Preliminary Approach to Calculate the Gamma Function without Numerical Integration

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Abstract:- We propose a semi-statistical technique to calculate the value of the Gamma function G(x) in the whole space of positive x, i.e. 0<x=<infinity.

The proposed method is rather simple and applies a second-degree polynomial fit to find accurate values for Gamma(x) in the interval 0<x<=2.

The precision of the numerical results of the new method is striking as presented in Table I where the absolute error is limited to less than 0.01 for the interval considered (x element of]0.5]).

We recommend the application of the proposed technique for practical purposes of calculating the gamma function in order to avoid the complication of its numerical integration.

I. INTRODUCTION

The Gamma function G(x) is well defined for any positive value of x,

G(x)= Integral from z=0 to infinity [Exp(e^-z^x)]dz. . . . . (1)

Needless to say, the special function Gamma as expressed by Equation 1 has great mathematical and physical significance.

Unfortunately, it is only for integer multiples of 1/2, that the integral (1) and therefore the values of Gamma have an exact analytic expression.

Many researches have been recently published to numerically evaluate the Gamma integral (1) using approximate numerical integration methods such as Simpson and trapezoidal rules...etc [1,2,3]

However, for practical purposes, it can also be calculated using a suitable closed-form polynomial expression to avoid going through complicated numerical integration.

This is the subject of this article.

Below we propose a special statistical technique to find a polynomial fit in quadrature (just of order 2) to avoid the complications of higher order polynomials of n up to 10 for example.

The numerical results of the proposed technique are surprisingly accurate. The absolute error Ea, defined here as the proposed value of G(x) (proposed formula) minus its reference value (presented in specialized tables) falls below 0.01 for all considered values of X. Therefore, the relative error Er is easily calculated by dividing the absolute error by the reference value.

II. THEORY

Generalized factorial functions must be able to receive and return non-integer values for non-integer inputs that are not yet defined. Below we show that a suitable polynomial expression for F(x) hence G(x) in the interval x element of]0.2.] can be found. This means that the problem of finding a closed form of non-integrative expression for G(x) is solved.

The required closed form polynomial-approximate expression should retain the mathematical and physical properties of G(x), namely the conditions i-iv defined below.

The process of finding a polynomial function that represent the numerical best fit is called the method of least squares and takes the form f(x) = c0 + c1 x + c2 x^2 ⋯ +cn.x^n where n is the degree of the polynomial and c,s is a set of coefficients.

The Gamma function is a generalization of the factorial function to non-integer numbers.

Therefore, this polynomial should represent the best fit to the numerical values of the Gamma function in the interval 0<X<=2. is subjected to the following conditions:

- minimum of Gamma occurs at x = 1.4616321 and the corresponding value of Gamma(x) is 0.8856032.
- Gamma(1.)=Gamma(2.)=1.
- Gamma(x)=(x-1.) !
- The recurrence relation,Gamma (x)=(x-1) . Gamma (x-1)

We propose a simple preliminary approach other than the classical method of least-squares. The proposed preliminary approach that satisfy the following conditions [4,5].

For this objective we divide the interval 0<x< infinity into three consecutive parts a,b and c.

A. x element of]0.1]

Here, the proposed second-order polynomial expression for the Gamma function is G(x)=F(x-1) where F(x) is the factorial function x!. F is approximated by,

F(x)=(1.-0.46163*x+0.46163*x*x) . . . . (2)

x element of [0,1].

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B. x element of [1,2]

The Gamma function is approximated via the expression,

G(x)=Done(x)+0.3333/X**1.5 . . . . (3)

Where 1/3 *1/X. Sqrt(x) is a correction factor.

Note that expressions (2) and (3) solve the difficulty of establishing an approximate polynomial expression for F(x) and therefore G(x) in the interval 0<X=<2.

C. x element of [2,infinity[

We can here use the expression (4) supplemented by the expression (2) for the remaining fraction,

G(x)=F(x-1) . . . . (4)

Equations 2, 3 and 4 were implemented in a suitable algorithm which produced the required numerical results for G(x) in the interval 0<x<infinity.

In order not to worry too much about the details of the theory let us go directly to numerical results.

III. NUMERICAL RESULTS

Numerical results are presented in Table I. It presents some examples of numerical results of the proposed technique (denoted by G1(x))compared to those of the numerical tables (denoted by G2(x)) that are obtained by numerical integration of Eq. 1.

The absolute error Ea is defined here as G1(x) minus G2(x) where G1(x) is the measured value calculated by the proposed technique and G2(x) is the reference value of Gamma function found in specialized tables.

The relative error Er= [(G1(x) -G2(x)] /G1(x) can be easily calculated.

	X	G1(X)	G2(X)	Absolute
	Proposed	Tables	Error technique	[1,2,3]Ea=G1-G2
0	0.00	Infinite	Infinite	--
1	0.05	30.8088	--	--
2	0.10	11.52668	--	--
3	0.15	6.7124	--	--
4	0.20	4.6891	--	--
5	0.25	3.6161	3.5798	0.0363
6	0.30	2.9652	--	--
7	0.35	2.5340	--	--
8	0.40	2.2304	--	--
9	0.45	2.0069	--	--
10	0.50	1.8373	1.7735	0.0638
11	0.55	1.7057	--	--
12	0.60	1.6024	--	
13	0.65	1.5209	--	
14	0.70	1.4571	--	
15	0.75	1.4081	--	
16	0.80	1.3719	--	
17	0.85	1.3471	--	
18	0.90	1.3328	--	
19	0.95	1.3283	--	--
20	1.00	1.0000	1.0000	0.0000
21	1.05	0.9781	0.9735	0.0046
22	1.10	0.9585	0.9514	0.0041
23	1.15	0.9411	0.9330	0.0081
24	1.20	0.9261	0.9182	0.0089
25	1.25	0.9134	0.9085	0.0049
26	1.30	0.9031	0.8975	0.0056
27	1.35	0.8950	0.8912	0.0038
28	1.40	0.8892	0.8873	0.0016
29	1.45	0.8857	0.8857	0.0000
30	1.50	0.8846	0.8862	-0.0016
31	1.55	0.8857	0.8889	-0.0032
32	1.60	0.8892	0.8935	-0.0043
33	1.65	0.8950	0.9001	-0.0051
34	1.70	0.9031	0.9086	-0.0035
35	1.75	0.9134	0.9191	-0.0043
36	1.80	0.9261	0.9314	-0.0047

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37	1.85	0.9411	0.9456	-0.0045
38	1.90	0.9585	0.9618	-0.0033
39	1.95	0.9781	0.9799	-0.0018
40	2.00	1.0000	1.0000	0.0000
41	2.05	1.0242		
42	2.10	1.0508		
43	2.15	1.0796		
44	2.20	1.1108		
45	2.25	1.1443		
46	2.30	1.1800		
47	2.35	1.2181		
48	2.40	1.2585		
49	2.45	1.3012		
50	2.50	1.3462	1.3293	0.0171
51	2.55	1.3935		
52	2.60	1.4432		
53	2.65	1.4951		
54	2.70	1.5493		
55	2.75	1.6059		
56	2.8	1.6647		
57	2.85	1.7259		
58	2.90	1.7894		
59	2.95	1.8552		
60	3.00	2.0000	2.0000	0.0000
61	3.05	2.1053		
62	3.10	2.2140		
63	3.15	2.3270		
64	3.20	2.4450		
65	3.25	2.5691		
66	3.3	2.7001		
67	3.35	2.8393		
68	3.40	2.9877		
69	3.45	3.1466		
70	3.50	3.3172	3.3234	-0.0062
71	3.55	3.5009		
72	3.60	3.6991		
73	3.65	3.91330		
74	3.70	4.1450		
75	3.75	4.3960		
76	3.8	4.6677		
77	3.85	4.9622		
78	3.90	5.2811		
79	3.95	5.6264		
80	4.00	6.0000	6.0000	0.0000
81	4.05	6.4212		
82	4.10	6.8635		
83	4.15	7.3300		
84	4.20	7.8240		
85	4.25	8.3495		
86	4.30	8.9105		
87	4.35	9.5117		
88	4.40	10.1583		
89	4.45	10.8558		
90	4.50	11.6101		

Table 1: Some examples of numerical results of the proposed technique compared to those of the numerical tables of ref.1,2,3

A portion of the numerical results of Table I is shown schematically in Figure 1 below.

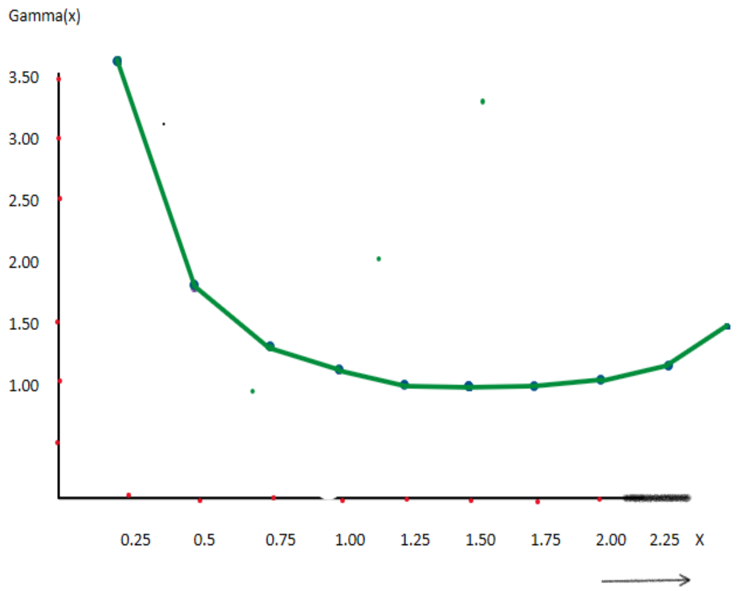

Fig 1: Part of the Gamma function G(x) vs x.

IV. CONCLUSION

We propose a semi-statistical technique to calculate the value of the Gamma function G(x) in the whole space of positive x, i.e. 0<x=<infinity.

The proposed method is simple and applies a second-degree polynomial fit to find accurate values for Gamma(x) in the interval 0<x<=2.

The precision of the numerical results of the new method is striking as shown in Table I where the absolute error is limited to less than 0.01 for the interval considered (x element of]0.5]).

We recommend the application of the proposed technique for practical cases of calculation of the gamma function in order to avoid the complication of its numerical integration.

NB. All calculations in this article were produced using the author's double-precision algorithm to ensure maximum accuracy, as follows by ref. 6 for example

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