the combined polynomial and logarithmic growth, providing a generalized form that adapts to various values n, k, and w.

[5:2] Analysis of Problem Set 5: Parameters $n, k, w \in \mathbb{Z}$

Conditions for values n, k, w:

Let $n = 1, k = w = 1$:

$$\int_0^1 \frac{x^1 ln^1(x)}{x^1+1} dx = \left[\frac{-1}{(2)^2}\right]\left(\Psi^{(1)}\left(\frac{2}{2}\right) - \Psi^{(1)}\left(\frac{3}{2}\right)\right) = \left[\frac{-1}{4}\right]\left(\Psi^{(1)}(1) - \Psi^{(1)}\left(\frac{3}{2}\right)\right)$$

Let $n = 2, k = w = 1$:

$$\int_0^1 \frac{x^1 ln^2(x)}{x^1+1} dx = \left[\frac{-1}{(2)^3}\right]\left(\Psi^{(2)}\left(\frac{2}{2}\right) - \Psi^{(2)}\left(\frac{3}{2}\right)\right) = \left[\frac{-1}{8}\right]\left(\Psi^{(2)}(1) - \Psi^{(2)}\left(\frac{3}{2}\right)\right)$$

Let $n = 3, k = w = 1$:

$$\int_0^1 \frac{x^1 ln^3(x)}{x^1+1} dx = \left[\frac{-1}{(2)^4}\right]\left(\Psi^{(3)}\left(\frac{2}{2}\right) - \Psi^{(3)}\left(\frac{3}{2}\right)\right) = \left[\frac{-1}{16}\right]\left(\Psi^{(3)}(1) - \Psi^{(3)}\left(\frac{3}{2}\right)\right)$$

The following method can be easily calculated for $n, k, w \in \mathbb{Z}$

[6:1] Problem Set 6: Integrals Involving Hyperbolic Inverses

➢ *Objective*

To compute integrals of the form:

$$\int_0^\infty \text{arcsinh}(csch(mx))\, dx, \quad where\ m \in \mathbb{Z}$$

➢ *Approach*

By expressing the inverse hyperbolic sine in logarithmic form and using substitutions involving exponential functions, the integral is simplified:

- *Consider*:

$$\int_0^\infty \text{arcsinh}(csch(mx))\, dx$$

By expressing the sine function in logarithmic form:

$$\text{arcsin}(x) = \log(x + \sqrt{1+x^2})$$

$$\text{csch}(mx) = \frac{2}{e^{mx} - e^{-mx}}$$

Hence:

$$\int_0^\infty \text{arcsinh}(csch(mx))\, dx = \int_0^\infty \log\left(\frac{2}{e^{mx}-e^{-mx}} + \sqrt{1 + \left(\frac{2}{e^{mx}-e^{-mx}}\right)^2}\right) dx$$

$$= \int_0^\infty \log\left(\frac{2}{e^{mx}-e^{-mx}} + \sqrt{1 + \frac{4}{e^{2mx} - 2 + e^{-2mx}}}\right) dx$$

$$= \int_0^\infty \log(\frac{2}{e^{mx} - e^{-mx}} + \sqrt{\frac{e^{2mx} + e^{-2mx} + 2}{e^{2mx} + e^{-2mx} - 2}})\, dx$$

$$= \int_0^\infty \log(\frac{2}{e^{mx} - e^{-mx}} + \sqrt{\frac{(e^{mx} + e^{-mx})^2}{(e^{mx} - e^{-mx})^2}})\, dx = \int_0^\infty \log\left(\frac{2}{e^{mx} - e^{-mx}} + \frac{e^{mx} + e^{-mx}}{e^{mx} - e^{-mx}}\right) dx$$

$$= \int_0^\infty \log(\frac{2 + e^{mx} + e^{-mx}}{e^{mx} - e^{-mx}})\ dx$$

$$\int_0^\infty \log\left(\frac{2 + e^{mx} + e^{-mx}}{e^{mx} - e^{-mx}}\right) dx = \int_0^\infty \log\left(\frac{e^{-mx}(2 + e^{mx} + e^{-mx})}{e^{-mx}(e^{mx} - e^{-mx})}\right) dx$$

$$= \int_0^\infty \log(\frac{2e^{-mx} + 1 + e^{-2mx}}{1 - e^{-2mx}})\ dx$$

Let:

$$u = e^{mx}$$

$$\therefore mx = \ln(u)$$

$$\therefore dx = \frac{du}{mu}$$

$$\int_0^\infty \log\left(\frac{2e^{-mx} + 1 + e^{-2mx}}{1 - e^{-2mx}}\right) dx = \int_0^1 \frac{1}{mu} \log\left(\frac{2u + 1 + u^2}{1 - u^2}\right) du$$

$$= \int_0^1 \frac{1}{mu} \log\left(\frac{(1+u)}{(1-u)}\right) du$$

$$= \int_0^1 \frac{1}{mu} \log(1+u)\, du - \int_0^1 \frac{1}{mu} \log(1-u) du$$

Consider the following identities:

$$\log(1+u) = -\sum_{k=1}^{\infty} \frac{(-1)^k u^k}{k}$$

$$\log(1-u) = -\sum_{k=1}^{\infty} \frac{u^k}{k}$$

Hence:

$$= \int_0^1 \frac{1}{mu} \log(1+u)\, du - \int_0^1 \frac{1}{mu} \log(1-u)\, du$$

$$= -\frac{1}{m}\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_0^1 u^{k-1}\, du + \frac{1}{m}\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 u^{k-1}\, du = -\frac{1}{m}\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{u^k}{k} |_0^1 + \frac{1}{m}\sum_{k=1}^{\infty} \frac{1}{k} \frac{u^k}{k} |_0^1$$

$$= -\frac{1}{m}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} + \frac{1}{m}\sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{1}{m}\left(\frac{-\pi^2}{6}\right) + \frac{1}{m}\frac{\pi^2}{12} = \frac{\pi^2}{4m}$$

Therefore, through algebraic identities from hyperbolic functions to exponential functions, the following can be expressed in terms of π for [6] Problem Set 6:

$$\int_0^\infty \text{arcsinh}(csch(mx))\,dx = \frac{\pi^2}{4m}$$

➢ *Results*

The general solution is to be found to be $\frac{\pi}{4m}$, reflecting the oscillatory behavior of the hyperbolic and inverse functions for arbitrary integer values of m.

[6:2] Continuation for all m arbitrary values, where $m \in \mathbb{Z}$

➢ *The Following Data was Compiled Through Wolfram Alpha:*

Let $m = 1$:

$$\int_0^\infty \text{arcsinh}(csch(x))\,dx = \frac{\pi^2}{4}$$

Let $m = 2$:

$$\int_0^\infty \text{arcsinh}(csch(2x))\,dx = \frac{\pi^2}{8}$$

Let $m = 3$:

$$\int_0^\infty \text{arcsinh}(csch(3x))\,dx = \frac{\pi^2}{12}$$

Let $m = 4$:

$$\int_0^\infty \text{arcsinh}(csch(4x))\,dx = \frac{\pi^2}{16}$$

...The following method can be easily calculated for $m \in \mathbb{Z}$

IV. DISCUSSION AND CONCLUSION

This work addresses complex integrals of functions that combine logarithmic, trigonometric, exponential, and hyperbolic elements, each parameterized by arbitrary integer values. The derived solutions leverage advanced mathematical tools, including transformations and identities associated with Beta, Gamma, poly-gamma, and Zeta functions. The successful application of these tools in solving integrals across different problem sets illustrates their versatility and highlights the structural relationships among special functions.

A. Discussion

The problem sets demonstrate specific techniques for handling integrals with unique structural forms:

➢ *Transformations and Substitutions*

Several problem sets employ variable substitutions to reframe integrals in terms of simpler, more manageable forms. For example, Problem Set 1 uses a trigonometric substitution to express the integral in terms of Beta and Gamma functions, illustrating how transformations simplify otherwise intricate expressions.

➢ *Special Functions and Identities*

The analysis applies identities from poly-gamma and Zeta functions in solving logarithmic and exponential integrals, as shown in Problem Set 2. This approach emphasizes the power of special functions in capturing the behavior of complex integrals across various domains.

➢ *Series Expansions and Approximations*

In certain cases, series expansions such as the binomial and Fourier series were used to approximate or simplify the expressions. Problem Set 5, which involves polynomial growth and logarithmic terms, shows how expansions can represent complex growth behaviors in compact forms.

These methods not only simplify the evaluation of challenging integrals but also provide insights into the broader mathematical relations among special functions, contributing to a deeper understanding of their computational and analytical properties.

B. Conclusion

The results obtained in this analysis illustrate how combining various mathematical techniques can yield closed-form or approximate solutions for integrals involving logarithmic, trigonometric, and hyperbolic terms. These solutions offer a framework for analyzing integrals with similar functional structures and could serve as a foundational tool in fields requiring precise integration of complex functions, such as theoretical physics, engineering, and computational mathematics. By providing general solutions for a wide range of integer values, this work contributes to the growing catalog of special function solutions and underscores the importance of symbolic computation and functional analysis in addressing advanced mathematical challenges. Future research could further explore the application of these techniques to other classes of integrals or examine the numerical stability of these solutions when applied in computational contexts.

REFERENCES

[1]. In compiling this work, standard mathematical references and computational tools were crucial in providing accurate derivations and validating results. Key resources include texts on special functions, integral transformations, and symbolic computation, as well as reputable mathematical software. The following references were instrumental in this analysis:

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These references collectively supported the computational and theoretical aspects of the problem sets, enabling accurate derivation and analysis of complex functions with arbitrary parameters.