A Proof of some Properties for the Matrices

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Abstract

To study of root of matrix it is necessary to know about properties and some results of matrix. We try to prove such known and unknown results and analyze stability property and convergence of proof.

Keywords:
a) $M_3(R)$ is a set of square matrices of order 3 with all real entries.
b) $B = A^{1/p}$ is matrix $A^{1/p}$ root of A.
c) t = Trace of matrix B. Whereas $tr(A)$ is trace of matrix A.
d) Δ = Determinant of B. Whereas $|A| = \det(A)$ is determinant of A.
e) $adj(B)$ = Adjoint of matrix B.
f) $\propto = tr(adj(B))$ = Trace of adjoint of matrix B.
g) I_n is identity matrix of order n. Though we use I as identity matrix of order 3.

I. INTRODUCTION

A be any square matrix of order n with real or complex entries. In this paper we observe proof of some properties of matrix. Also verify such known and unknown results for analyze stability property and convergence of proof.s.

➢ *Some Results Defined and Proved:*

- If $A, B \in M_n(R))$ then $tr(A) + tr(B) = tr(A + B)$.
- If$A \in M_n(R))$ then, $A * adj(A) = adj(A) * A = |A| * I_n$.
- If$A, B \in M_n(R))$ and $|A| \neq 0$ then $adj(A * B) = adj(B) * adj(A)$.
- If$A, B \in M_n(R))$ then $\det(A * B) = \det(A) * \det(B)$.
- If$A \in M_n(R))$ then, $(adj(A))^n = adj(A^n)$.
- If$A \in M_n(R))$, then $adj\big(adj(A)\big) = |adj(A)| * \big(adj(A)\big)^{-1}$.
- If$A \in M_n(R))$ then $tr\big(adj(k * A)\big) = k * tr\big(adj(A)\big)$.

II. PROOF OF RESULTS

➢ *Result:*
If $A, B \in M_n(R))$ then $tr(A) + tr(B) = tr(A + B)$.

- *Proof:*

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ a_{21} & a_{22}\ldots\ldots & a_{2n} \\ a_{n1} & a_{n2}\ldots\ldots & a_{nn} \end{pmatrix} \quad \& \quad B = \begin{pmatrix} b_{11} & b_{12}\ldots\ldots & b_{1n} \\ b_{21} & b_{22}\ldots\ldots & b_{2n} \\ b_{n1} & b_{n2}\ldots\ldots & b_{nn} \end{pmatrix}$$

Therefore,

$$tr(A) + tr(B) = (a_{11} + a_{22} + \ \ldots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}\ )$$

Also,

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \ldots\ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} \ldots\ldots & a_{2n} + b_{2n} \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} \ldots\ldots & a_{nn} + b_{nn} \end{pmatrix}$$

$$tr(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \ldots + (a_{nn} + b_{nn})$$

$$= (a_{11} + a_{22} + \ldots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn})$$

$$= tr(A) + tr(B)$$

$tr(A) + tr(B) = tr(A + B)$ ……………………………………………………………………………...…………(1)

- *Result :*

$$If\ A \in M_n(R) then, A * adj(A) = |A| * I_n = adj(A) * A$$

- *Proof :*

$$Let\ A = \begin{pmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{pmatrix} \ \&\ adj(A) = \begin{pmatrix} A_{11} & A_{12} \ldots\ldots & A_{1n} \\ A_{21} & A_{22} \ldots\ldots & A_{2n} \\ A_{n1} & A_{n2} \ldots\ldots & A_{nn} \end{pmatrix}$$

$(i, j)^{th}$ entry of $A * adj(A)$

$$= (a_{1i} * A_{1j} + a_{i2} * A_{2j} + \cdots + a_{in} * A_{nj})$$

= 0 If $i \neq j$

$= |A|$ If $i = j$

Therefore,

$$A * adj(A) = \begin{bmatrix} |A| & 0 & \ldots & 0 \\ 0 & |A| & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & |A| \end{bmatrix}$$

$$= |A| * I_n.$$

Similarly,

$$adj(A) * A = |A| * I_n.$$

Therefore,

$A * adj(A) = adj(A) * A = |A| * I_n.$ ……………………………………………………………………………...….(2)

- *Result:*

If $A, B \in M_n(R)$, and $|A * B| \neq 0$ then,

$$adj(A * B) = adj(B) * adj(A).$$

- *Proof:*

We have

$$(A * B)^{-1} = B^{-1} * A^{-1}.$$

As we know $(A)^{-1} = \frac{1}{|A|} * adj(A)$.

=> $(A * B)^{-1} = \frac{1}{|A*B|} * adj(A * B)$. ……………………………(I)

Also

$$B^{-1} * A^{-1} = \left(\frac{1}{|B|} * adj(B)\right) * \left(\frac{1}{|A|} * adj(A)\right).$$

$= \frac{1}{|A|*|B|} * adj(B) * adj(A)$. ……………………………………………….….(II)

By (I) & (II) we have,

$$\frac{1}{|A * B|} * adj(A * B) = \frac{1}{|A| * |B|} * adj(B) * adj(A).$$

Therefore,

$$\frac{1}{|A| * |B|} * adj(A * B) = \frac{1}{|A| * |B|} * adj(B) * adj(A).$$

$adj(A * B) = adj(B) * adj(A)$. ……………………………………………………………………………………(3)

- *Result:*

If $A, B \in M_n(R)$, then $det(A * B) = det(A) * det(B)$.

That is

$$|A * B| = |A| * |B|.$$

- *Proof :*

Consider, $det(A * B) * I_n$.

$det(A * B) * I_n = (A * B) * adj(A * B)$. (by Result 2)

$= (A * B) * adj(B) * adj(A). (by\ Result\ 3)$

$= A * \big(B * adj(B)\big) * (adj(A))$.

$= A * det\ (B) * adj(A)$

$= det\ (B) * (A * adj(A))$

$= det\ (B) * det\ (A) * I_n$.

Therefore $det\ (A * B) = det(A) * det(B)$. ……………………………………………………………….,……. (4)

That is

$$|A * B| = |A| * |B|.$$

- *Result:*

If $A \in M_n(R)$ then, $(adj(A))^n = adj((A)^n)$.

- *Proof:*
  Proof by Mathematical Induction

✓ Step 1:- Prove that result is true for $n = 1$.

$$(adj(A))^1 = adj((A)^1)$$

✓ Step 2:- Assume that result is true for $n = k$, $\forall\ k \in N$.

$$\left(adj(A)\right)^k = adj((A)^k)$$

✓ Step 3 :- prove that the result is true for n = k+1,

i.e $(adj(A))^{k+1} = adj((A)^{k+1})$

Consider $(adj(A))^{k+1}$

$= (adj(A))^k * adj(A)$ (By Result 2.14)

$= adj(A * A^k)$

$= adj(A)^{k+1}$

Therefore $(adj(A))^n = adj((A)^n)$, ……………………………………………………………………………………….(5)

Hence result is true for $n = k + 1$.

Therefore by Mathematical Induction result is true for all $n \in N$.

➢ *Result:*
  If $A \in M_n(R)$, then $adj\left(adj(A)\right) = |adj(A)| * adj(A^{-1})$.

- *Proof:*

Consider, $(adj(A))^{-1} = \frac{1}{|adj(A)|} * adj\left(adj(A)\right)$.

Therefore

$adj\left(adj(A)\right) = |adj(A)| * (adj(A)^{-1})$. ...……………………………………………………..…………………………… (6)

➢ *Result:*
  If $A \in M_n(R)$, then $tr\left(adj(k * A)\right) = k * tr(adj(A))$.

- *Proof:*

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ a_{21} & a_{22}\ldots\ldots & a_{2n} \\ a_{n1} & a_{n2}\ldots\ldots & a_{nn} \end{pmatrix} \ \&\ adj(A) = \begin{pmatrix} A_{11} & A_{12}\ldots\ldots & A_{1n} \\ A_{21} & A_{22}\ldots\ldots & A_{2n} \\ A_{n1} & A_{n2}\ldots\ldots & A_{nn} \end{pmatrix}$$

Consider $tr(adj(k * A))$

$= (k * A_{11}) + (k * A_{22}) + (k * A_{33}) + \cdots + (k * A_{nn})$.

$= k * (A_{11} + A_{22} + A_{33} + \cdots A_{nn})$.

$= k * tr(adj(A))$.

Therefore

$tr\big(adj(k * A)\big) = k *$
$tr(adj(A))$……... (7)

III. SOME ADDITIONAL RESULTS

- *Lemma 1:*
  Let $A, B \in M_3(R)$. If $A = B^2$, then $A = B^2 = tr(B) * B + adj(B) - tr\big(adj(B)\big) * I$.

$A = B^2 = t * B + adj(B) - \propto * I$. and $tr(A) = tr(B^2) = t^2 - 2 * \propto$.

- *Proof:*

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \; and \; adj(B) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

$$\text{Where } B_{ij} = (-1)^{i+j} * \big(b_{ji} * b_{kk} - b_{jk} * b_{ki}\big). for\ all\ i, j, k \ \in \{1, 2, 3\}.$$

$$\text{Here } A = B^2$$

Therefore,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{pmatrix} b^2{}_{11} + b_{12}b_{21} + b_{13}b_{31} & b_{12}(b_{11} + b_{22}) + b_{13}b_{32} & b_{13}(b_{11} + b_{33}) + b_{12}b_{23} \\ b_{21}(b_{22} + b_{11}) + b_{23}b_{31} & b^2{}_{22} + b_{21}b_{12} + b_{23}b_{32} & b_{23}(b_{22} + b_{33}) + b_{21}b_{13} \\ b_{31}(b_{33} + b_{11}) + b_{32}b_{21} & b_{32}(b_{33} + b_{22}) + b_{31}b_{12} & b^2{}_{33} + b_{31}b_{13} + b_{32}b_{23} \end{pmatrix}.$$

We have for i $\neq$ j,

$$a_{ij} = b_{ij} * \big(b_{ii} + b_{jj}\big) + b_{ik} * b_{kj}.$$

$$= b_{ij} * \big(b_{ii} + b_{jj} + b_{kk} - b_{kk}\big) + b_{ik} * b_{kj}.$$

$$= b_{ij} * tr(B) - \big(b_{ij} * b_{kk} - b_{ik} * b_{kj}\big).$$

$$= b_{ij} * tr(B) - B_{ji}.$$

$$for\ i = 1, 2, 3\ and\ i \neq j, i \neq k..$$

$$a_{ii} = b_{ii}^2 + \big(b_{ij} * b_{ji}\big) + b_{ik} * b_{ki}.$$

$$= b_{ii} * \big(b_{ii} + b_{jj} + b_{kk}\big) - b_{ii} * b_{jj} + b_{ij} * b_{ji} - b_{ii} * b_{kk} + b_{ik} * b_{ki}.$$

$$= b_{ii} * tr(B) - B_{kk} - B_{jj}.$$

$$= b_{ii} * tr(B) + B_{ii} - \big(B_{ii} + B_{jj} + B_{kk}\big).$$

$$= b_{ii} * tr(B) + B_{ii} - tr\big(adj(B)\big).$$

Therefore,

$A = B^2 = tr(B) * B + adj(B) - tr\big(adj(B)\big) * I.$ ……………………………………………………………………..(8)

$A = B^2 = t * B + adj(B) - \propto * I$ ..…………………………………………………………………(9)

Where

$$t = tr(B), \propto = tr\big(adj(B)\big).$$

Therefore,

$$tr(B^2) = tr(t * B + adj(B) - \propto * I).$$

$$= t * tr(B) + tr\big(adj(B)\big) - 3 * \propto.$$

$$= t^2 + \propto - 3 * \propto.$$

$$tr(B^2) = t^2 - 2 * \propto.$$

$tr(B^2) = tr(A) = t^2 - 2 * \propto$.………………………………………………………………………………………...… (10)

- *Lemma 2:*

Let $A \in M_3(R)$, then characteristic equation of A is given by,

$$\det(\lambda * I - A) = \lambda^3 - tr(A) * \lambda^2 + tr\big(adj(A)\big) * \lambda - \det(A).$$

And

$$\det(\lambda * I + A) = \lambda^3 + tr(A) * \lambda^2 + tr\big(adj(A)\big) * \lambda + \det(A).$$

- *Proof:*

The characteristic equation of A is given by

$$\det(\lambda * I - A) = \det\left(\begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{bmatrix}\right).$$

$$= (\lambda - a_{11}) * [(\lambda - a_{22}) * (\lambda - a_{33}) - a_{23} * a_{32}].$$

$$+ a_{12} * [-a_{21} * (\lambda - a_{33}) - (a_{31} * a_{23})]$$

$$- a_{13} * [a_{21} * a_{32} + a_{31} * (\lambda - a_{22})].$$

$$= \lambda^3 - (a_{11} * a_{22} + a_{33}) * \lambda^2$$

$$+ [(a_{11} * a_{22} + a_{12} * a_{21}) + (a_{11} * a_{33} + a_{13} * a_{31}) + (a_{22} * a_{33} + a_{32} * a_{23})] * \lambda + (a_{11} * A_{11} + a_{12} * A_{12} + a_{13} * A_{13}).$$

$$= \lambda^3 - tr(A) * \lambda^2 + (A_{11} + A_{22} + A_{33}) * \lambda - \det(A).$$

$$= \lambda^3 - tr(A) * \lambda^2 + tr(adj(A)) * \lambda - \det(A).$$

Therefore,

$\det(\lambda * I - A) = \lambda^3 - tr(A) * \lambda^2 + tr\big(adj(A)\big) * \lambda - \det(A)$. ……………………………………………………....…….(11)

Similarly,

$\det(\lambda * I + A) = \lambda^3 + tr(A) * \lambda^2 + tr\big(adj(A)\big) * \lambda + \det(A)$. ……………………………………………………….......(12)

- *Remark:*

Every Matrix$B \in M_3(R)$satisfies it's characteristic equation.

$\lambda^3 - tr(B) * \lambda^2 + tr\big(adj(B)\big) * \lambda - \det(B) = 0$. …………………………………………………………………………(13)

Consider 4.1 eq^n.1,

$$B^2 = tr(B) * B + adj(B) - tr\big(adj(B)\big) * I.$$

Multiply everywhere by B.

$$B^3 = tr(B) * B^2 + adj(B) * B - tr\big(adj(B)\big) * B.$$

$$B^3 - tr(B) * B^2 + tr(adj(B)) * B - (\det(B)) * I = 0 \quad \ldots\ldots\ldots\ldots..(14)$$

Equation in λ (6) and equation in B (6') are similar.

Thus B satisfies characteristic equation.

IV. CONCLUSION AND OPEN PROBLEMS

Variety of formulas for expressing properties of A in varies forms. They help to study of matrix. Formulae help to prove characterization theorem, that every matrix satisfy it's characteristic equation and reduce the computation of $A^{1/p}$ to numerical integration on the unit circle, to computing the matrix sign function of a block companion matrix, to inverting a matrix Laurent polynomial, to computing a Wiener–Hopf factorization, and to applying a fixed point iteration.

REFERENCES

[1]. P. Benner, R. Byers, V. Mehrmann and H. Xu, A unified deflating subspace approach for classes of polynomial and rational matrix equations, Preprint SFB393/00-05, ZentrumfürTechnomathematik, Universität Bremen, Bremen, Germany (January 2000).

[2]. M.A. Hasan, J.A.K. Hasan and L. Scharenroich, New integral representations and algorithms for computing nth roots and the matrix sector function of nonsingular complex matrices, in: Proc. of the 39th IEEE Conf. on Decision and Control, Sydney, Australia (2000) pp. 4247–4252.

[3]. N.J. Higham, Newton's method for the matrix square root, Math. Comp. 46(174) (1986) 537–549.

[4]. W.D. Hoskins and D.J. Walton, A faster, more stable method for computing the pth roots of positive definite matrices, Linear Algebra Appl. 26 (1979) 139–163.

[5]. L.-S. Shieh, Y.T. Tsay and R.E. Yates, Computation of the principal nth roots of complex matrices, IEEE Trans. Automat. Control 30(6) (1985) 606–608.

[6]. M.I. Smith, ASchur algorithm for computing matrix pth roots, SIAM J. Matrix Anal. Appl. 24(4) (2003) 971–989.

[7]. J.S.H. Tsai, L.S. Shieh and R.E. Yates, Fast and stable algorithms for computing the principal nth root of a complex matrix and the matrix sector function, Comput. Math. Appl. 15(11) (1988) 903–913.

[8]. Y.T. Tsay, L.S. Shieh and J.S.H. Tsai, A fast method for computing the principal nth roots of complex matrices, Linear Algebra Appl. 76 (1986) 205–221).