Relative Controllability of Nonlinear Implicit Fractional Integro-Differential Systems in Banach Spaces with Distributive Delays in the Control

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Abstract:- In this work, Nonlinear Implicit Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control were presented for Relative Controllability analysis. General argument was found which was used to establish the relationship between the relative controllability and the intersection of the two compact and convex set functions derived from the mild solution of the system. The establishment of the relationship gives impetus to the existence of optimal control for the system as it becomes self-evidence that the intersection of the two compact and convex set functions be non-void to establish relative controllability. Thus we have established relative controllability of our system. Uses were made of the notion of the measure of non-compactness of a set and Dabos' fixed point theorem, as well as the unsymmetric Fubinis' theorem to establish the mild solution of the system .Necessary and sufficient conditions for the existence of computable criterion for the relative controllability of our system were established. The establishment was built on the usage of definition of properness of the system and the effects of the existence of zero in the interior of a reachable set of any dynamical control system.

Keywords:- *Maximization, Dabos' Fixed Point Theorem, Mild Solution, Optimal Control Relative Controllability, Set Function, Measure of Non-Compactness, Calculus of Variation.*

I. INTRODUCTION

Integrodifferential Equations arise in many fields of Science and Engineering such as Fluid dynamics,Biological models,and Chemical kinetics.A detailed investigation of Integro differential Equations have been used to model various physical phenomena such as heat conduction in materials with memory, combined conduction, convection and Radiation problems(See Caputo (1967),Olmstead and Handels man (1976),Oraekie (2018)), and numerical methods for such equations can be found in the works of Mittal and Nigam(2008) as well as Rawashdeh(2011).It is interesting to introduce a fractional derivative for these models and study their qualitative behaviours.

Controllability is one of the fundamental concepts in control theory and plays a major role in many control problems such as stablization of unstable systems by feedback or optimal control (Klamka(1993 ;Oraekie(2012).This problem can be studied by using different techniques,among which are the Fixed Point Theorem Techniques (See Balachandran and Dauer(1987),(2008)).Dacka (1980) introduced a method based on the measure of non compactness of a set and Darbos^' Fixed Piont Theorem for studing the controllabilityof a nonlinear systems with an implicit derivative.

Anichini etal (1986) addresssed the controllability problem for nonlinear systems through the notion of the measure of noncompactness,the condensing operator and the Sadovskii Fixed Point Theorem (See Sadovskii(1972),where as Balachandran and Balasubramaniam(1992) considered the same problem for Nonlinear Volterra Integro-differential systems with an Implicit derivative.Oraekie (2018) discussed the Impulsive Quasi-Linear Fractional Mixed Volterra-Fredholm-Type Integro-dfferential Systems in Banach Spaces with Multiple Delays in the Control and established Necessary and Sufficient Conditions for the existence of Optimal Control of such Systems.

Recently, Balachandran etal (20212a ,2012b and 2012c) studied the Controllability problem for various types of Nonlinear Fractional Dynamical Systems by using Fixed Point Theorems .While Oraekie(2019) studied Fractional Integro-Differential Systems with Distrbuted Delays in the Control and established Necessary and Sufficient Conditions for such Systems to be Null-Controllable.

However, no work has been reported on the Relative Controllability of Nonlinear Implicit Fractional Integro-Dfferential Systems in the existing Literature. Optimality Conditions for Relative Controllability of Nonlinear Implicit Fractional Integro-dfferential Systems with Distributed Delays in the Control is yet to be reported. In this work, therefore, we shall consider the Nonlinear Implicit Fractional Integro-dfferential Systems with Distributed Delays in the Control of the form:

$$1^{C}D^{\lambda}x(t) = Ax(t) + \int_{0}^{t} H(t-s)x(s)ds + \int_{-h}^{0} d_{\alpha}\, G(t,\alpha)u(t+\alpha)$$

$$+ f\big(t, x(t), 1^{C}D^{\lambda}x(t), u(t)\big) \qquad (1.1)$$

$$x(0) = x_0$$

With the main objective of investigating the Relative Controllability of the system and to establish the relationship between the Relative Controllability of the System(1.1) and the Intersection of the two Compact and Convex Set Functions derivable from the Mild Solution of the system(1.1)

II. NOTATION AND PRELIMINARIES

Let E denote the real line.For any integer n,E^n is the Euclidean space of n-tuples with the Euclidean norm denoted by |.| .Let J=[a,b] be any sub-interval of E, where a,b are numbers such that a<b .

A. DESCRIPTION OF SYSTEM

In this section,we consider the fractional system represented by the fractional integro-differential system with an implicit fractional derivative and distributed delays in the control given by

$$\mathbf{2}1^{C}D^{\lambda}x(t) = Ax(t) + \int_{0}^{t} H(t-s)x(s)ds + \int_{-h}^{0} d_{\alpha}\, G(t,\alpha)u(t+\alpha)$$

$$+ f\big(t, x(t), 1^{C}D^{\lambda}x(t), u(t)\big) \qquad (1.1)$$

$$x(0) = x_0$$

With the initial condition $(0) = x_0$.

Here, $0 < \lambda < 1; \in \;; x \in E^{n}; u \in E^{m}; A, H$ *are respectively nxn , nxm matrices. H is an nxn continuous matrix and the nonlinear function* $\boldsymbol{f} : \boldsymbol{J x E^n\; x\; E^n x\; E^m \to E^n}$ *is continuous, and u is an admissible square integrable* $m -$ *dimensional vector function ; with* $[\boldsymbol{u_j}] \le \mathbf{1}\; ;\boldsymbol{j} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{m}$. $(\boldsymbol{t}, \boldsymbol{\alpha})$ *is an nxm matri , continuous in t and of bounded variation in* α *on the delay interval* $[-h, 0]$; $h > 0$ *for each* $t \in [0, t_1]$, $t_1 > 0$.

In order to study this problem, we need some basic facts or concepts about Measure of noncompactness and the related fixed point theorem due toDarbo as it is contained in Krishnam Balachandran and Shanmugam Divny (2014).

➢ ***DEFINITION (2, 1. 1)***

Let $(\boldsymbol{X}, \|.\|)$*be a Banach space and S be a bounded subset of* $\boldsymbol{X}$. *Then the Measure of noncompactness of a set* $\boldsymbol{S}$ *is defined by*

$\boldsymbol{\mu(S) = inf\ \{\ r > 0 : S\ can\ be\ covered\ by\ a\ finite\ number\ of\ balls\ whose\ radii\ are\ smaller\ than\ r\ \}}$

➢ ***THEOREM 2. 1. 1. (DARBOS' FIXED POINT THEOREM)***

If M is a nonempty bounded closed convex subset of X andP is a map such that $\boldsymbol{P : M \to M}$ ***is a continuous mapping such that for any set*** $\boldsymbol{B \sqsubset M}$ ***,We have***

$$\boldsymbol{\mu(PB) \le k\mu(B)},$$

where k is a constant , $\boldsymbol{0 \le k \le 1}$ ***, then P has a fixed point. (See Dacka (1980))*** .

B. ***VARIATION OF CONSTANT FORMULA***

Consider the Fractional Integrodifferential syst(1.1) *given as*

$$1^{C}D^{\lambda}x(t) = Ax(t) + \int_{0}^{t} H(t-s)x(s)ds + \int_{-h}^{0} d_{\alpha}\, G(t,\alpha)u(t+\alpha)$$

$$+ f\big(t, x(t), 1^{C}D^{\lambda}x(t), u(t)\big) \qquad (2.2.1)$$

$$x(0) = x_0\,,$$

With its standing hypothes .

***Using Balachandran and Kokila* (2013 *a*) *like arguments as it is contained in K. Balachandran and S. Divya* (2014) , *the solution of the above system*(2. 2. 1) *can be written a*s**

$$x(t) = E_{\lambda}(t)x_0 \ + \ \int_0^t (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-1) \left[\int_{-h}^{0} d_{\alpha}\, G(t\,,\alpha)\, u(t+\ \alpha) \right]$$

$$+ \ \int_0^t (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-1) \left[f\big(s, x(s), 1^C D^{\lambda} x(s), u(s)\big) \right] ds \qquad (2.2.2)$$

A careful observation of the solution of syst(2.2.1) *given as system*(2.2.2) *shows that the values of the control* $u(t)$ *for* $t \in [-h\,, t1\,]$ *enter the definition of the complete state thereby creating the need for an explicit variation of constant formula. The control in the* 2*nd term of the Right Hand Side of yst*(2.2.2),*therefore, has to be separated in the intervals* $[-h\,,0]$*and* $[0\,, t1\,]$. *To do this,he* 2*nd term has to be transformed into the required form by applying the method of klamka*(1976) *as it is contained in Oraekie* (2018).

Firstly, we interchange the order of integration by using the Unsymmetric Fubinis' Theorem to getting :

$$x(t) = \ E_{\lambda}(t)x_0 \ + \ \int_{-h}^{0} dG_{\alpha} \int_{0+\alpha}^{t+\alpha} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) G(s-\alpha\,,\alpha) u(s-\alpha+\alpha)$$

$$+ \ \int_{\mathbf{0}}^{\boldsymbol{t}} [(\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\lambda}-\mathbf{1}} \boldsymbol{E}_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(\boldsymbol{t}-\mathbf{1})] \left[\boldsymbol{f}\big(\boldsymbol{s}, \boldsymbol{x}(\boldsymbol{s}), \mathbf{1}^{\boldsymbol{C}} \boldsymbol{D}^{\boldsymbol{\lambda}} \boldsymbol{x}(\boldsymbol{s}), \boldsymbol{u}(\boldsymbol{s})\big) \right] \boldsymbol{ds} \qquad (2.2.3)$$

***Simplifying system*(2. 2. 3), *we hav*e**

$$x(t) = \ E_{\lambda}(t)x_0 \ + \ \int_{-h}^{0} dG_{\alpha} \int_{0+\alpha}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) G(s-\alpha\,,\alpha) u_0(s) ds$$

$$+ \ \int_{-h}^{0} dG_{\alpha} \int_{0}^{t+\alpha} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) G(s-\alpha\,,\alpha) u(s)\, ds$$

$$+ \ \int_0^t [(t-s)^{\lambda-1} E_{\lambda,\lambda}(t-1)] \left[f\big(s, x(s), 1^C D^{\lambda} x(s), u(s)\big) \right] ds \qquad (2.2.4)$$

Using again the Unsymmetric Fubunis' Theorem on the change of the order of integration and incor porating $\boldsymbol{G}*$ *as defined below* :

$$\boldsymbol{G}^{*}(s-\alpha\,,\alpha) = \ \begin{cases} G(s-\alpha\,,\alpha) \; for \; s < t \\ \mathbf{0}\,, \qquad \boldsymbol{for} \quad \boldsymbol{s} \geq \boldsymbol{t}\,. \end{cases}$$

***Then, formula* (2. 2. 4) *become*s**

$$x(t) = \ E_{\lambda}(t)x_0 \ + \ \int_{-h}^{0} dG_{\alpha} \int_{\alpha}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) G(s-\alpha\,,\alpha) u_0(s) ds$$

$$+ \int_0^t [\int_{-h}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_{\alpha} G^{*}(s-\alpha\,,\alpha)] u(s)\, ds$$

$$+ \int_0^t [(t-s)^{\lambda-1} E_{\lambda,\lambda}(t-1)][f(s, x(s), 1^C D^\lambda x(s), u(s))]\, ds \qquad (2.2.4)$$

Integration is still in the Lebesgue Stieltjes sense in the variable α. For brevity, let

$$\boldsymbol{\eta(t)} = E_\lambda(t)x_0 + \int_{-h}^0 dG_\alpha \int_\alpha^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha)G(s-\alpha, \alpha)u_0(s)ds \qquad (\mathbf{2.2.5})$$

$$\boldsymbol{\beta(t)} = \int_0^t [(t-s)^{\lambda-1} E_{\lambda,\lambda}(t-1)][f(s, x(s), 1^C D^\lambda x(s), u(s))]\, ds \qquad (2.2.6)$$

$$\mathrm{Z}(t) = [\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha, \alpha)] \qquad (2.2.7)$$

Substituting systems $(\mathbf{2.2.5})$, $(\mathbf{2.2.6})$, $(\mathbf{2.2.7})$ ***into systm*** $(\mathbf{2.2.4})$,***we have***

$$x(t, x_0, u) = \eta(t) + \beta(t) + \int_0^t [\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha, \alpha)]u(s)ds$$

C. *BASIC SET FUNCTIONS and PROPERTIES*

In this section ,we define the set functions upon which our study hinges.

- ***DEFINITION 2. 3. 1 (ATTAINABLE SET)***

The attainable set for the system (1.1) *, or system*(2.2.1) *is given as* :

$$A(t_1, t_0) = \{x(t, x_0, u) : u \in U\}$$

Where $U = \{u \in L_2([0, t_1], E^m) : |u_j| \le 1 ; j = 1, 2, \dots n\}$

- ***DEFINITION 2. 3. 2 (REACHABLE SET)***

The reachable set for the system (1.1) *, or system*(2.2.1) *is given as* :

$$R(t_1, t_0) = \left\{ \int_0^{t_1} [\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha, \alpha)]u(s)ds \right\}$$

Where $u \in U$, *and* $U = \{u \in L_2([0, t_1], E^m) : |u_j| \le 1 ; j = 1, 2, \dots n\}$

- ***DEFINITION 2. 3. 3 (CONTROLLABILITY GRAMMIAN)***

The controllability grammian for the system (1.1) *, or system*(2.2.1) *is given as* :

$$\boldsymbol{W(t_1, t_0)} = \int_0^{t_1} [\int_{-h}^0 \boldsymbol{(t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha, \alpha)}] [\int_{-h}^0 \boldsymbol{(t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha, \alpha)}]^{\boldsymbol{T}}$$

Where T denotes matrix transpose.

- ***DEFINITION 2. 3. 4 (TARGET SET)***

The target set for the system (1.1) *, or system*(2.2.1) *is given as* :

$$G(t_1, t_0) = \{ x(t_1, x_0, u) : t_1 \ge \tau > t_0 = 0, \text{for some fixed } \tau \text{ and } u \in U\}$$

III. RELATIONSHIP BETWEEN SET FUNCTIONS

Let us consider the relationship between the attainable set $\boldsymbol{A}(\boldsymbol{t1}, \boldsymbol{t0})$ and the reachable set R($t1$,$t0$) firstly , with a view to establish that, once a property is proved for one set , say reachable set, then it is app*licable to the others. From syst*(2.2. 4), *attainable set* $\boldsymbol{A}(\boldsymbol{t1}, \boldsymbol{t0})$ ***is given as*** :

$$\boldsymbol{A}(\boldsymbol{t_1}, \boldsymbol{t_0}) = \boldsymbol{\mu}(\boldsymbol{t}) + \int_0^t [\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha)] u(s) ds$$

$$where\ \boldsymbol{\mu}(\boldsymbol{t}) = \eta(t) + \beta(t), \mu \in E^n$$

This means that the attainable set is the translation of the reachable set through the origin , , $\mu \in E\, n$.

Using the attainable set,therefore, it is easy to show that the set functions possess the properties of Convexity, Closedness and Compactness. and Continuous on the interval [0 , ∞)*to the Metric Subsets of* $E\, n$.

- ***LEMMA 3. 1***

If the constraint set $U \in L_2$ ([0 ,t_1], $E\, m$) , *where* $U = \{u \in L_2$ ([0 ,t_1], $E\, m$) *such that* $|uj| \leq 1$;$j = 1, 2, \ldots n\}$ *is convex so is the reachable set* $R(t_1, t_0)$.

- ***Proof.***

Recall that the solution of the system (1.1) , *or system*(2.2.1) *is* given as

$$x(t, x_0, u) = \eta(t) + \beta(t) + \int_0^t [\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha)] u(s) ds$$

While the reachable set ($t1$,0) *is given a*s

$$R(t_1, t_0) = \left\{ \int_0^t [\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha)] u(s) ds \right\}$$

We need to prove that ***if x, y*** ∈ (***t*** , ***t₀***), ***the***n $[\boldsymbol{tx} + (\boldsymbol{1}-\boldsymbol{t})\boldsymbol{y}] \in R(t_1, t_0)$ *and* $0 \leq t \leq 1$. (*by definition of convexity*).

Now,t $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}(\boldsymbol{t_1}, \boldsymbol{t_0})$, ***the***n

$$\boldsymbol{x} = \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha) \right] u_1(s) ds\ ,\ u_1 \in U. and$$

$$y = \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha) \right] u_2(s) ds, \qquad u_2 \in U$$

$$FOR\ t \in [0,1], then$$

$$\boldsymbol{tx} + (\boldsymbol{1}-\boldsymbol{t})\boldsymbol{y} = \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha) \right] \boldsymbol{t} u_1(s) ds\ +$$

$$\int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha, \alpha) \right] (\boldsymbol{1}-\boldsymbol{t}) u_2(s) ds$$

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$$= \int_0^t \left[\int_{-h}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] [\boldsymbol{tu_1(s)} + (\boldsymbol{1-t})\boldsymbol{u_2(s)}] ds$$

But U is convex*, *then there exists $\boldsymbol{v \in U}$ ***such that***

$$\boldsymbol{v} = [\boldsymbol{tu_1} + (\boldsymbol{1-t})\boldsymbol{u_2}] \in \boldsymbol{U}$$

$$\therefore, \boldsymbol{tx} + (\boldsymbol{1-t})\boldsymbol{y} = \int_0^t \left[\int_{-h}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] v(s) ds$$

The proof is made. Hence $(t_1, {}_0)$ *is convex*

- ***THEOREM 3.1***

If U is Convex and Compact,hen the Reachable set $R(t1, t0)$ is Convex, Closed and Compact

- ***PROOF***

We have established the Convexity of the Reachable Set $(\boldsymbol{t}, \boldsymbol{t_0})$ *from LEMMA*3.1.

- ***Recall***: *tha Compactness implies Closed and Bounded.*

Then we need to show tha the Reachable Set $\boldsymbol{R(t_1}, \boldsymbol{t_0})$ *is both* ***closed*** *and* ***bounded***. *Snce reachable set is asubset* E^n i. e. $(\boldsymbol{t}, \boldsymbol{t0}) \subset \boldsymbol{E^n}$, let $\boldsymbol{x_k}$ be a sequence of points in $\boldsymbol{R(t_1}, \boldsymbol{t_0})$ such that $x_k \to$ x as k $\to \infty$. *we need to show that* $x \in \boldsymbol{R(t_1}, \boldsymbol{t_0})$. Let M be a set of functions u from the interval [0 ,t] into E^m suc that u ∈ U ⊂ $\boldsymbol{L_2}$ ([0 ,t],E^m)

$$\text{i.e.}\,\mathbf{M} = \{\text{u}: [\,0\,, t\,] \to E^m\,, \quad \text{u} \in \text{U} \subset \boldsymbol{L_2}([\,0\,, t\,], E^m) \text{ such that } \|u\| \le 1\}.$$

$$\text{Then,}\,\mathbf{M} \subset \boldsymbol{L_2}([\,0\,, t\,], E^m) \text{ and so M is a set given as :}$$

$$\mathbf{M} = \{u \in U \subset \boldsymbol{L_2}([\,0\,, t\,], E^m) : \|u\| \le \beta, \beta > 0 \text{ and } \beta \in [0,1]\}$$

Thus, $\boldsymbol{M}$ *is also* ***convex*** , ***closed*** *and* ***bounded unit ball***. *But any closed ball in* $\boldsymbol{L_2}$ Space is weakly compact .Hence , any sequence $\{\boldsymbol{u_k}\} \subset \boldsymbol{M}$ *has a subsequence which converges weakly to a point* $\boldsymbol{u} \in \boldsymbol{M}$.

Hence,he reachable set $\boldsymbol{R(t_1}, \boldsymbol{t_0})$ given as

$$\boldsymbol{R(t_1, t_0)} = \int_0^t \left[\int_{-h}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u_k(s) ds$$

$$\to \int_0^t [\int_{-h}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha)] u(s) ds \text{ as k} \to \infty$$

$$Therefore, x = \int_0^t \left[\int_{-h}^{0} (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u(s) ds.$$

The proof is made. Hence $(t_1, {}_0)$ *is closed andcompact.*

- ***CONTINUITY OF SET FUNCTIONS*** (***REACHABLE SET***) ***The set functions are continuous on the interval*** [$\boldsymbol{0}$, ∞) *to the metric space of Compact subset of* E^n.

$$\text{Set}\, \boldsymbol{\mathcal{P}} = \{\, \mathbf{p} \in \boldsymbol{E^n}: \mathbf{p \text{ is compact}}, \mathbf{p} \ne \mathbf{0}\}$$

It is necessary to make the set of all non empty compact subset of E^n into ametric space by defining the distance function **d** for each x ∈ , the distance of x to p ∈ $\boldsymbol{\mathcal{P}}$ by

$$\boldsymbol{d(x,p)} = \boldsymbol{max}\{\|\boldsymbol{x-p}\| : \boldsymbol{p} \in \boldsymbol{\mathcal{P}}\}$$

where ‖x‖ is the Euclidean lenth of x ∈ .

The distance between two sets $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *is defined as :*

$$\boldsymbol{d(P,Q)} = \max_{p \in P} \boldsymbol{d(p,Q)} + \max_{q \in Q} \boldsymbol{d(q,P)}$$

If we equip $\mathcal{P}$ with the metric ,then ($\mathcal{P}$,**d**) is a metric space

➢ ***THEOREM* 3. 2**

The Reachable set (t^1 ,^{0})*is a continuous function from the interval* [0, ∞) *into* ($\mathcal{P}$, d). **i. e.** $\boldsymbol{R(t^1 , t^0)}$: [**0**, ∞) → ($\mathcal{P}$, **d**) ***is continuous***.

➢ ***PROOF***

We need to prove that given any $\varepsilon > 0$,*there exists* $\delta(\varepsilon) > 0$ *such that* $\boldsymbol{d(R(t_1 , t_0) ,R(t_2 , t_0)) < \varepsilon}$, ***whenever*** $\boldsymbol{|t_1 - t_2 | < \delta(\varepsilon)}$; ***for*** $\boldsymbol{t_1 , t_2 > 0}$, *If* $x(t , u) \in R(t ,t_0)$,*then*

$$x(t ,u) = \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha ,\alpha) \right] u(s)ds.$$

$$\Rightarrow, \qquad \|x(t_1 ,u) - x(t_2 ,u)\| = \left\| \int_{t_1}^{t_2} Y(s)u(s)\, ds \right\|,$$

$$\textbf{where } Y(s) = \int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha ,\alpha).$$

Recall that our control set U is given by

$U = \{u \in L_2 ([0 ,t_1], E^m)$ *such that* $|u_j | \leq 1$;$j = 1 , 2 , . . . n$ }.

⇒ *U is closed and bounded* ; *hence compact*.

Since there exists a positive number $\omega > 0$ *such that* $\|u\| <$, ($u \in U$, *where U is Compact set*), *it follows that*

$$\|x(t_1 ,u) - x(t_2 ,u)\| \leq \int_{t_1}^{t_2} \omega \|Y(s)\| u(s)\, ds \leq \int_{t_1}^{t_2} \omega \|Y(s)\| \|u(s)\|\, ds$$

$$\textbf{where } ,Y(s)\boldsymbol{u(s)ds} = \int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha ,\alpha) u(s)ds.$$

Since an integral is absolutely continuous, *for every* $\varepsilon > 0$, ∃ (ε) ∋ ,f

$$|t_1 - t_2| < \delta ,then \left| \omega \int_{t_1}^{t_2} \|Y(s)\| u(s)\, ds \right| < \varepsilon .$$

Consequently, *for every* $\varepsilon > 0$, *there exists* (ε) *such that if* $|t_1 - t_2 | <$,*hen* $\|x(t_1 , u) - x(t_2 , u)\| \leq \varepsilon$.

This completes proof.

IV. CONTROLLABILITY

➢ ***DEFINITION 4. 1*.** (***Relative Controllability***)

The system (1.1) , *or system*(2.2.1) *is relatively controllable on the interval* [0 ,t_1] *if*

$$\boldsymbol{A(t_1 ,t_0) \cap G(t_1 ,t_0) \neq \emptyset}, \qquad (\emptyset = \textbf{empty}) ,\boldsymbol{t_1 > t_0 = 0}.$$

➢ ***DEFINITION 4. 2*.** (***Controllability Index***)

The function g : [0 , ∞) → E^n *defined by*

$$g(t) = \boldsymbol{C^T} \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha ,\alpha) \right], \qquad C \in E^n$$

Is *called* ***the Index of the control*** *of the system* (1.1) , *or system*(2.2.1)

- ***DEFINITION. 4. 3. (PROPERNESS)***

The system (1.1) , *or system*(2.2.1) *is Proper in E n on the interval* $[0, t_1]$ *if span* $\boldsymbol{R}(\boldsymbol{t_1}, \boldsymbol{t_0}) = \boldsymbol{E}^n$.

$$\boldsymbol{i.e. if\ C^T}\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right] = \boldsymbol{0}\ \ \boldsymbol{a.e.} \Longrightarrow \boldsymbol{C} = \boldsymbol{0}, \boldsymbol{C} \in \boldsymbol{E^n}, \boldsymbol{t_1} > \boldsymbol{0}.$$

- ***PROPOSITION 4. 1.***

The system (1.1) , *or system*(2.2.1) *is Proper in E n on the interval* $[0, t_1]$ *iff*

$$\boldsymbol{C^T}\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right] = \boldsymbol{0}\ \ \boldsymbol{a.e.}$$

$$\Longrightarrow \boldsymbol{C} = \boldsymbol{0}, \boldsymbol{C} \in \boldsymbol{E^n}, \boldsymbol{t_1} > \boldsymbol{0}.$$

- ***PROOF.***

Suppose that The system (1.1) , *or system*(2.2.1) *is Proper in E n on* $[0, t_1]$, then *span* $R(t_1, t_0) = E^n, t_1 > 0$.

Since $(\boldsymbol{t}, \boldsymbol{t_0})$ is a subset of E^n , *an inner product space that is finite dimensional, we have*

$$\boldsymbol{R}(\boldsymbol{t_1}, \boldsymbol{t_0}) = \boldsymbol{R}^{\perp\perp}(\boldsymbol{t_1}, \boldsymbol{t_0}),$$

Where ⊥ *denotes orthogonal complement.*

$$\therefore, \boldsymbol{R}^{\perp\perp}(\boldsymbol{t_1}, \boldsymbol{t_0}) = E^n$$

$$Hence, \boldsymbol{R}^{\perp}(\boldsymbol{t_1}, \boldsymbol{t_0}) = \{0\}$$

Then $\{\ : \langle C, x\rangle = 0, \forall x \in \boldsymbol{R}(\boldsymbol{t_1}, \boldsymbol{t0})\} = \{\boldsymbol{0}\}$. ***From the definition of reachable set*** $(\boldsymbol{t}, \boldsymbol{t_0})$, ***this is equivalent to***

$$C^T\int_0^t\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right]u(s)ds = 0,\ \ \forall\ u \in U \Longrightarrow \boldsymbol{C} = \boldsymbol{0}$$

Since the integral is a nonnegative, we have

$$C^T\int_0^t\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right]u(s)ds = 0,\ \ a.e, s \in [0,t_1] \Longrightarrow \boldsymbol{C} = \boldsymbol{0}$$

$$Therefore, C^T\int_0^t\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right] = 0,\ \ a.e,\ on\ [0.t_1]$$

$\Longrightarrow \boldsymbol{C} = \boldsymbol{0}$.

$\Longrightarrow$ *the system* (1.1) , *or system*(2.2.1) *is Proper in E n on* $[0, t_1]$,

- ***THEOREM 4. 1***

$\boldsymbol{Consider}\ the\ system\ (1.1)\ ,\ or\ system(2.2.1)\ given\ as$

$$1^{C}D^{\lambda}x(t) = Ax(t) + \int_{0}^{t} H(t-s)x(s)ds + \int_{-h}^{0} d_{\alpha}\, G(t,\alpha)u(t+\alpha) + f\big(t,x(t),1^{C}D^{\lambda}x(t),u(t)\big) \qquad (4.1)$$

$$x(0) = x_0\ ,$$

$with\ its\ standing\ hypothes\ ,hen\ the\ following\ statements\ are\ equivalent.$

- $Syst(4.1)\ is\ relatively\ controllable\ on\ [0,t1\ ].$
- $The\ contrllability\ grammian\ (t_1\ ,_0\ )\ of\ System(4.1)\ is\ nonsingular.$
- $Syst(4.1)\ is\ Proper\ in\ E\ n\ on\ [0\ ,t_1\ ].$

- ***PROOF.***

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$

- $\boldsymbol{Recall\ that}$. $The\ contrllability\ grammian\ W(t_1\ ,t_0\ )\ of\ System(4.1)\ is\ given\ by$

$$\boldsymbol{W(t_1\,,t_0) = \int_{0}^{t_1}\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_{\alpha}G^{*}(s-\alpha\,,\alpha)\right]x}$$

$$\boldsymbol{\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_{\alpha}G^{*}(s-\alpha\,,\alpha)\right]^{T} ds}$$

$\boldsymbol{W(t_1\ ,\ t_0\ )\ is\ non-singular\ is\ equivalent\ to\ say\ that\ W(t_1\ ,\ t_0\ )\ is\ positive\ definite,which\ in\ turn\ is\ equivalent\ to\ saying\ that\ the\ transpose\ of\ some\ constant\ square\ matrix\ C\ ^T\ multiplied\ by\ the\ controllability\ index\ of\ the\ system(4.\ 1)\ is\ equal\ to\ zero\ almost\ everywhere(a.\ e),plies\ that\ C} = 0$

$$\boldsymbol{i.e.}\quad C^{T}\int_{0}^{t}\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_{\alpha}G^{*}(s-\alpha\,,\alpha)\right]u(s)ds = 0,\ \ a.e\,, \Longrightarrow \boldsymbol{C} = \boldsymbol{0}.$$

$\boldsymbol{Which\ is\ properness\ of\ system(4.\ 1)\ visa\ vis\ system(1.\ 1)\ andthis\ situation\ implies\ controllability\ the\ system(4.\ 1)\ visa\ vis\ system(1.\ 1).\ Oraekie(2019)}$

$Thus\ (ii) \Longrightarrow (\boldsymbol{i})$

$\boldsymbol{Now,it\ remains\ to\ prove\ that\ the\ statement\ (iii)\ implies\ statement\ (i)\ of\ the\ theorem\ 4.\ 1}$

$Consider\ (iii) – Proerness\ of\ syst(4.1)\ /(1.1)$

$$\boldsymbol{i.e.}\quad C^{T}\int_{0}^{t}\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_{\alpha}G^{*}(s-\alpha\,,\alpha)\right]u(s)ds = 0,\ \ a.e\,,$$

$$\boldsymbol{s} \in [0\,,t_1\,].\quad \forall\, s.then$$

$$\int_{0}^{t} C^{T}\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_{\alpha}G^{*}(s-\alpha\,,\alpha)\right]u(s)ds$$

$$= C^T \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u(s) ds = 0,$$

I**t follows from this consideration that C is orthogonal to the reachable set* $(t_1, _0)$. *If we assume the relative controllability of syst*(4.1)*visa vis system*(1.1)/(2.2.1).*then SpanR*(t_1 ,t0) = E^n *sothat C* = 0. *Showing that* (iii***) ⟹ (***i***).

- ***Conversely***,

Assume the syst(4.1) *is not controllable sothat SpanR*(t_1 ,t_0) = E^n , *for* $t_1 > 0$. *Then there exists* $C \neq 0$, *and* $C \in E^n$ *such that* $C(t_1, t_0) = 0$. It implies that for all the admissible controls u ∈ U ⊏ $E^m \subset L_2$ ([0 ,t],E n) *so that*

$$C^T \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u(s) ds = 0,$$

$$Hence, C^T R(t_1, t_0) = 0. \text{ a.e}, s \in [0, t_1] \Longrightarrow \boldsymbol{C} = \boldsymbol{0}.$$

Tis, by definition of properness of systems,his implies that systm(4.1) *or system* (1.1)*is not proper*, , *since* $C \neq 0$.

Hence the sys(4.1), *or system* (1.1)/*system*(2.2.1) *is relatively controllable. Having established that* (**ii**) ⟹ (***i***),*and* (***iii***) ⟹ (***i***), *we conclude that* (***i***) = (***ii***) = (***iii***)

This completes the proof.

- ***Theorem 4. 2***

Consider the syst(4.1)*with its standing hypothesis. we state that the syst*(4.1) *is relatively controllable iff zero is in the interior of the reachable set*

$$\boldsymbol{i.e. 0} \in \boldsymbol{Interior\ of\ R(t_1, t_0)}$$

- ***PROOF***

From ***Theorem 3. 1*** , *we realized that the reachable set* $\boldsymbol{R(t_1, t_0)}$ *is a closed and convexed subset of* E^n.

Therefo , a point $y_1 \in E\ n$ *on the boundary implies there is a* ***support plane Z*** *of* $\boldsymbol{R(t_1, t_0)}$ *through* y_1

$$i.e. C^T(y - y_1) \leq 0, \forall y \in R(t_1, t_0)$$

where $C \neq 0$ *is an outward normal to the* ***support plane Z***.

If u_1 *is the corresponding control to* y_1 *we have*

$$C^T \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u(s) ds$$

$$\leq C^T \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u_1(s) ds \qquad (4.2)$$

For every $u \in U$, *and since U is a unit sphere,he inequality* (4.2)*becomes*

$$\left| C^T \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] u(s) ds \right|$$

$$\leq C^T \int_0^t \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] |u(s)| ds$$

$$= \int_0^t C^T \left[\int_{-h}^0 (t-s)^{\lambda-1} E_{\lambda,\lambda}(t-s-\alpha) d_\alpha G^*(s-\alpha,\alpha) \right] ds$$

$$\boldsymbol{sgn}\ \ C^T\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right] \qquad (4.3)$$

Comparing the Inequalities (4.2)*d* (4.3), *we hav*e

$$u_1(s)\ \ =\boldsymbol{sgn}\ \ C^T\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right] \qquad (4.3)$$

*More so, as y*1 *is on the boundary and we always have* $0 \in$ *Interior of* $(\boldsymbol{t}\ ,\ \boldsymbol{t_0}\ )$. *If we have zero not in the interior of reachable set* $\boldsymbol{R}(\boldsymbol{t_1}\ ,\ \boldsymbol{t_0}\ )$ *,then it is on the boundary. Hence, from the preeding argument , it implies that*

$$\int_0^t C^T\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right]ds\ \ =0$$

So that

$$C^T\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right]=0\ \ ,a.e.$$

This, by the definition of properne , it implies that of the system (4.1) *is not proper since* $C^T \neq 0$.

However, if zero is in the Interior of the reachable set $(\boldsymbol{t}\ ,\ \boldsymbol{t_0}\ )$, *for* $\boldsymbol{t_1} > 0$

$$C^T\left[\int_{-h}^{0}(t-s)^{\lambda-1}E_{\lambda,\lambda}(t-s-\alpha)d_\alpha G^*(s-\alpha,\alpha)\right]=0\ \ ,a.e. \Longrightarrow \boldsymbol{C}=\boldsymbol{0}$$

which is the properness of the syst(4.1) *and by equivalence in our Theorem* 4.1,*the relative controllability of the system*(4.1)/(1.1) *On the interval* $[0,t_1\ ], t_1 > 0$ *is thus established.*

V. CONCLUSION

In this work,therefore, we have investigated and established that The Nonlinear Implicit Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control is Relatively ontrollability.

The relationship between the set functions of the systems are also established. Not alone, we have established that the reachable set of the system is a continuous function.

*It was also established that if zero is in the interior of the reachable set of the syst , then the system is relatively controllabl*e.

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