Controllability Criterion for Nonlinear Neutral Type Fractional Order Differential Systems with State Delays and Distributed Delays in the Control, and Impulsive Effects in Banach Spaces

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Abstract:- Nonlinear Neutral Type Fractional Order Differential Systems with State Delays and Distributed Delays in the Control, and Impulsive Effects in Banach Space is investigated for Controllability Criterion. A set of necessary and sufficient criteria for the system to be controllable were established. Uses were made of some Controllability Standards and the Properties of Controllability Grammian or mapping. The Mild Solution of the System was established, and the Set Functions upon our study hinged were extracted from the Mild Solution,using Sundara like Arguments,and the Unsymmetric Fubini's theorem.

Keywords:- *Positive Definite, Non-singularity, Properness, Controllability, Orthogonality.*

I. INTRODUCTION

Recently the study of fractional differential systems have emerged as a new area of research in the field of applied mathematics which have been used to model any practical systems in science and engineering (Sheen and Cao,2012) and (Huang,2017).

The fractional differential expressions have been used in engineering since 1930's to describe the viscoelastic materials, electric circuits and fractal geometry involving non integer spatial dimensions. Also fractional derivatives and integrals can be applied to real systems characterized by power laws, critical phenomena and scale free processes.

Moreover, controllability is one of the fundamental concepts in mathematics and fractional control theory and is the generalization of the classical control theory (Ammour, 2009). It is noted in (Oraekie,2012) that any control systems is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals.

Furthermore, it's well known that neutral differential equation is a very special class of ordinary differential equation and it arises in compartmental models in which the system can be divided into separate compartments, marking the path ways of material flow between compartments and the possible outflow into the inflow from the environment of the system; (Gyori and Wu,1991) as it is contained in (Oraekie,2012; Oraekie,2014). Such models are usually used in theoretical epidemiology, physiology and population dynamics to describe the evolution of systems.

The above said models can be remodified as a neutral fractional differential equation or neutral fractional volterra integrodifferential equation.

At the same time, time delay is very commonly experienced in diverse scientific systems such as electric, pneumatic and hydraulic networks, chemical process, long transmission lines etc. Because the subsistence of pure time delay, nevertheless if it is available either in the control or the state may result in unacceptable system momentary response or even instability. Also, time delay is one of the inevitable problems in practical engineering applications, which has an important effect on the stability and performance of the system (Li and Song,2017).

With the interest from the above fact, in the last few years, several studies have been done on the fractional delay differential systems.(Chen and Zhou,2011) (Kaslik and Sivabundaram,2012),(Oraekie,2018).

Generally, most of the dynamical systems are analyzed in either continuous or discrete time domain, many real systems in physics, biology, chemistry, engineering and information science may experience abrupt changes as certain instants during the continuous dynamical systems (Sundara,2018) and (Li and Wu 2016).According to (Sundara,2018),there has been a somewhat new category of dynamical system; which is neither purely continuous time nor purely discrete time ones, these are called impulsive control system.

This 3rd category of system displays a combination of characteristics of both the continuous and discrete-time systems.

The significance of this system is to control the hasty changes in nature adversity.

Numerous advancements procedure are subject to short term perturbations which act instantaneously in the form of impulses. For instance, the existence of impulses can be seen in the biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics.

Consequently the impulsive differential equations provide a natural description of observed advancement procedure of several real world problems. Recently (Zhang, 2013) have derived the controllability criteria for linear fractional order differential systems with state delay and impulses. (Sundara,2018),studied the controllability problem of nonlinear neutral type fractional differential systems with state delay and impulsive effects and established a new set of sufficient conditions for the system to be controllable using the controllability grammian and laplace transformation.

To the best of our knowledge there are no relevant reports on the controllability of Nonlinear neutral-type fractional- order differential systems with State delays and distributed delays in the control , and impulsive effects in Banach spaces in the existing literature. Hence the research.

II. VARIATION OF CONSTANT FORMULA

We reemphasize that we denote $C_\rho([O.T],E^n)$ the space of all piecewise left continuous functions mapping the interval$[O,T]$ $into\ E^n.$ $Let\ \alpha;\beta>0$ with $n-1<\alpha,\beta<n$ and $n\in N, D$ is the usual differential operator; E^m is the $m-$ dimensional Euclidean space, $R^+=[0,\infty)$ and suppose $f\in L_1(R^+.)$ The following definitions, properties and theorems are familiar and helpful in establishing our main results.

➢ $Definition$ 2.1
The Riemann-Liouville fractional integral operator of order $\alpha>0$ with the lower limit zero for a function $f: R^+\to R^n\ is\ defined\ as$

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^\alpha (t-s)^{\alpha-1}f(s)\,ds,\qquad t>0$$

Where $\Gamma(.)$ is the euler gamma function

➢ $Definition$ 2.2
The Riemann-fractional derivative of the order $\alpha>0$, with the lower limit zero for a function $f;\ R^+\to R^n, n-1<\alpha<n, n\in N$ is defined as

$$(D_0^\alpha, f)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}f(s)\,ds,$$

Where the function f has absolutely continuous derivatives up to order $(n-1)$

➢ $Definition$ 2.3
The caputo fractional derivative of order $\alpha>0, n-1<\alpha<n$ is defined as

$$(\ {}^cD_0^\alpha, f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^n(s)\,ds,$$

Where the function f has absolutely continuous derivatives up to order $(n-1)$.

If $0<\alpha<1$, then

$$(\ {}^cD_0^\alpha f)(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(s)}{(t-s)^\alpha}ds,$$

Consider the controllability of nonlinear fractional order type differential systems with **state delay** and **distributed delays in the control**, and impulses in Banach spaces as follows:

$${}^cD^\propto\Big(x(t)-g\big(t,x(t)\big)\Big)=Ax(t)+\int_0^t B(t-s)x(s-h)ds+\int_{-h}^0\big(d_\theta H(t,\delta)u(t+\delta)\big)$$

$$t\in[0,T]-\{t_1,t_2,\dots,t_k\},\qquad \Delta x(t_j)=x(t_i^+)-x(t_i^-)=I_i\big(x(t_i)\big),\qquad i=1,2,3\dots,k,$$

$$x(t)=\phi(t),\ t\in[-h,0]\qquad (1.1)$$

Where ${}^cD^\propto x(t)$ denotes an α order caputo's fractional derivative of $x(t).$ $0<\alpha<1$, A is a constant matrix and satisfies $A\in E^{n\times n}$, where B is a continuous matrix in their argument with initial condition $x(t_0)=x_0=x(0), where\ x\in E^n$ is the state space and $u\in E^m$ is the control function, $H(t,\delta)\ is\ an\ n\times m$ matrix continuous at t and of bounded variation in $\delta\ on\ [-h,0]$; $h>0$ for each $t\in[0,T]$; $0<T.\phi\in([-h,0],E^n)$ denotes the initial function while , $C([-h,0],E^n)$ denotes the space of all continuous functions mapping the interval $[-h,0]\ into\ E^n$;

$I_i: E^n\to E^n$ is continuous for $i=1,2,3,4,\dots,k$, $and\ x(t_i^+)=\lim_{\varepsilon\to0^+}x(I_i+\varepsilon)$

$$x(t_i^-)=\lim_{\varepsilon\to0^-}x(I_i+\varepsilon)\qquad (1.2)$$

Represent the right and left limits of $x(t)$ at $t = t_i$ and the discontinuous points

$$t_1 < t_2 < t_3 < \cdots < t_k \tag{1.3}$$

$$Where \quad 0 = t_0 < t_1, t_k < t_{k+1} = T < \infty, and \;\; x(t_i^+) = x(t_i^-)$$

Which implies that the solution of the **system (1.1)** is left continuous.

- 2.1. $The\ Mild\ Solution$

In order to obtain **the mild solution of system (1.1**), we first consider the representation of solution for nonlinear fractional State delay differential systems without impulses as follows:

$$^{C}D^{\propto}\left(x(t) - g\big(t, x(t)\big)\right) = Ax(t) + \int_0^t B(t-s)x(s-h)ds$$

$$+ \; f\left(t, x(t), \int_0^t k(s, x(s)ds)\right), t \in [0.T]$$

$$x(t) = \phi(t), t \in [-h, 0] \tag{1.4}$$

- $Theorem$ 2.1 ($B.Sundara\ etal$, 2018)

Let $0 < \alpha < 1, if\ f: [0, T] \to E^n$ is continuous and exponential bounded ,then the solution of the system (1.4) can be represented as

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \int_0^t (t-s)^{\alpha-1} E_\alpha, \alpha(A(t-s)^\alpha\,)$$

$$X\left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big) + \int_0^s B(s-m)x(m-h)dm \;+ f\left(s, x(s), \int_0^s K\big(s, \tau, x(\tau)\big)\right)d\tau\right] ds\,, t\epsilon[0, T]$$

$$x(t) = \phi(t), t\epsilon[-h, 0]\,.$$

(See B. Sundara, etal, 2018 for the proof).

- $Theorem$ 2.2

Let $0 < \alpha < 1$ and $\mathrm{u} \in \mathrm{C_p}([0, \mathrm{T}], \mathrm{E^m})$, then the state response of the system (1.1) can be represented as follows;

For $\boldsymbol{t\epsilon[-h, 0]}$**, then, we have**

$$x(t) = \phi(t).$$

For $\boldsymbol{t\epsilon[0, t_1]}$**, then we have**

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \int_0^t (t-s)^{\alpha-1} E_\alpha, \alpha[A(t-s)^\alpha\,]$$

$$\times\left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big) + \int_0^s B(s-m)x(m-h)dm \;+ \int_{-h}^0 d_\delta H(s, \delta)u(t+s)\right] ds.$$

For $\boldsymbol{t\epsilon[t_1, t_2]}$**, then we have**

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \boldsymbol{I_1}\big(\boldsymbol{x}(\boldsymbol{t_1}^-)\big) + \int_0^t (t-s)^{\alpha-1} E_\alpha, \alpha[A(t-s)^\alpha\,]$$

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$$\times\left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big) + \int_0^s B(s-m)x(m-h)dm \; + \int_{-h}^{0} d_\delta H(s,\delta)u(t+s)\right]ds$$

For $\boldsymbol{t\epsilon[t_j, t_{j+1}]}$, $\boldsymbol{j = 1, 2, \dots, k}$. **We have**

$$x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) \; + \sum_{j=1}^{i} \boldsymbol{I_j}\left(\boldsymbol{x}\big(\boldsymbol{t_j^-}\big)\right) + \int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha].$$

$$\left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big) + \int_0^s B(s-m)x(m-h)dm \; + \int_{-h}^0 d_\delta H(s,\delta)u(s+\delta)\right]ds \qquad (1.5)$$

System (1.5) implies:

$$x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) \; + \sum_{j=1}^{i} \boldsymbol{I_j}\left(\boldsymbol{x}\big(\boldsymbol{t_j^-}\big)\right)$$

$$+ \int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha]. \; \left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\right]ds$$

$$+ \int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha]. \int_0^s B(s-m)x(m-h)dmds$$

$$+ \int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha]. \int_{-h}^0 d_\delta H(s,\delta)u(s+\delta)\,ds \qquad (1.6)$$

(1.6) $\Rightarrow$

$$x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) \; + \sum_{j=1}^{i} \boldsymbol{I_j}\left(\boldsymbol{x}\big(\boldsymbol{t_j^-}\big)\right)$$

$$+ \int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha]. \; \left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\right]ds$$

$$+ \int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha] \int_0^s B(s-m)x(m-h)dmds$$

$$\int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha] \int_{-h}^0 d_\delta H(s,\delta)u(s+\delta)\,ds \qquad (1.7)$$

A careful observation of the solution of system (1.1) given as system (1.7) shows that the values of the control $\boldsymbol{u(t)}$ ***for*** $\boldsymbol{t\epsilon[-h, T]}$ **enter the definition of the initial complete state, thereby creating the need for an explicit variation of constant formula. The control in the last term of formula (1.7), therefore, has to be separated in the intervals** $\boldsymbol{[-h, 0]}$ ***and*** $\boldsymbol{[0, T]}$.

To achieve this, that term has to be transformed by applying the method of (klamka, 1978) as it is contained in (Oraekie ,2019). Finally, we interchange the order of integration using the Unsymmetric Fubini's Theorem to have

$$x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) \; + \sum_{j=1}^{i} I_j\left(x\big(t_j^-\big)\right)$$

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$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha},\alpha[A(t-s)^{\alpha}].\ \left[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\right]ds$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha},\alpha[A(t-s)^{\alpha}]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_0^t \left[\int_{-h}^0 d_{\delta}H\int_{0+\delta}^{t+\delta}(t-s-\delta)^{\alpha-1}E_{\alpha},\alpha[A(t-s-\delta)^{\alpha}H(s-\delta,\delta)u(s-\delta+\delta)]\right]ds$$

$$\Rightarrow x(t)=\ \phi(0)-g\big(0,x(0)\big)+g\big(t,x(t)\big)\ +\sum_{j=1}^{i} I_j\left(x\big(t_j^-\big)\right)$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha},\alpha[A(t-s)^{\alpha}]\left[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\right]ds$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha},\alpha[A(t-s)^{\alpha}]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_0^t \left[\int_{-h}^0 d_{\delta}H\int_{0+\delta}^{t+\delta}(t-s-\delta)^{\alpha-1}E_{\alpha},\alpha[A(t-s-\delta)^{\alpha}H(s-\delta,\delta)u(s)]\right]ds \qquad (1.8)$$

Simplifying system (1.8), we have

$$x(t)=\ \phi(0)-g\big(0,x(0)\big)+g\big(t,x(t)\big)\ +\sum_{j=1}^{i} I_j\left(x\big(t_j^-\big)\right)$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha},\alpha[A(t-s)^{\alpha}]\left[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\right]ds$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha},\alpha[A(t-s)^{\alpha}]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_{\delta}\int_{\delta}^0 (t-s-\delta)^{\alpha-1}E_{\alpha},\alpha[A(t-s-\delta)^{\alpha}H(s-\delta,\delta)u_0(s)]ds$$

$$+\int_{-h}^0 dH_{\delta}\int_0^{t+\delta} (t-s-\delta)^{\alpha-1}E_{\alpha},\alpha[A(t-s-\delta)^{\alpha}H(s-\delta,\delta)u(s)]ds \qquad (1.9)$$

Using again the Unsymmetric Fubini's Theorem on the change of order of integration and incorporating H^* as defined below

$$H^*(s-\delta,\delta)=\begin{cases}H(s-\delta,\delta)\ for\ s\le t\\ 0\ for\ s\ge t\end{cases} \qquad (1.10)$$

Formula (1.9) becomes

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \sum_{j=1}^{i} I_j\left(x\big(t_j^{-1}\big)\right)$$

$$+\int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha]\big[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big)\big]\, ds$$

$$+\int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha\,]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_\delta \int_\delta^0 (t-s-\delta)^{\alpha-1}E_\alpha, \alpha[A(t-s-\delta)^\alpha H(s-\delta,\delta)u_0(s)]ds$$

$$+\int_0^t \left[\int_{-h}^0 (t-s-\delta)^{\alpha-1}E_\alpha, \alpha[A(t-s-\delta)^\alpha\,]d_\delta H^*(s-\delta,\delta)u(s)\right] ds \qquad (1.11)$$

Integration is still in the lebesque stieltjes sense in the variable $\delta\ in\ H$.

For brevity, let

$$\boldsymbol{\beta(t)} = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \sum_{j=1}^{i} I_j\left(x\big(t_j^{-}\big)\right)$$

$$+\int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha\,]\big[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big)\big]\, ds$$

$$+\int_0^t (t-s)^{\alpha-1}E_\alpha, \alpha[A(t-s)^\alpha\,]\int_0^s B(s-m)x(m-h)dmds \qquad (1.12)$$

$$\boldsymbol{\mu(t)} = \int_{-h}^0 dH_\delta \int_\delta^0 (t-s-\delta)^{\alpha-1}E_\alpha, \alpha[A(t-s-\delta)^\alpha H(s-\delta,\delta)u_0(s)]ds \qquad (1.13)$$

$$\boldsymbol{z(t,s)} = \int_{-h}^0 (t-s-\delta)^{\alpha-1}E_\alpha, \alpha[A(t-s-\delta)^\alpha\,]d_\delta H^*(s-\delta,\delta)u(s) \qquad (1.14)$$

Substituting (1.12), (1.13) and (1.14) into (1.11),we have a precise variation of constant formula for system (1.1) as:

$$\boldsymbol{x(t, x_0, u) = \beta(t) + \mu(t) + \int_0^t z(t,s)u(s)ds} \qquad (1.15)$$

III. MAIN RESULTS

➢ *3.1. Basic Set Functions and Properties with Appropriate Terminologies*

- $Definition$ 3.1.1. $(Complete\ State)$

The complete state for the system (4.1) is given by $z(t) = \{x_t, u_t\}$

- $Definition$ 3.1.2. $(Controllability)$

The system (1.1) is said to be controllable on the interval $J = [0,T]$ if every function ϕ and every state $x_1 \in R^n$, there exists an admissible control function u such that a solution $x(t, 0, \phi, u)$of system (1.1) satisfies

$$x(t, 0, \phi, u) = x_1\big(t, 0, \phi, u(t_1)\big)$$

- $Definition$ 3.1.3. $(Attainable\ Set)$

Here, we define the attainable set $A(t, 0)$ of the system (1.1) as the set of all the possible solutions of the system(1.1). We denote it by

$$A(t, 0) = \{x(t, 0, \phi, u); u \in U \subset L_2([0, T], R^m)\}$$

$$\text{Where, } U = \{u \in L_2([0, T], R^m): |u_j| \leq 1\}, j = 01,2, \dots, m.$$

- $Definition$ 3.1.4. $(Reachable\ Set)$

The reachable set for the system (1.1) denoted by $R(t, 0)$ is part of the solution that takes the system to the target in finite time through the control. It is given as:

$$R(t, 0) = \{\int_0^t \left[\int_{-h}^0 (t - s - \delta)^{\propto -1} E_{\propto}, \propto [A(t - s - \delta)^{\propto}] d_\delta H^*(s - \delta, \delta) u(s)\right] ds : : u \in U\}.$$

- $Definition$ 3.1.5. $(Controllability\ Grammian)$

The controllability grammian or map of the system (1.1) is given as

$$W(t, 0) = \int_0^t \left[\int_{-h}^0 (t - s - \delta)^{\propto -1} E_{\propto}, \propto [A(t - s - \delta)^{\propto}] d_\delta H^*(s - \delta, \delta)\right]$$

$$\times \left[\int_{-h}^0 (t - s - \delta)^{\propto -1} E_{\propto}, \propto [A(t - s - \delta)^{\propto}] d_\delta H^*(s - \delta, \delta)\right]^T$$

- $Definition$ 3.1.6. $(Target\ Set)$

The target set for the system (1.1) denoted by system $G(t, 0)$ is given as:

$$G(t, 0) = \{x(\tau, x_0, u): t \geq \tau \geq 0\ for\ some\ fixed\ \tau\ and\ u \in U\}$$

- $Definition$ 3.1.7. $(Properness)$

The system (1.1) is proper on the interval $J = [0, t]$, if span $R(t, 0) = R^n$

$$i.e$$

$$C^T \left[\int_{-h}^0 (t - s - \delta)^{\propto -1} E_{\propto}, \propto [A(t - s - \delta)^{\propto}] d_\delta H^*(s - \delta, \delta)\right] = 0 \quad a.e, \qquad t > 0,$$

$$\Rightarrow C = 0; C \in R^n$$

- *Theorem 3.1. (Necessary and sufficient condition for the System (1.1) to be Controllable).*

Consider the system (1.1) with its standing hypotheses given as the system below:

$$^C D^{\alpha}\Big(x(t) - g\big(t, x(t)\big)\Big) = Ax(t) + \int_0^t B(t - s) x(s - h) ds + \int_{-h}^0 [d_\delta H(t, \delta)]\, u(t + \delta)$$

$$t \in [0, T] - \{t_1, t_2, \dots t_k\},$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i\big(x_i(t_i)\big)\ i = 1,2,3, \dots, k,$$

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$$x(t) = \phi(t), \qquad t \in [-h, 0] \quad . \tag{1.1}$$

If the system (1.1) is controllable then the following statements are equivalent:

- ✓ *System (1.1) is relatively controllable on the interval* $[0, T]$
- ✓ *The controllability grammian* $W(t, 0)$ *of the system (1.1) is nonsingular*
- ✓ *The system (1.1) is proper.*

- *Proof*

From the controllability standard, we realized that the controllability grammian or map is invertible square matrix, thus it is nonsingular. This non singularity of controllability grammian is equivalent to saying that the controllability grammian $W(t, 0)$ is positive definite, this is in turn is equivalent to saying that there exists some constant matrix $C \in E^n such\ that\ C^T$ multiplied by the controllability index of the system (1.1) is equal to zero, almost everywhere, implies that

$$C = 0$$

$$i.e\ \ C^T\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha}, \alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^*(s-\delta, \delta)\right] = 0 \quad a.e \Rightarrow C = 0, \ \ t > 0,$$

$C \in R^n$ (properness).Thus (2) and (3) are equivalent.

It remains to show that (1) and (2) are equivalent.

Assume that the system (1.1) is relatively controllable on the interval $[0, T]$.

$Let\ C \in R^n = E^n$ and that

$$C^T\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha}, \alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^*(s-\delta, \delta)\right] = 0 \ \ a.e$$

$t \in [0, T]$ for every t, then

$$C^T\int_0^t\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha}, \alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^*(s-\delta, \delta)\right]u(s)ds$$

$$= \int_0^t C^T\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha}, \alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^*(s-\delta, \delta)\right]u(s)ds$$

$$for\ u \in U \subset R^m$$

It follows that C is orthogonal to the reachable set $R(t, 0)$, where

$$R(t,0) = \left\{\int_0^t\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha}, \alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^*(s-\delta, \delta)\right]u(s)ds\right\}$$

$$such\ that\ u \in U \subset R^m.$$

If we assume the relative controllability of system (1.1) then the span of the reachable set equals the whose space $R^n. i.e\ span\ R(t, 0) = R^n.$ So that $C = 0$, showing that (1) implies (3).

Conversely, Assume the system (1.1) is not relatively controllable, so that $R(t, 0) \neq 0,$ for $t > 0.$ Then there exists $C \neq 0, C \in R^n\ such\ that\ \ C^T R(t, 0) = 0.$

It follows here, that for every admissible control $u \in U \subset R^m$ that,

$$0 = C^T \int_0^t \left[\int_{-h}^0 (t-s-\delta)^{\alpha-1} E_\alpha, \alpha[A(t-s-\delta)^\alpha] d_\delta H^*(s-\delta,\delta) \right] u(s)ds$$

$$= C^T \int_0^t \left[\int_{-h}^0 (t-s-\delta)^{\alpha-1} E_\alpha, \alpha[A(t-s-\delta)^\alpha] d_\delta H^*(s-\delta,\delta) \right] u(s)ds.$$

Hence

$$C^T \int_0^t \left[\int_{-h}^0 (t-s-\delta)^{\alpha-1} E_\alpha, \alpha[A(t-s-\delta)^\alpha] d_\delta H^*(s-\delta,\delta) \right] u(s)ds = 0, a.e$$

$\Rightarrow C \neq 0$ (No properness)

IV. CONCLUSION

We have established the necessary and sufficient conditions for the controllability of the nonlinear neutral-type fractional order differential systems with state delays and distributed delays in the control, and impulsive effects in Banach spaces. Thus, we have extended the controllability of Nonlinear Neutral-Type Fractional-order differential systems with delays in the state and impulsive effects to the same systems with state delays and distributed delays in the control in Banach spaces.

REFERENCES

[1]. Shankar, S. (2002). The evolution of the concept of controllability. *Mathematical and Computer Modelling of Dynamical Systems*, *8*(4), 397-406.
[2]. Underwood, R. G., & Young, D. F. (1979). Null controllability of nonlinear functional differential equations. *SIAM Journal on Control and Optimization*, *17*(6), 753-772.
[3]. Klamka, J. (2013). Controllability of dynamical systems. A survey. *Bulletin of the Polish Academy of Sciences. Technical Sciences*, *61*(2).
[4]. Klamka, J. (2008A). Controllability of dynamical systems. *Mathematica Applicanda*, *36*(50/09).
[5]. Klamka, J. (2009). Stochastic controllability of systems with multiple delays in control. *International Journal of Applied Mathematics and Computer Science*, *19*(1), 39-47.
[6]. Chukwu, E. N., & Lenhart, S. M. (1991). Controllability questions for nonlinear systems in abstract spaces. *Journal of Optimization theory and Applications*, *68*, 437-462..
[7]. Oraekie P.A. (2019A),' Null controllability of sobolev type integrodifferential systems in Banach spaces with distributed delays in the control; COOU Journal of physical sciences 1(2).
[8]. Triggiani, R. (1975). Controllability and observability in Banach space with bounded operators. *SIAM Journal on Control*, *13*(2), 462-491.
[9]. Oraekie, P. A. (2017); Null Controllability of Nonlinear Infinite Space of Neutral Differential Systems with Distributed Delays in the Control. *Journal of the Nigerian Association of Mathematical Physics*, *41*.
[10]. Chyung, D. H. (1970, August). On the problem of controlling certain discrete systems with delays. In *IFAC Symp. Systems Engineering Approach to Computer Control*
[11]. Leela, S., McRae, F. A., & Sivasundaram, S. (1993). Controllability of impulsive differential equations. *Journal of Mathematical Analysis and Applications*, *177*(1), 24-30.
[12]. Oraekie, P. A. (2018); Relative Controllability of Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control. *International Journal of New Technology and Research*, *4*(2), 263138.
[13]. Oraekie P.A (2019B);Null controllability of linear t-invariant delay systems with delay in both state and the limited control powers.
[14]. Zill, D. G. (2012). *A first course in differential equations with modeling applications*. Cengage Learning.
[15]. Balachandran, K., & Dauer, J. P. (1996). Null controllability of nonlinear infinite delay systems with time varying multiple delays in control. *Applied Mathematics Letters*, *9*(3), 115-121.
[16]. Oraekie, P.A (2012);Ph.D THESIS on Controllability results for nonlinear neutral functional differential equations.
[17]. Chukwu, E. N. (1992). Stability and time-optimal control of hereditary systems, vol. 188 of. *Mathematics in Science and Engineering*.
[18]. Balachandran, K., & Anandhi, E. R. (2003). Neutral functional integrodifferential control systems in Banach spaces. *Kybernetika*, *39*(3), 359-367.
[19]. Umana, R. A. (2008). Null Controllability Of Nonlinear Infinite Neutral Systems With Multiple Delays In Control. *Journal of Computational Analysis & Applications*, *10*(1).

[20]. Shi, H., Xie, G., & Luo, W. (2012, January). Controllability analysis of linear discrete time systems with time delay in state. In *Abstract and Applied Analysis* (Vol. 2012). Hindawi.
[21]. Atmania, R., & Mazonzi, S. (2005). Controllability of semilinear integrodifferential equations with nonlocal conditions. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, *2005*, Paper-No.
[22]. Sikora, B. (2016). Controllability criteria for time-delay fractional systems with a retarded state. *International Journal of Applied Mathematics and Computer Science*, *26*(3), 521-531.
[23]. Oraekie P.A, Laisin M, Ugwunze N.B (2019); Relative controllability of linear time varying systems with multiple delays in the control. COOU Journal of physical science 2(8)2019.
[24]. Oraekie P.A, Laisin M, Benson G (2019). Controllability and Null controllability of the semilinear differential system with distributed delays in the control. COOU Journal of physical science 2(8)2019
[25]. Oraekie P.A (2019C);Note on the null controllability of semilinear integrodifferential systems in banach spaces with distributed delays in control.International Journal of new technology and research(IJNTR).ISSN;2454-4116,volume 5,issue 3,march 2019 pages 61-69.
[26]. Dauer, J. P., Balachandran, K., & Anthoni, S. M. (1998). Null controllability of nonlinear infinite neutral systems with delays in control. *Computers & Mathematics with Applications*, *36*(2), 39-50.
[27]. Nse, C. A. (2007). Global Relative Controlability for Nonlinear Neutral Systems with Delays in the Control. *Research Journal of Applied Sciences*, *2*(7), 807-809.
[28]. Bonilla, B., Rivero, M., Rodríguez-Germá, L., & Trujillo, J. J. (2007). Fractional differential equations as alternative models to nonlinear differential equations. *Applied Mathematics and computation*, *187*(1), 79-88.
[29]. Ahmed, H. M. (2012). Controllability for Sobolev type fractional integro-differential systems in a Banach space. *Advances in Difference Equations*, *2012*, 1-10.
[30]. Oraekie P.A (2018). Null controllability of fractional integrodifferential systems in banach spaces with distributed delays in the limited control powers.Journal of the Nigerian Association of mathematical physics volume 48(Sept and Nov.,2018 issue) pp1-10
[31]. Zhou, H. X. (1983). Approximate controllability for a class of semilinear abstract equations. *SIAM Journal on Control and Optimization*, *21*(4), 551-565.
[32]. Oraeke (2014). Relative controllability of linear time varying systems with delay in the control . COOU interdisplinary Research Journal vol.1,No 1.December 2014
[33]. Onwuatu, J. U. (1984). On the null-controllability in function space of nonlinear systems of neutral functional differential equations with limited controls. *Journal of optimization theory and applications*, *42*(3), 397-420.
[34]. Asuquo B and Usah E (2008). Control of linear systems using pure time delay. Journal of NAMP Vol.12,pp 55-62
[35]. Cheban, D. N. (2000). Uniform exponential stability of linear almost periodic systems in Banach spaces.
[36]. Cameron, R. H., & Martin, W. T. (1941). An unsymmetric Fubini theorem.
[37]. Klamka, J. (2008 B). Controllability of dynamical systems. *Mathematica Applicanda*, *36*(50/09).
[38]. Stroud, K. A. & Dexter (2001). Engineering Mathematics, with additions by Dexter J. *Booth, Fifth Edition, International Edition. Hampshire, England: Palgrave Publishers Ltd*, 403.
[39]. Frank, S. A. (2018). *Control theory tutorial: basic concepts illustrated by software examples*. Springer Nature.
[40]. Chyung, D. H., & Lee, E. B. (1970). Delayed action control problems. *Automatica*, *6*(3), 395-400.
[41]. Oraekie, P. A. (2017). Necessary and Sufficient Conditions for the Target Set of Nonlinear Infinite Space of Neutral Functional Differential Systems with Distributed Delay in the Control to be on the Boundary of the Attainable Set. *Journal of the Nigerian Association of Mathematical Physics*, *41*, 21-26.
[42]. Banks, H. T., & Jacobs, M. Q. (1970). A differential calculus for multifunctions. *Journal of Mathematical Analysis and Applications*, *29*(2), 246-272.
[43]. Davison, E. J., & Kunze, E. G. (1970). Some sufficient conditions for the global and local controllability of nonlinear time-varying systems. *SIAM Journal on Control*, *8*(4), 489-497.
[44]. Sikora, B., & Klamka, J. (2017). Constrained controllability of fractional linear systems with delays in control. *Systems & Control Letters*, *106*, 9-15.
[45]. Nawaz, M., Wei, J., & Jiale, S. (2020). The controllability of fractional differential system with state and control delay. *Advances in Difference Equations*, *2020*, 1-11.
[46]. Oraekie, P.A (2015A). Relative controllability of neutral functional integrodifferential systems in abstract space with distributed delays in the control. *American Academic & Scholarly Research Journal*, *7*(6).
[47]. Oraekie P.A. (2015B). Optimality conditions for the relative controllability of neutral functional integrodifferential systems in banach spaces with distributed delays in the control. *American Academic & Scholarly Research Journal*, *7*(6).
[48]. Balachandran, K., & Leelamani, A. (2006). Null controllability of neutral evolution integrodifferential systems with infinite delay. *Mathematical Problems in Engineering*, *2006*.
[49]. Balachandran, K., & Sakthivel, R. (1998). Controllability of delay integrodifferential systems in Banach spaces. *LIBERTAS MATHEMATICA (vol. I-XXXI)*, *18*, 119-128.

[50]. Oraekie, P. A. (2017). Necessary and Sufficient Conditions for the Target Set of Nonlinear Infinite Space of Neutral Functional Differential Systems with Distributed Delay in the Control to be on the Boundary of the Attainable Set. *Journal of the Nigerian Association of Mathematical Physics*, *41*, 21-26.
[51]. Iyai, D. (2007). Euclidean null controllability of perturbed infinite delay systems with limited control. *Journal of the Nigerian Association of Mathematical Physics*, *11*, 571-576.
[52]. Oraekie, P. A. (2018). Euclidean Null Controllability of Nonlinear Infinite Neutral Systems with Multiple Delays in Control. *Journal of the Nigerian Association of Mathematical Physics*, *44*.
[53]. Guo, D., Lakshmikantham, V., & Liu, X. (2013). *Nonlinear integral equations in abstract spaces* (Vol. 373). Springer Science & Business Media.
[54]. Sundara, V. B., Raja, R., Agarwal, R. P., & Rajchakit, G. (2018). A novel controllability analysis of impulsive fractional linear time invariant systems with state delay and distributed delays in control. *Discontinuity, Nonlinearity, and Complexity*, *7*(3), 275-290.
[55]. Si-Ammour, A., Djennoune, S., & Bettayeb, M. (2009). A sliding mode control for linear fractional systems with input and state delays. *Communications in Nonlinear Science and Numerical Simulation*, *14*(5), 2310-2318.
[56]. Györi, I. (1991). On approximation of the solutions of delay differential equations by using piecewise constant arguments. *International Journal of Mathematics and Mathematical Sciences*, *14*(1), 111-126.
[57]. Brenner, S. C., & Sung, L. Y. (2017). A new convergence analysis of finite element methods for elliptic distributed optimal control problems with pointwise state constraints. *SIAM Journal on Control and Optimization*, *55*(4), 2289-2304.
[58]. Kaslik, E., & Sivasundaram, S. (2012). Nonlinear dynamics and chaos in fractional-order neural networks. *Neural Networks*, *32*, 245-256.
[59]. Li, X., & Wu, J. (2016). Sufficient stability conditions of nonlinear differential systems under impulsive control with state-dependent delay. *IEEE Transactions on Automatic Control*, *63*(1), 306-311.
[60]. Zhang, C. K., Jiang, L., Wu, Q. H., He, Y., & Wu, M. (2013). Delay-dependent robust load frequency control for time delay power systems. *IEEE Transactions on Power Systems*, *28*(3), 2192-2201.
[61]. Aestract, Y. (2007). Null controllability of non-linear infinite delay systems with impicit derivative. *Global Journal of Pure and Applied Sciences*, *13*(2), 243-247.
[62]. Sun, N.K (1996); Unified approach for constained approximate controllability for the heat equation and retarded equations,Journal of mathematical analysis and application Vol 150,pp 1-19.
[63]. Klamka, J(1991);Control of Dynamical Systems,Dondiecht.Wawer academic publishers.