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Comparison of Areas Formed from Angle Trisectors on Any Nonconvex Quadrilateral

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Abstract:- Angular trisectors were originally discussed for Morleey's theorem on the existence of equilateral triangles formed from the intersection of trisectors. This trisector has been developed by several authors on convex quadrilaterals. However, no one has developed in detail on nonconvex quadrilaterals. In this paper, the author will determine the side lengths of angle trisectors on nonconvex quadrilaterals in various cases. At the end, a comparison of the area of triangles formed from the construction of angle trisectors on nonconvex quadrilaterals is also given, which will be described in several cases.

Keywords:- *Area Comparison, Angle Trisectors, Nonconvex Quadrilaterals.*

I. INTRODUCTION

One of the discussions about flat buildings is about angle trisectors in triangles that divide each angle of a triangle into three equal parts. In [1 - 3] discussed how to divide the angle into three equal parts. The result obtained is that there are various methods to divide the angle into three equal parts. Morley's theorem is the most interesting and surprising result of the 20th century in the field of geometry because it divides every angle in an arbitrary triangle into three equal parts. In [5 - 16], states that if there is an arbitrary triangle formed by a trisector at each angle, then there are three intersections of two adjacent trisectors to form a morley triangle. The result obtained is that there is an equilateral triangle formed from the intersection of the angle trisectors. In addition to angle trisectors in triangles, [17] discussed the outer angle trisector in any triangle by dividing the outer angle in any triangle into three equal parts. The result obtained is that there is an equilateral triangle formed from the intersection of the outer angle trisector.

In addition to triangles, Morley's theorem is also developed on quadrilaterals. In [18], the development of Morley's theorem on special quadrilaterals namely square, rectangle, rhombus, kite, and isosceles trapezoid is discussed. The results obtained are Morley quadrilaterals formed from square flat shapes are square flat shapes. Morley's quadrilateral formed from a rectangular flat is a rhombus flat. Morley's quadrilateral formed from a rhombus flat is a rectangular flat. A Morley quadrilateral formed from a kite is an isosceles trapezoid. The Morley quadrilateral formed from an isosceles trapezoid is a kite.

Another discussion of angle trisectors is to determine the area ratio formed from angle trisectors. In [19], it is discussed how to determine the area ratio of the shapes formed from angle trisectors on any triangle. In addition, it also discusses the comparison of the area of the shape formed from the angle trisector on any convex quadrilateral. The results obtained are there are three triangles that are made area comparison. The discussion only discusses the area comparison formed from angle trisectors on a convex quadrilateral but not on a nonconvex quadrilateral where this nonconvex quadrilateral, there are several cases that make the area comparison different.

II. ANGLE TRISECTOR ON A CONVEX QUADRILATERAL

Some development of angular trisectors was done by [13] which is about the length of two trisector lines and the area ratio formed from any convex quadrilateral. Suppose there is an arbitrary quadrilateral ABCD, the length of the side AB is denoted by a, the length of the side BC is denoted by b, the length of the side CD is denoted by c, the length of the side DA is denoted by d. Furthermore, ∠A is denoted by α, ∠B is denoted by β, ∠C is denoted by γ, and ∠D is denoted by δ. Then if a trisector is formed from the four corners, there will be two trisector lines from each corner that divide the corner into three equal parts. In Figure 1, it can be seen that lines AA_1 and AA_2 divide angle A, lines BB_1 and BB_2 divide angle B, lines CC_1 and CC_2 divide angle C, and lines DD_1 and DD_2 divide angle D, each of these trisector lines divides each angle of the ABCD quadrilateral into three equal parts. The following theorems are given about the length of two angle trisector lines on any nonconvex quadrilateral and the area ratio formed from any convex quadrilateral.

- *Theorem 1.*

In any convex quadrilateral $ABCD$, if a trisector is formed at angle A, the length of the trisector side is

$$AA_1 = \frac{a\sin\beta}{\sin\left(\frac{\alpha}{3}+\beta\right)}$$

And

$$AA_2 = \frac{d\sin\delta}{\sin\left(\frac{\alpha}{3}+\delta\right)}$$

Proof: see [12] ■

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By using the same method in determining the side lengths of the trisectors AA_1 and AA_2, the side lengths of the trisectors at other arbitrary angles are obtained, namely

$$BB_1 = \frac{b\sin\gamma}{\sin\left(\frac{\beta}{3}+\gamma\right)},\ BB_2 = \frac{a\sin\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)},\ CC_1 = \frac{c\sin\delta}{\sin\left(\frac{\gamma}{3}+\delta\right)},$$

$$CC_2 = \frac{b\sin\beta}{\sin\left(\frac{\gamma}{3}+\beta\right)},\ DD_1 = \frac{d\sin\alpha}{\sin\left(\frac{\delta}{3}+\alpha\right)},\ \text{and } DD_2 = \frac{c\sin\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)}.$$

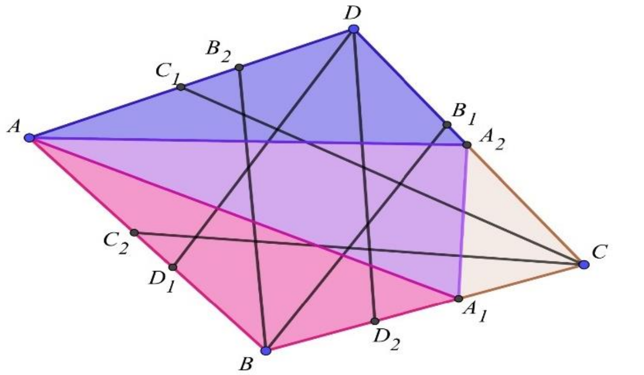

Fig 1 Trisector of Each Angle of an Arbitrary Quadrilateral $ABCD$

➢ *Teorema 2*

In an arbitrary quadrilateral $ABCD$ if a trisector is formed at $\angle A$ then the ratio of the triangle area of the trisector $L \triangle BAA_1 : L \triangle AA_1A_2 : L \triangle DAA_2$ is

$$\frac{a^2\sin\beta}{\sin\left(\frac{\alpha}{3}+\beta\right)} : \frac{ad\sin\beta\ \sin\delta}{\sin\left(\frac{\alpha}{3}+\beta\right)\sin\left(\frac{\alpha}{3}+\delta\right)} : \frac{d^2\sin\delta}{\sin\left(\frac{\alpha}{3}+\delta\right)}$$

Proof: see [12] ■

III. SIDE LENGTH NONCONVEX QUADRILATERAL TRISECTOR

Suppose there is a nonconvex quadrilateral $ABCD$, with side length $AB = a$, side length $BC = b$, side length $CD = c$, and side length $AD = d$. Suppose $\angle DAB = \alpha$, $\angle ABC = \beta$, $\angle BCD = \gamma$, and $\angle CDA = \delta$. Then if a trisector is formed from the four angles, there will be two trisector lines from each angle that divide the angle into three equal parts. In Figure 2, line AA_1 and line AA_2 divide angle DAB into three equal parts so that $\angle DAA_1 = \frac{\alpha}{3}$, $\angle A_1AA_2 = \frac{\alpha}{3}$, and $\angle A_2AB = \frac{\alpha}{3}$. Line BB_1 and line BB_2 divide angle ABC into three equal parts so that $\angle ABB_1 = \frac{\beta}{3}$, $\angle B_1BB_2 = \frac{\beta}{3}$, and $\angle B_2BC = \frac{\beta}{3}$. Line CC_1 and line CC_2 divide angle BCD into three equal parts so that, $\angle BCC_1 = \frac{\gamma}{3}$, $\angle C_1CC_2 = \frac{\gamma}{3}$, and $\angle C_2CD = \frac{\gamma}{3}$. Line DD_1 and line DD_2 divide angle CDA into three equal parts so that $\angle CDD_1 = \frac{\delta}{3}$, $\angle D_1DD_2 = \frac{\delta}{3}$, and $\angle D_2DA = \frac{\delta}{3}$. The lengths of AA_1 and AA_2 are presented in the following theorem.

➢ *Theorem 3.*

In any nonconvex quadrilateral $ABCD$ in Figure 2, if a trisector is formed at angle A, the length of the trisector side is

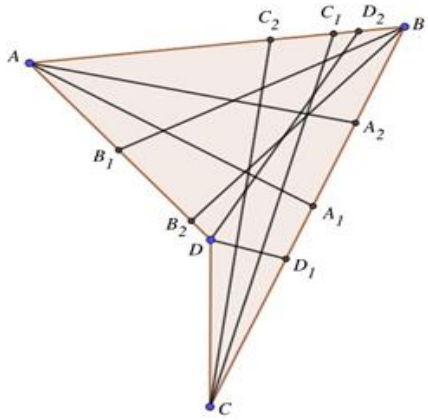

Fig 2 Trisector of Each Corner of an Arbitrary Nonconvex Quadrilateral $ABCD$

$$AA_1 = \frac{a\sin\beta}{\sin\left(\frac{2\alpha}{3}+\beta\right)}$$

and

$$AA_2 = \frac{a\sin\beta}{\sin\left(\frac{\alpha}{3}+\beta\right)}$$

- *Proof:*

The length of the trisector side of each angle of an arbitrary quadrilateral is shown using trigonometric comparisons. Consider $\triangle AA_1B$, because the sum of the angles in the triangle is 180°, it is obtained

$$\angle AA_1B = 180 - (\angle A_1AB + \angle ABA_1)$$

Then by substituting the value of $\angle A_1AB = \frac{2\alpha}{3}$ and $\angle ABA_1 = \beta$, we obtain

$$\angle AA_1B = 180 - \left(\frac{2\alpha}{3}+\beta\right)$$

By using the sine rule on $\triangle AA_1B$, we obtain

$$\frac{AB}{\sin(\angle AA_1B)} = \frac{AA_1}{\sin(\angle ABA_1)}$$

Furthermore, we get

$$AA_1 = \frac{a\sin\beta}{\sin\left(\frac{2\alpha}{3}+\beta\right)}$$

Next, the length of side AA_2 will be shown with respect to $\triangle BAA_2$, because the sum of the angles in the triangle is 180°, we obtain

$$\angle AA_2B = 180 - (\angle A_2AB + \angle ABA_2)$$

Then by substituting the value of $\angle A_2AB = \frac{\alpha}{3}$ and $\angle ABA_2 = \beta$, then

$$\angle AA2B = 180 - \left(\frac{\alpha}{3} + \beta\right)$$

By using the sine rule on $\Delta\, AA_2B$, we obtain

$$\frac{AB}{\sin(\angle AA_2B)} = \frac{AA_2}{\sin(\angle ABA_2)}$$

Furthermore, we get

$$AA_2 = \frac{a \sin \beta}{\sin\left(\frac{\alpha}{3} + \beta\right)}$$

So Theorem 3 is proven. ■

In the same way of determining the lengths of AA_1 and AA_2, the lengths of the side trisectors at angle C are $CC_1 = \frac{b \sin \beta}{\sin\left(\frac{\gamma}{3}+\beta\right)}$ and $CC_2 = \frac{b \sin \beta}{\sin\left(\frac{2\gamma}{3}+\beta\right)}$. If $\angle D$ is greater than 90°, it raises the case to determine the length of BB_1, BB_2, DD_1 and DD_2. As for AA_1, AA_2, CC_1 and CC_2, it does not cause a case. The following theorem determines the length of BB_1 and BB_2 in some cases.

➢ *Theorem 4.*

In any nonconvex quadrilateral $ABCD$ with $\angle D$ greater than 90°, if a trisector is formed at angle B, the length of the trisector side is

$$BB_1 = \frac{a \sin \alpha}{\sin\left(\frac{\beta}{3} + \alpha\right)}$$

$$BB_2 = \frac{a \sin \alpha}{\sin\left(\frac{2\beta}{3} + \alpha\right)}$$

Or

$$BB_2 = \frac{b \sin \gamma}{\sin\left(\frac{\beta}{3} + \gamma\right)}$$

- *Proof:*

In determining the length of BB_1 and BB_2, there are several cases to prove it. The following is an explanation of each case in determining the length of BB_1 and BB_2.

- *Case 1*

Suppose there is a nonconvex quadrilateral $ABCD$ with $\angle D$ greater than 90°, an angle trisector will be formed from its four angles. The two sides of the trisector of angle B are on the side of AD illustrated in Figure 2. Next, the length of the trisector side of angle B will be determined using trigonometric comparison. Consider $\Delta\, AB_1B$, because the sum of the angles in the triangle is 180°, the following is obtained

$$\angle AB_1B = 180 - (\angle ABB_1 + \angle B_1AB)$$

Then by substituting the value of $\angle ABB_1 = \frac{\beta}{3}$ and $\angle B_1AB = \alpha$, we obtain

$$\angle AB_1B = 180 - \left(\frac{\beta}{3} + \alpha\right)$$

By using the sine rule on $\Delta\, AB_1B$, we obtain

$$\frac{AB}{\sin(\angle AB_1B)} = \frac{BB_1}{\sin(\angle BAB_1)}$$

Furthermore, we get

$$BB_1 = \frac{a \sin \alpha}{\sin\left(\frac{\beta}{3} + \alpha\right)}$$

Next, the length of side BB_2 will be shown by considering $\Delta\, AB_2B$, because the sum of the angles in the triangle is 180°, we obtain

$$\angle AB_2B = 180 - (\angle ABB_2 + \angle B_2AB)$$

Then by substituting the value of $\angle ABB_2 = \frac{2\beta}{3}$ and $\angle B_2AB = \alpha$, we obtain

$$\angle AB_2B = 180 - \left(\frac{2\beta}{3} + \alpha\right)$$

By using the sine rule on $\Delta\, AB_2B$, , we obtain

$$\frac{AB}{\sin(\angle AB_2B)} = \frac{BB_2}{\sin(\angle BAB_2)}$$

Furthermore, we get

$$BB_2 = \frac{a \sin \alpha}{\sin\left(\frac{2\beta}{3} + \alpha\right)}$$

After obtaining the length of BB_1 and BB_2 in case 1, the length of BB_1 and BB_2 in case 2 will be determined. In case 2, the length of BB_1 is the same as the length of BB_1 in case 1, but the length of BB_2 is different. The following explanation is given in determining the length of BB_2 in case 2.

- *Case 2*

Suppose there is a nonconvex quadrilateral $ABCD$, an angle trisector will be formed from its four corners. The two sides of the angle trisector B are AD and CD, respectively, as illustrated in Figure 3. Next, the lengths of the sides of the angle trisector B, BB_1 and BB_2, will be determined. The length of BB_1 in case 2 is the same as the length of BB_1 in case 1 so it will be continued to determine the length of BB_2.

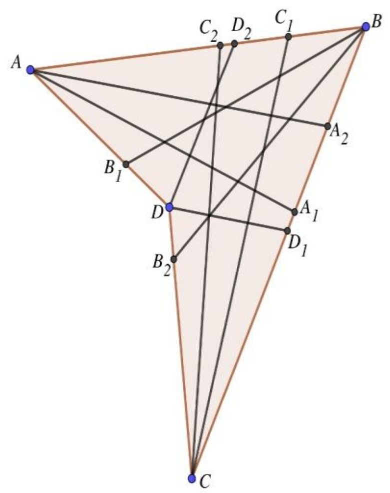

Fig 3 Trisector Angle B on the Nonconvex Quadrilateral $ABCD$ Case 2.

Consider $\triangle BCB_2$, because the sum of the angles in the triangle is 180°, the following is obtained

$$\angle BB_2C = 180 - (\angle CBB_2 + \angle B_2CB)$$

Then by substituting the value of $\angle CBB_2 = \frac{\beta}{3}$ and $\angle B_2CB = \gamma$, we obtain

$$\angle BB_2C = 180 - \left(\frac{\beta}{3} + \gamma\right)$$

By using the sine rule on $\triangle BCB_2$, we obtain

$$\frac{BC}{\sin(\angle BB_2C)} = \frac{BB_2}{\sin(\angle B_2CB)}$$

Furthermore, we get

$$BB_2 = \frac{b \sin\gamma}{\sin\left(\frac{\beta}{3} + \gamma\right)}$$

After obtaining the length of BB_1 and BB_2 in case 2, the length of BB_1 and BB_2 will be determined in case 3. In case 3, the length of BB_1 is the same as the length of BB_1 in case 1. The following explanation is given in determining the length of BB_2 in case 3.

- *Case 3*

Suppose there is a nonconvex quadrilateral $ABCD$, an angle trisector will be formed from its four corners. The two sides of the angle trisector B are AD and point D, respectively, illustrated in Figure 4. Next, the lengths of the sides of the angle trisector B will be determined, namely BB_1 and BD. The length of BB_1 in case 2 is the same as the length of BB_1 in case 1 so it will be continued to determine the length of BD.

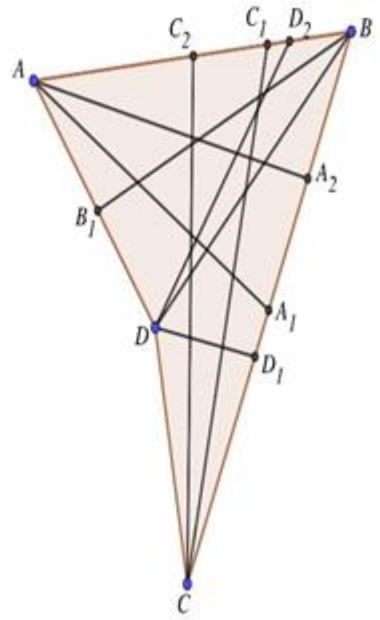

Fig 3 Trisector Angle B on the Nonconvex Quadrilateral $ABCD$ Case 3.

Consider $\triangle BDC$, because the sum of the angles in the triangle is 180°, the following is obtained

$$\angle BDC = 180 - (\angle CBD + \angle DCB)$$

Then by substituting the value of $\angle CBD = \frac{\beta}{3}$ and $\angle DCB = \gamma$, we obtain

$$\angle BDC = 180 - \left(\frac{\beta}{3} + \gamma\right)$$

By using the sine rule on $\triangle BDC$, we obtain

$$\frac{BC}{\sin(\angle BDC)} = \frac{BD}{\sin(\angle DCB)}$$

Furthermore, we get

$$BD = \frac{b \sin\gamma}{\sin\left(\frac{\beta}{3} + \gamma\right)}$$

So Theorem 4 is proven. ■

After getting the length of the side trisector at angle B, the length of the side trisector at angle D, DD_1 and DD_2, will be determined. The following theorem determines the length of DD_1 and DD_2 in some cases.

- *Theorem 5.*

In any nonconvex quadrilateral $ABCD$ with $\angle D$ greater than 90°, if a trisector is formed at angle D, the length of the trisector side is

$$DD_1 = \frac{c \sin\gamma}{\sin\left(\frac{\delta}{3} + \gamma\right)}$$

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$$DD_2 = \frac{c \sin\gamma}{\sin\left(\frac{2\delta}{3} + \gamma\right)}$$

Or

$$DD_2 = \frac{d \sin\alpha}{\sin\left(\frac{\delta}{3} + \alpha\right)}$$

- *Proof:* In determining the length of DD_1 and DD_2, there are several cases to prove it. The following is an explanation of each case in determining the length of DD_1 and DD_2.

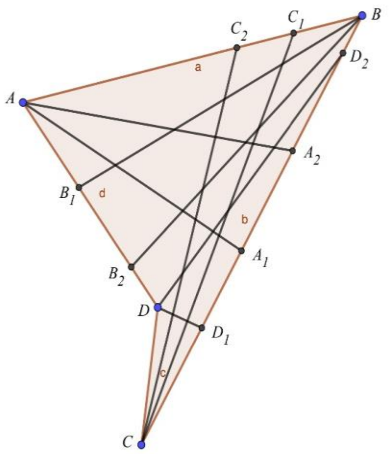

Fig 5 Trisector Angle D on the Nonconvex Quadrilateral $ABCD$ Case 1.

- *Case 1*

Suppose there is a nonconvex quadrilateral $ABCD$ with $\angle D$ greater than 90°, an angle trisector will be formed from its four angles. The two sides of the trisector of angle D are on the BC side illustrated in Figure 5. Next, the length of the trisector side of angle D will be determined using trigonometric comparison. Consider $\triangle CD_1D$, since the sum of the angles in the triangle is 180°, the following is obtained

$$\angle CD_1D = 180 - (\angle CDD_1 + \angle D_1CD)$$

Then by substituting the value of $\angle CDD_1 = \frac{\delta}{3}$ and $\angle D_1CD = \gamma$, we obtain

$$\angle CDD_1 = 180 - \left(\frac{\delta}{3} + \gamma\right)$$

By using the sine rule on $\triangle CD_1D$, we obtain

$$\frac{CD}{\sin(\angle CDD_1)} = \frac{DD_1}{\sin(\angle D_1CD)}$$

Furthermore, we get

$$DD_1 = \frac{c \sin\gamma}{\sin\left(\frac{\delta}{3} + \gamma\right)}$$

Next, the length of side DD_2 will be shown by considering $\triangle CD_2D$, because the sum of the angles in the triangle is 180°, we obtain

$$\angle CD_2D = 180 - (\angle CDD_2 + \angle D_2CD)$$

Then by substituting the value of $\angle CDD_2 = \frac{2\delta}{3}$ and $\angle D_2CD = \gamma$, we obtain

$$\angle CD_2D = 180 - \left(\frac{2\delta}{3} + \gamma\right)$$

By using the sine rule on $\triangle CD_2D$, we obtain

$$\frac{CD}{\sin(\angle CD_2D)} = \frac{DD_2}{\sin(\angle D_2CD)}$$

Furthermore, we get

$$DD_2 = \frac{c \sin\gamma}{\sin\left(\frac{2\delta}{3} + \gamma\right)}$$

After obtaining the length of DD_1 and DD_2 in case 1, the length of DD_1 and DD_2 in case 2 will be determined. In case 2, the length of DD_1 is the same as the length of DD_1 in case 1, but the length of DD_2 is different. The following explanation is given in determining the length of DD_2 in case 2.

- *Case 2*

Suppose there is a nonconvex quadrilateral $ABCD$, an angle trisector will be formed from its four corners. The two sides of the angle trisector D are respectively on the sides AB and BC illustrated in Figure 2. Next, the length of the sides of the angle trisector D will be determined, namely DD_1 and DD_2. The length of DD_1 in case 2 is the same as the length of DD_1 in case 1 so it will be continued to determine the length of DD_2. Consider $\triangle DAD_2$, because the sum of the angles in the triangle is 180° then obtained

$$\angle DD_2A = 180 - (\angle ADD_2 + \angle D_2AD)$$

Then by substituting the value of $\angle ADD_2 = \frac{\delta}{3}$ and $\angle D_2AD = \alpha$, we obtain

$$\angle DD_2A = 180 - \left(\frac{\delta}{3} + \alpha\right)$$

By using the sine rule on $\triangle DAD_2$, we obtain

$$\frac{AD}{\sin(\angle DD_2A)} = \frac{DD_2}{\sin(\angle D_2AD)}$$

Furthermore, we get

$$DD_2 = \frac{d \sin\alpha}{\sin\left(\frac{\delta}{3} + \alpha\right)}$$

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After obtaining the length of DD_1 and DD_2 in case 2, the length of DD_1 and DD_2 will be determined in case 3. In case 3, the length of DD_1 is the same as the length of DD_2 in case 1. The following explanation is given in determining the length of DD_2 in case 3.

- *Case 3*

Suppose there is a nonconvex quadrilateral $ABCD$, an angle trisector will be formed from its four corners. The two sides of the angle trisector D are respectively on the BC side and point B illustrated in Figure 6. Next, the length of the sides of the angle trisector D will be determined, namely DD_1 and DB. The length of DD_1 in case 3 is the same as the length of DD_1 in case 1 so it will be continued to determine the length of DB.

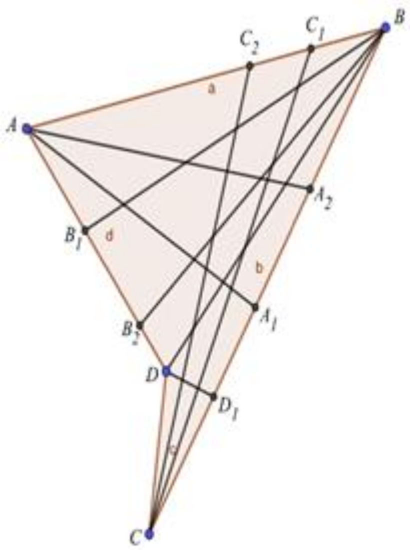

Fig 6 Trisector Angle D on the Nonconvex Quadrilateral $ABCD$ Case 3.

Consider $\triangle DBC$, because the sum of the angles in the triangle is 180° then obtained

$$\angle DBC = 180 - (\angle CDB + \angle DCB)$$

Then by substituting the value of $\angle CDB = \frac{2\delta}{3}$ and $\angle DCB = \gamma$, we obtain

$$\angle DBC = 180 - \left(\frac{2\delta}{3} + \gamma\right)$$

By using the sine rule on $\triangle DBC$, we obtain

$$\frac{CD}{\sin(\angle DBC)} = \frac{DB}{\sin(\angle DCB)}$$

Furthermore, we get

$$DB = \frac{c \sin \gamma}{\sin\left(\frac{2\delta}{3} + \gamma\right)}$$

So Theorem 5 is proven. ■

IV. AREA COMPARISON IN NONCONVEX QUADRILATERALS

There is an arbitrary nonconvex quadrilateral $ABCD$, if an angle trisector is formed at each corner, then several shapes are formed at each corner, namely at $\angle A$ formed $\triangle ABA_2$, $\triangle A_2AA_1$ and $\triangle ADA_1$, at $\angle B$ formed $\triangle ABB_1$, $\triangle B_1BA_2$ and quadrilateral B_2BCD, at $\angle C$ formed $\triangle BCC_1$, $\triangle C_1CC$_2 and $\triangle CDC_2$, and then at $\angle D$ formed $\triangle CDD_1$, $\triangle ADD_2$ and quadrilateral D_1DD_2B. The following will show the comparison of the area of each angle of the nonconvex quadrilateral $ABCD$, by showing the comparison of the triangular area of one of the angles, namely at angle A, as in the following theorem.

➢ *Theorem 6.*

In an arbitrary nonconvex quadrilateral $ABCD$, if a trisector is formed at angle A then the area ratio $L \triangle ABA_2 : L \triangle A_2AA_1 : L \triangle ADA_1$ is

$$a \sin\left(\frac{2\alpha}{3} + \beta\right) : a \sin\beta : d \sin\left(\frac{\alpha}{3} + \beta\right)$$

- *Proof:*

Suppose there is a nonconvex quadrilateral $ABCD$, an angle trisector will be formed from its four corners. The two sides of the angle trisector B are on the AD side illustrated in Figure 2. Next, the area ratio of the trisector triangle at angle A will be determined, namely determining the area of triangles ABA_2, A_2AA_1 and ADA_1. The area of triangle ABA_2 is shown using the triangle area formula in trigonometry, resulting in

$$L \triangle ABA_2 = \frac{1}{2} \cdot AB \cdot AA_2 \cdot \sin(\angle BBA_2)$$

Then by substituting the value of AA_1 in Theorem 3, we obtain

$$L \triangle ABA_2 = \frac{a^2 \sin\beta \sin\left(\frac{\alpha}{3}\right)}{2\sin\left(\frac{\alpha}{3} + \beta\right)}$$

Next, $L \triangle A_2AA_1$ is shown using the triangle area formula, resulting in

$$L \triangle A_2AA_1 = \frac{1}{2} \cdot AA_1 \cdot AA_2 \cdot \sin(\angle A_2AA_1)$$

Then by substituting the values of AA_1 and AA_2 in Theorem 3, we obtain

$$L \triangle A_2AA_1 = \frac{a^2 \sin^2\beta \sin\left(\frac{\alpha}{3}\right)}{2\sin\left(\frac{2\alpha}{3} + \beta\right)\sin\left(\frac{\alpha}{3} + \beta\right)}$$

Next, $L \triangle ADA_1$ is shown using the triangle area formula, resulting in

$$L \triangle ADA_1 = \frac{1}{2} \cdot AD \cdot AA_1 \cdot \sin(\angle DAA_1)$$

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Then by substituting the value of AA_1 in Theorem 3, we obtain

$$L\triangle ADA_1=\frac{ad\sin\beta\sin\left(\frac{\alpha}{3}\right)}{2\sin\left(\frac{2\alpha}{3}+\beta\right)}$$

By comparing $L\triangle ABA_2$, $L\triangle A_2AA_1$ and $L\triangle ADA_1$, the triangle area comparison is obtained as follows:

$$L\triangle ABA_2 : L\triangle A_2AA_1 : L\triangle ADA_1$$

$$=a\sin\left(\frac{2\alpha}{3}+\beta\right) : a\sin\beta : d\sin\left(\frac{\alpha}{3}+\beta\right)$$

Therefore, Theorem 5 is proven ■

Using the same method, we can determine the area ratio of the trisector at angle C as follows.

$$L\triangle BCC_1 : L\triangle C_1CC_2 : L\triangle CDC_2$$

$$=b\sin\left(\frac{2\gamma}{3}+\beta\right) : b\sin\beta : c\sin\left(\frac{\gamma}{3}+\beta\right)$$

If $\angle D$ is greater than 90°, then a case arises to determine the area ratio at angle B and angle D. While the area ratio at angles A and C does not cause a case. The following theorem determines the length of the area ratio of the shape formed at angle B in some cases.

➢ *Theorem 7.*

In an arbitrary nonconvex quadrilateral $ABCD$, if a trisector is formed at angle B then the area ratio formed is

$$L\triangle ABB_1 : L\triangle B_1BB_2=\sin\left(\frac{2\beta}{3}+\alpha\right) : \sin\alpha$$

Or

$$L\triangle ABB_1 : L\triangle CBB_2=\frac{a^2\sin\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)} : \frac{b^2\sin\gamma}{\sin\left(\frac{\beta}{3}+\gamma\right)}$$

Or

$$L\triangle ABB_1 : L\triangle B_1BD : L\triangle CBD$$

$$=\frac{a^2\sin\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)} : \frac{a^2\sin^2\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)\sin\left(\frac{2\beta}{3}+\alpha\right)} : \frac{b^2\sin\gamma}{\sin\left(\frac{\beta}{3}+\gamma\right)}$$

- *Proof:*

In determining the area ratio at angle B, there are several cases to prove it. The following is an explanation of each case in determining the area ratio of angle B.

- *Case 1*

Suppose there is a nonconvex quadrilateral $ABCD$ in Figure 2, an angle trisector will be formed from its four corners. Both sides of the angle trisector B are on the AD side. Next, the area ratio of the trisector triangle at angle B will be determined by determining the area of triangle ABB_1 and the area of triangle B_1BB_2. The area of triangle ABB_1 is determined by using the triangle area formula in trigonometry, resulting in

$$L\triangle ABB_1=\frac{1}{2}\cdot AB\cdot BB_1\cdot\sin(\angle ABB_1)$$

Then by substituting the value of BB_1 in Theorem 4, we obtain

$$L\triangle ABB_1=\frac{a^2\sin\alpha\sin\left(\frac{\beta}{3}\right)}{2\sin\left(\frac{\beta}{3}+\alpha\right)}$$

Next, $L\triangle B_1BB_2$ is shown by using the triangle area formula, resulting in

$$L\triangle B_1BB_2=\frac{1}{2}\cdot BB_1\cdot BB_2\cdot\sin(\angle B_1BB_2)$$

Then by substituting the values of BB_1 and BB_2 in Theorem 4, we obtain

$$L\triangle B_1BB_2=\frac{a^2\sin^2\alpha\sin\left(\frac{\beta}{3}\right)}{2\sin\left(\frac{\beta}{3}+\alpha\right)\sin\left(\frac{2\beta}{3}+\alpha\right)}$$

By comparing $L\triangle ABB_1$ and $L\triangle B_1BB_2$, the following triangular area comparison is obtained:

$$L\triangle ABB_1 : L\triangle B_1BB_2=\sin\left(\frac{2\beta}{3}+\alpha\right) : \sin\alpha.$$

Figure 2 explains that the area comparison between $L\triangle ABB_1$ and $L\triangle B_1BB_2$ only applies to the figure, so the area of $L\triangle ABB_1$ and $L\triangle B_1BB_2$ only applies if the angle trisector B is on the AD side. Next, we will discuss case 2 which occurs if the angle trisector B is on the AD and CD sides, respectively.

- *Case 2*

Suppose there is a nonconvex quadrilateral $ABCD$ in Figure 3, an angle trisector will be formed from its four corners. The two sides of the angle trisector B are on the sides of AD and CD. Next, the area ratio of the trisector triangle at angle B will be determined by determining the area of triangle ABB_1 and the area of triangle CBB_2. The area of triangle ABB_1 has been obtained in the previous discussion. The area of triangle CBB_2 is determined by using the triangle area formula in trigonometry, resulting in

$$L\triangle CBB_2=\frac{1}{2}\cdot CB\cdot BB_2\cdot\sin(\angle CBB_2)$$

Then by substituting the value of BB_2 in Theorem 4, we obtain

$$L \triangle CBB_2 = \frac{b^2 \sin\gamma \sin\left(\frac{\beta}{3}\right)}{2\sin\left(\frac{\beta}{3}+\gamma\right)}$$

By comparing $L \triangle ABB_1$ and $L \triangle CBB_2$, the comparison of the areas of the triangles is obtained as follows:

$$L \triangle ABB_1 : L \triangle CBB_2 = \frac{a^2 \sin\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)} : \frac{b^2 \sin\gamma}{\sin\left(\frac{\beta}{3}+\gamma\right)}$$

Figure 3 explains that the area comparison between $L \triangle ABB_1$ and $L \triangle CBB_2$ only applies to the figure, so the areas of $L \triangle ABB_1$ and $L \triangle CBB_2$ only apply if the angle trisectors B are on sides AD and CD. Next, we will discuss case 3 which occurs if the trisector of angle B is on side AD and point D, respectively.

- *Case 3*

Suppose there is a nonconvex quadrilateral $ABCD$ in Figure 4, an angle trisector will be formed from the four corners. The two sides of the angle trisector B are on the side of AD and and point D. Next, we will determine the ratio of the triangle area of the trisector at angle B by determining the area of triangle ABB_1, the area of triangle B_1BD and the area of triangle CBD. The area of triangle ABB_1 has been obtained in the previous discussion. The area of triangle B_1BD is equal to the area of triangle B_1BB_2 and the area of triangle CBD is equal to and the area of triangle CBB_2. By comparing the area of triangle ABB_1, the area of triangle B_1BD and the area of triangle CBD obtained

$$L \triangle ABB_1 : L \triangle B_1BD : L \triangle CBD$$

$$= \frac{a^2 \sin\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)} : \frac{a^2 \sin^2\alpha}{\sin\left(\frac{\beta}{3}+\alpha\right)\sin\left(\frac{2\beta}{3}+\alpha\right)} : \frac{b^2 \sin\gamma}{\sin\left(\frac{\beta}{3}+\gamma\right)}$$

Therefore, Theorem 6 is proven. ■

After getting the area ratio of the shape formed at angle B, we will determine the area ratio of the shape formed at angle D. The following theorem is given which determines the length of the area ratio of the shape formed at angle D in some cases.

- *Theorem 8.*

In an arbitrary nonconvex quadrilateral $ABCD$, if a trisector is formed at angle D then the area ratio formed is

$$L \triangle CDD_1 : L \triangle D_1DD_2 = \sin\left(\frac{2\delta}{3}+\gamma\right) : \sin\gamma$$

Or

$$L \triangle CDD_1 : L \triangle ADD_2 = \frac{c^2 \sin\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)} : \frac{d^2 \sin\alpha}{\sin\left(\frac{\delta}{3}+\alpha\right)}$$

Or

$$L \triangle CDD_1 : L \triangle D_1DB : L \triangle ADB$$

$$= \frac{c^2 \sin\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)} : \frac{c^2 \sin^2\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)\sin\left(\frac{2\delta}{3}+\gamma\right)} : \frac{d^2 \sin\alpha}{\sin\left(\frac{\delta}{3}+\alpha\right)}$$

- *Proof:*

In determining the area ratio of angle D, there are several cases to prove it. The following is an explanation of each case in determining the area ratio of angle D.

- *Case 1*

Suppose there is a nonconvex quadrilateral $ABCD$ in Figure 5, an angle trisector will be formed from its four corners. Both sides of the angle trisector D are on the BC side. Next, the area ratio of the trisector triangle at angle D will be determined by determining the area of triangle CDD_1 and the area of triangle D_1DD_2. The area of triangle CDD_1 is determined by using the triangle area formula in trigonometry, resulting in

$$L \triangle CDD_1 = \frac{1}{2} \cdot CD \cdot DD_1 \cdot \sin(\angle CDD_1)$$

Then by substituting the value of DD_1 in Theorem 5, we obtain

$$L \triangle CDD_1 = \frac{c^2 \sin\gamma \sin\left(\frac{\delta}{3}\right)}{2\sin\left(\frac{\delta}{3}+\gamma\right)}$$

Next, $L \triangle D_1DD_2$ is shown by using the triangle area formula, resulting in

$$L \triangle D_1DD_2 = \frac{1}{2} \cdot DD_1 \cdot DD_2 \cdot \sin(\angle D_1DD_2)$$

Then by substituting the values of DD_1 and DD_2 in Theorem 5, we obtain

$$L \triangle D_1DD_2 = \frac{c^2 \sin^2\gamma \sin\left(\frac{\delta}{3}\right)}{2\sin\left(\frac{\delta}{3}+\gamma\right)\sin\left(\frac{2\delta}{3}+\gamma\right)}$$

By comparing $L \triangle CDD_1$ and $L \triangle D_1DD_2$, the ratio of the areas of the triangles is obtained as follows:

$$L \triangle CDD_1 : L \triangle D_1DD_2 = \sin\left(\frac{2\delta}{3}+\gamma\right) : \sin\gamma$$

Figure 5 explains that the area comparison between $L \triangle CDD_1$ and $L \triangle D_1DD_2$ only applies to the figure, so the area of $L \triangle CDD_1$ and $L \triangle D_1DD_2$ only applies if the angle trisector D is on the BC side. Next we will discuss case 2 which occurs if the angle trisector D is on the AB and BC sides respectively.

- *Case 2*

Suppose there is a nonconvex quadrilateral $ABCD$ in Figure 2, an angle trisector will be formed from its four corners. The two sides of the angle trisector D are on the AB and BC sides. Next, the area ratio of the trisector triangle at angle D will be determined by determining the area of triangle CDD_1 and the area of triangle ADD_2. The area of triangle CDD_1 has been obtained in the previous discussion. The area of triangle ADD_2 is determined by using the triangle area formula in trigonometry, resulting in

$$L \triangle ADD_2 = \frac{1}{2} \cdot AD \cdot DD_2 \cdot \sin(\angle ADD_2)$$

Then by substituting the value of DD_2 in Theorem 5, we obtain

$$L \triangle ADD_2 = \frac{d^2 \sin\alpha \sin\left(\frac{\delta}{3}\right)}{2\sin\left(\frac{\delta}{3}+\alpha\right)}$$

By comparing $L \triangle CDD_1$ and $L \triangle ADD_2$, the area of the triangles is compared as follows:

$$L \triangle CDD_1 : L \triangle ADD_2 = \frac{c^2 \sin\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)} : \frac{d^2 \sin\alpha}{\sin\left(\frac{\delta}{3}+\alpha\right)}$$

Figure 2 explains that the area comparison between $L \triangle CDD_1$ and $L \triangle ADD_2$ is only valid for that figure, so the areas of $L \triangle CDD_1$ and $L \triangle ADD_2$ are only valid if the angle trisectors D are on sides AB and BC. Next, we will discuss case 3 which occurs if the corner trisector D is on side BC and point B, respectively.

- *Case 3*

Suppose there is a nonconvex quadrilateral $ABCD$ in Figure 6, angle trisectors will be formed from its four corners. The two sides of angle trisector D are on side BC and point B. Next, the area ratio of the trisector triangle at angle D will be determined by determining the area of triangle CDD_1, the area of triangle D_1DB and the area of triangle ADB. The area of triangle CDD_1 has been obtained in the previous discussion. The area of triangle D_1DB is equal to the area of triangle D_1DD_2 and the area of triangle ADB is equal to and the area of triangle ADD_2. By comparing the area of triangle CDD_1, the area of triangle D_1DB and the area of triangle ADB we obtain

$$L \triangle CDD_1 : L \triangle D_1DB : L \triangle ADB$$

$$= \frac{c^2 \sin\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)} : \frac{c^2 \sin^2\gamma}{\sin\left(\frac{\delta}{3}+\gamma\right)\sin\left(\frac{2\delta}{3}+\gamma\right)} : \frac{d^2 \sin\alpha}{\sin\left(\frac{\delta}{3}+\alpha\right)}$$

Therefore, Theorem 7 is proven ■

V. CONCLUSIONS

This research discusses the length of the side of an arbitrary angle trisector of a nonconvex quadrilateral in some cases and the comparison of the area of the shape formed from the angle trisector of a nonconvex quadrilateral in some cases. As for calculating the comparison of the area of the shape formed from the angle trisector on the non-convex quadrilateral, it is calculated using the sine rule and the triangle area in trigonometry. Calculating the area of shapes formed from angle trisectors on non-convex quadrilaterals can be an enrichment material at the senior high school level.

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