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Best Rational Approximation of πe Using Simple Continued Fractions

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Abstract:- The best rational approximation of a real number are rational numbers that are closest to the real number compared to other rational numbers with the same or smaller denominator. One of methods to find the best rational approximation is using simple continued fractions. Simple continued fraction expansion is a representation of a number that is produced through an iterative process of adding the largest integer that is less than that number, with the inverse multiplication of other numbers, with the other number is also the result of adding the largest integer that is less than it, with the inverse multiplication of another number. This research will focus on finding the best rational approximation of πe by using multiplication of simple continued fraction. The result obtained is best rational approximation of πe with 15 decimal places precision.

Keywords: *Best Rational Approximation, Simple Continued Fraction, Irrational Number, Multiplication.*

I. INTRODUCTION

Continued fractions are expressions of numbers into a stack of fractions using a special algorithm, making continued fractions called fractions expanded by fractions. Continued fractions have been used for representing irrational numbers since ages, but were only introduced in the 16th century by two mathematicians, Rafael Bombelli and Pietro Antonio Cataldi. Continued fractions were first used by Christiaan Huygens, a Dutch mathematician, physicist and astronomer, for designing a planetarium, which was displaying the motion of the planets around the sun. In designing the model of the solar system, Huygens used toothed wheels in moving the model of the planets. Huygens was then faced with the problem of determining the number of teeth, so that the ratios of the two interconnected wheels equal the ratios of the periods of the two planets. Due to the finite number of teeth on the wheels, Huygens used continued fractions to approximate the ratio of the periods of two planets so that he could find a ratio that satisfied the number of wheels.[1]

As approximating the ratio of two gears, finding rational approximation using continued fractions is the most efficient way. In 2017, N. A. Carella proved that the product e and π is irrational, with its proof is also using simple continued fractions. In 2019, S. Mugassabi and S. M. Amsheri researched the multiplication and division of two simple continued fractions. Because of that, the writer is interested in finding the rational approximation of the multiplication πe.

II. LITERATURE REVIEW

- *Simple Continued Fraction:*

Continued fractions are mathematical expressions in the form of

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}}} \tag{1}$$

where $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$are real numbers. If $b_n = 1$ for $n = 0, 1, 2, \ldots$, $a_0$is an integer and a_n is a positive integer for $n = 1, 2, 3, \ldots$, then it is called a simple continued fraction. $a_0, a_1, a_2, \ldots$ are then called the partial quotient of the continued fraction.[9]

- *The Algorithm of Simple Continued Fraction Expansion:*

Suppose α is a real number, there exists uniquely an integer, which is the integer part of , and a real number $x_0 \in [0, 1)$, such that,

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$$\alpha = [\alpha] + x_0 \tag{2}$$

If α is not an integer, such that $x_0 \neq 0$ and $y_1 := \frac{1}{x_0}$, then

$$\alpha = [\alpha] + \frac{1}{y_1} \tag{3}$$

Furthermore, if $y_1 = [y_1] + x_1$ is not an integer, so that $x_1 \neq 0$ and $y_2 := \frac{1}{x_1}$, we get

$$\alpha = [\alpha] + \cfrac{1}{[y_1] + \cfrac{1}{y_2}} \tag{4}$$

This process repeats until y_i is an integer and $x_i = 0$. By replacing $a_0 := [\alpha]$ and $a_i := [y_i], i = 1, 2, ...$, we get simple continued fraction expansion of α ,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{5}$$

which can then be denoted as $\alpha = [a_0;\ a_1, a_2, a_3, ...]$.[11]

- *Convergent:*

Suppose α is a real number, where $\alpha = [a_0;\ a_1, a_2, a_3, ...]$, $C_k := \frac{p_k}{q_k}$ is called convergent of α , if it satisfies

$$\frac{p_k}{q_k} = [a_0;\ a_1,\ a_2,\ \ldots,\ a_k] \tag{6}$$

where $k = 0, 1, 2, ...$[10]

- *Intermediate Fraction:*

Suppose α is a real number, the intermediate fraction of α, which is denoted by $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$, is defined by

$$p_{i+1}^{(j)} = j \cdot p_i + p_{i-1} \tag{7}$$

$$q_{i+1}^{(j)} = j \cdot q_i + q_{i-1} \tag{8}$$

where $j \in \{1, 2, ..., a_{i+1} - 1\}$.[4]

➢ Best Rational Approximation:

There are two types of best rational approximation, i.e., best rational approximation of the first kind and best rational approximation of the second kind.

- *Best Rational Approximation of The First Kind*:

Suppose α is a real number, $\frac{p}{q}$ is the best rational approximation of the first kind if for every $\frac{r}{s}$ satisfies

$$\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{r}{s}\right| \tag{9}$$

where $\frac{r}{s} \neq \frac{p}{q}, 0 < s \leq q$.[4]

- *Best Rational Approximation of The Second Kind:*
  Suppose α is a real number, $\frac{p}{q}$ is the best rational approximation of the second kind if for every $\frac{r}{s}$ satisfies

$$\left|q\alpha - p\right| < \left|s\alpha - r\right| \tag{10}$$

  where $\frac{r}{s} \neq \frac{p}{q}, 0 < s \leq q$.[4]

Theorem

Every best approximation of a number is a convergent or an intermediate fraction of the continued fraction representing that number.[3]

➢ The Multiplication of Simple Continued Fractions:

Suppose $[a_0;\ a_1, a_2, \dots, a_n\ ]$, an are simple continued fractions, $K_i(a_j) = a_{i+(j-1)}K_{i-1}(a_j) + K_{i-2}(a_j)$ where $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$, and $K_{-i}(a_j) = 0, K_0(a_j) = 1$, then $[a_0;\ a_1, a_2, \dots, a_n\ ]$, an can be defined as $[a_0;\ a_1, a_2, \dots, a_n\ ] = a_0 + \frac{K_{n-1}(a_2)}{K_n(a_1)}$ or $[a_0;\ a_1, a_2, \cdots, a_n\ ] = \frac{K_{n+1}(a_0)}{K_n(a_1)}$. From the definitions, convergents as well can be written in $K_i(a_j)$, that is, $K_i(a_0) = p_{i-1}$and $K_i(a_1) = q_i$.[8]

Suppose $[a_0;\ a_1, a_2, a_3, \dots, a_m]$ and $[b_0;\ b_1, b_2, b_3, \dots, b_n]$ are simple continued fractions with non-negative a_0 and b_0, so the multiplication of the two simple continued fractions is defined:

- if $m = n$, then

$$[a_0;\ a_1, a_2, a_3, \dots, a_n] \times [b_0;\ b_1, b_2, b_3, \dots, b_n] = [d_0;\ d_1, d_2, d_3, \dots, d_n], \text{ where}$$

$$d_0 = a_0 b_0$$

$$d_1 = \left[\frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1}\right]$$

$$d_2 = \left[\frac{a_0 b_2(a_1 a_2 + 1) + a_2 b_0(b_1 b_2 + 1) + a_2 b_2}{(a_1 a_2 + 1)(b_1 b_2 + 1) - d_1\left[a_0 b_2(a_1 a_2 + 1) + a_2 b_0(b_1 b_2 + 1) + a_2 b_2\right]}\right]$$

$$\vdots$$

$$d_i = \begin{cases} \left[\dfrac{K_i(a_1)K_i(b_1)K_{i-3}(d_2) - \left[a_0 K_i(a_1)K_{i-1}(b_2) + b_0 K_{i-1}(a_2)K_i(b_1) + K_{i-1}(a_2)K_{i-1}(b_2)\right]K_{i-2}(d_1)}{\left[a_0 K_i(a_1)K_{i-1}(b_2) + b_0 K_{i-1}(a_2)K_i(b_1) + K_{i-1}(a_2)K_{i-1}(b_2)\right]K_{i-1}(d_1) - K_i(a_1)K_i(b_1)K_{i-2}(d_2)}\right], & \text{if } i \text{ odd} \\ \left[\dfrac{\left[a_0 K_i(a_1)K_{i-1}(b_2) + b_0 K_{i-1}(a_2)K_i(b_1) + K_{i-1}(a_2)K_{i-1}(b_2)\right]K_{i-2}(d_1) - K_i(a_1)K_i(b_1)K_{i-3}(d_2)}{K_i(a_1)K_i(b_1)K_{i-2}(d_2) - \left[a_0 K_i(a_1)K_{i-1}(b_2) + b_0 K_{i-1}(a_2)K_i(b_1) + K_{i-1}(a_2)K_{i-1}(b_2)\right]K_{i-1}(d_1)}\right], & \text{if } i \text{ even} \end{cases}$$

for $i = 2, 3, \dots, n$.

- if $m \neq n$ $(m < n)$, then $[a_0;\ a_1, a_2, a_3, \dots, a_m] \times [b_0;\ b_1, b_2, b_3, \dots, b_m, b_{m+1}, \dots, b_n] = [d_0;\ d_1, d_2, d_3, \dots, d_n]$, where d_j for $j = 0, 1, 2, \dots, m$ is the same as d_j for case $m = n$, and for $j = m + 1, m + 2, \dots, n$,

$$d_j = \begin{cases} \left[\dfrac{K_m(a_1)K_j(b_1)K_{j-3}(d_2) - \left[a_0 K_m(a_1)K_{j-1}(b_2) + b_0 K_{m-1}(a_2)K_j(b_1) + K_{m-1}(a_2)K_{j-1}(b_2)\right]K_{j-2}(d_1)}{\left[a_0 K_m(a_1)K_{j-1}(b_2) + b_0 K_{m-1}(a_2)K_j(b_1) + K_{m-1}(a_2)K_{j-1}(b_2)\right]K_{j-1}(d_1) - K_m(a_1)K_j(b_1)K_{j-2}(d_2)}\right], & \text{if } j \text{ odd} \\ \left[\dfrac{\left[a_0 K_m(a_1)K_{j-1}(b_2) + b_0 K_{m-1}(a_2)K_j(b_1) + K_{m-1}(a_2)K_{j-1}(b_2)\right]K_{j-2}(d_1) - K_m(a_1)K_j(b_1)K_{j-3}(d_2)}{K_m(a_1)K_j(b_1)K_{j-2}(d_2) - \left[a_0 K_m(a_1)K_{j-1}(b_2) + b_0 K_{m-1}(a_2)K_j(b_1) + K_{m-1}(a_2)K_{j-1}(b_2)\right]K_{j-1}(d_1)}\right], & \text{if } j \text{ even} \end{cases}$$

For the two cases above, $d_1, d_2, \dots, d_{n-1}$ are rounded off using integer part, while d_n is expanded into a simple continued fraction.[6]

III. RESULTS

➢ *Simple Continued Fraction of π and e:*

Irrational numbers π and e are $\mathbf{3.141592653589793...}$ and $\mathbf{2.718281828459045...}$ consecutively. Using the algorithm of simple continued fraction expansion, we get

$$3.141592653589793\ldots$$

$$= 3 + 0.141592653589793\ldots$$

$$= 3 + \cfrac{1}{7.062513305931045\ldots}$$

$$= 3 + \cfrac{1}{7 + 0.062513305931045\ldots}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15.996594406685719\ldots}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + 0.996594406685719\ldots}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1.003417231013372\ldots}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + 0.003417231013372\ldots}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292.634591014395472\ldots}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + 0.634591014395472\ldots}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + 0.634591014395472\ldots}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1.575818089628365\ldots}}}}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + 0.575818089628365\ldots}}}}}$$

$$2.718281828459045\ldots$$

$$= 2 + 0.718281828459045\ldots$$

$$= 2 + \cfrac{1}{1.392211191177332\ldots}$$

$$= 2 + \cfrac{1}{1 + 0.392211191177332\ldots}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2.549646778303844\ldots}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + 0.549646778303844\ldots}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1.819350243598079\ldots}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + 0.819350243598079\ldots}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1.220479285645437\ldots}}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + 0.220479285645437\ldots}}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4.535573476086741\ldots}}}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + 0.535573476086741\ldots}}}}}$$

$\pi = [3;\ 7, 15, 1, 292, 1, \ldots]$ and $e = [2;\ 1, 2, 1, 1, 4, \ldots]$.

➢ *Multiplication of Simple Continued Fractions of π and e:*

Multiplication of simple continued fractions only multiplies finite continued fractions. Hence, irrational numbers π and e will be chopped to 15 decimal places, with the expansion of their simple continued fractions are

$$[3;\ 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 4, 2, 3, 1, 12, 5, 1, 5, 20, 1, 11, 1, 1, 1, 2] \text{ and } [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 10, 2, 1, 3, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 7, 4, 1, 8].$$

Suppose $[3;\ 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 4, 2, 3, 1, 12, 5, 1, 5, 20, 1, 11, 1, 1, 1, 2] = [a_0;\ a_1, a_2, \ldots, a_{27}]$ and $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 10, 2, 1, 3, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 7, 4, 1, 8] = [b_0;\ b_1, b_2, \ldots, b_{37}]$, so $27 = m < n = 37$.

$[a_0;\ a_1, a_2, \ldots, a_{27}] \times [b_0;\ b_1, b_2, \ldots, b_{37}] = [d_0;\ d_1, d_2, \ldots, d_{37}]$,

where

$$d_0 = a_0 b_0 = 3 \cdot 2 = 6$$

$$d_1 = \left[\frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1}\right] = \left[\frac{7 \cdot 1}{3 \cdot 7 + 2 \cdot 1 + 1}\right] = \left[\frac{7}{24}\right] = 0,$$

$$d_2 = \left[\frac{a_0 b_2 (a_1\, a_2 + 1) + a_2 b_0 (b_1\, b_2 + 1) + a_2 b_2}{(a_1\, a_2 + 1)(b_1\, b_2 + 1) - d_1 [a_0 b_2 (a_1\, a_2 + 1) + a_2 b_0 (b_1\, b_2 + 1) + a_2 b_2]}\right] = \left[\frac{756}{318}\right] = 2$$

$$d_3 = \left[\frac{K_3(a_1)K_3(b_1) - [a_0 K_3(a_1)K_2(b_2) + b_0 K_2(a_2)K_3(b_1) + K_2(a_2)K_2(b_2)]d_1}{[a_0 K_3(a_1)K_2(b_2) + b_0 K_2(a_2)K_3(b_1) + K_2(a_2)K_2(b_2)]K_2(d_1) - K_3(a_1)K_3(b_1)d_2}\right] = \left[\frac{452}{289}\right] = 1$$

$$d_4 = \left[\frac{[a_0 K_4(a_1)K_3(b_2) + b_0 K_3(a_2)K_4(b_1) + K_3(a_2)K_3(b_2)]K_2(d_1) - K_4(a_1)K_4(b_1)d_2}{K_4(a_1)K_4(b_1)K_2(d_1) - [a_0 K_4(a_1)K_3(b_2) + b_0 K_3(a_2)K_4(b_1) + K_3(a_2)K_3(b_2)]K_3(d_1)}\right] = \left[\frac{122155}{109559}\right] = 1$$

$$d_5 = \left[\frac{K_5(a_1)K_5(b_1)K_2(d_2) - [a_0 K_5(a_1)K_4(b_2) + b_0 K_4(a_2)K_5(b_1) + K_4(a_2)K_4(b_2)]K_3(d_1)}{[a_0 K_5(a_1)K_4(b_2) + b_0 K_4(a_2)K_5(b_1) + K_4(a_2)K_4(b_2)]K_4(d_1) - K_5(a_1)K_5(b_1)K_3(d_2)}\right] = \left[\frac{487644}{87592}\right] = 5$$

$$d_6 = \left[\frac{[a_0 K_6(a_1)K_5(b_2) + b_0 K_5(a_2)K_6(b_1) + K_5(a_2)K_5(b_2)]K_4(d_1) - K_6(a_1)K_6(b_1)K_3(d_2)}{K_6(a_1)K_6(b_1)K_4(d_1) - [a_0 K_6(a_1)K_5(b_2) + b_0 K_5(a_2)K_6(b_1) + K_5(a_2)K_5(b_2)]K_5(d_1)}\right] = \left[\frac{200121}{192516}\right] = 1$$

$$d_7 = \left[\frac{K_7(a_1)K_7(b_1)K_4(d_2) - [a_0 K_7(a_1)K_6(b_2) + b_0 K_6(a_2)K_7(b_1) + K_6(a_2)K_6(b_2)]K_5(d_1)}{[a_0 K_7(a_1)K_6(b_2) + b_0 K_6(a_2)K_7(b_1) + K_6(a_2)K_6(b_2)]K_6(d_1) - K_7(a_1)K_7(b_1)K_5(d_2)}\right] = \left[\frac{437821}{125009}\right] = 3$$

$$d_8 = \left[\frac{[a_0 K_8(a_1)K_7(b_2) + b_0 K_7(a_2)K_8(b_1) + K_7(a_2)K_7(b_2)]K_6(d_1) - K_8(a_1)K_8(b_1)K_5(d_2)}{K_8(a_1)K_8(b_1)K_6(d_1) - [a_0 K_8(a_1)K_7(b_2) + b_0 K_7(a_2)K_8(b_1) + K_7(a_2)K_7(b_2)]K_7(d_1)}\right] = \left[\frac{2030293}{1683655}\right] = 1$$

$$d_9 = \left[\frac{K_9(a_1)K_9(b_1)K_6(d_2) - [a_0 K_9(a_1)K_8(b_2) + b_0 K_8(a_2)K_9(b_1) + K_8(a_2)K_8(b_2)]K_7(d_1)}{[a_0 K_9(a_1)K_8(b_2) + b_0 K_8(a_2)K_9(b_1) + K_8(a_2)K_8(b_2)]K_8(d_1) - K_9(a_1)K_9(b_1)K_7(d_2)}\right] = \left[\frac{2545336}{704744}\right] = 3$$

$$d_{10} = \left[\frac{[a_0 K_{10}(a_1)K_9(b_2) + b_0 K_9(a_2)K_{10}(b_1) + K_9(a_2)K_9(b_2)]K_8(d_1) - K_{10}(a_1)K_{10}(b_1)K_7(d_2)}{K_{10}(a_1)K_{10}(b_1)K_8(d_1) - [a_0 K_{10}(a_1)K_9(b_2) + b_0 K_9(a_2)K_{10}(b_1) + K_9(a_2)K_9(b_2)]K_9(d_1)}\right] = \left[\frac{4403329}{4906103}\right] = 0$$

$$d_{11} = \left[\frac{K_{11}(a_1)K_{11}(b_1)K_8(d_2) - [a_0 K_{11}(a_1)K_{10}(b_2) + b_0 K_{10}(a_2)K_{11}(b_1) + K_{10}(a_2)K_{10}(b_2)]K_9(d_1)}{[a_0 K_{11}(a_1)K_{10}(b_2) + b_0 K_{10}(a_2)K_{11}(b_1) + K_{10}(a_2)K_{10}(b_2)]K_{10}(d_1) - K_{11}(a_1)K_{11}(b_1)K_9(d_2)}\right] = \left[\frac{51817007}{48009249}\right] = 1$$

$$d_{12} = \left[\frac{[a_0 K_{12}(a_1)K_{11}(b_2) + b_0 K_{11}(a_2)K_{12}(b_1) + K_{11}(a_2)K_{11}(b_2)]K_{10}(d_1) - K_{12}(a_1)K_{12}(b_1)K_9(d_2)}{K_{12}(a_1)K_{12}(b_1)K_{10}(d_1) - [a_0 K_{12}(a_1)K_{11}(b_2) + b_0 K_{11}(a_2)K_{12}(b_1) + K_{11}(a_2)K_{11}(b_2)]K_{11}(d_1)}\right] = \left[\frac{792572266}{65740966}\right] = 12$$

$$d_{13} = \left[\frac{K_{13}(a_1)K_{13}(b_1)K_{10}(d_2) - [a_0 K_{13}(a_1)K_{12}(b_2) + b_0 K_{12}(a_2)K_{13}(b_1) + K_{12}(a_2)K_{12}(b_2)]K_{11}(d_1)}{[a_0 K_{13}(a_1)K_{12}(b_2) + b_0 K_{12}(a_2)K_{13}(b_1) + K_{12}(a_2)K_{12}(b_2)]K_{12}(d_1) - K_{13}(a_1)K_{13}(b_1)K_{11}(d_2)}\right] = \left[\frac{496460573}{154296889}\right] = 3$$

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$$d_{14}=\left[\frac{[a_0K_{14}(a_1)K_{13}(b_2)+b_0K_{13}(a_2)K_{14}(b_1)+K_{13}(a_2)K_{13}(b_2)]K_{12}(d_1)-K_{14}(a_1)K_{14}(b_1)K_{11}(d_2)}{K_{14}(a_1)K_{14}(b_1)K_{12}(d_1)-[a_0K_{14}(a_1)K_{13}(b_2)+b_0K_{13}(a_2)K_{14}(b_1)+K_{13}(a_2)K_{13}(b_2)]K_{13}(d_1)}\right]=\left[\frac{3498891240}{1253615760}\right]=2$$

$$d_{15}=\left[\frac{K_{15}(a_1)K_{15}(b_1)K_{12}(d_2)-[a_0K_{15}(a_1)K_{14}(b_2)+b_0K_{14}(a_2)K_{15}(b_1)+K_{14}(a_2)K_{14}(b_2)]K_{13}(d_1)}{[a_0K_{15}(a_1)K_{14}(b_2)+b_0K_{14}(a_2)K_{15}(b_1)+K_{14}(a_2)K_{14}(b_2)]K_{14}(d_1)-K_{15}(a_1)K_{15}(b_1)K_{13}(d_2)}\right]=\left[\frac{4578789100}{4090913104}\right]=1$$

$$d_{16}=\left[\frac{[a_0K_{16}(a_1)K_{15}(b_2)+b_0K_{15}(a_2)K_{16}(b_1)+K_{15}(a_2)K_{15}(b_2)]K_{14}(d_1)-K_{16}(a_1)K_{16}(b_1)K_{13}(d_2)}{K_{16}(a_1)K_{16}(b_1)K_{14}(d_1)-[a_0K_{16}(a_1)K_{15}(b_2)+b_0K_{15}(a_2)K_{16}(b_1)+K_{15}(a_2)K_{15}(b_2)]K_{15}(d_1)}\right]=\left[\frac{9686506201}{1793948059}\right]=5$$

$$d_{17}=\left[\frac{K_{17}(a_1)K_{17}(b_1)K_{14}(d_2)-[a_0K_{17}(a_1)K_{16}(b_2)+b_0K_{16}(a_2)K_{17}(b_1)+K_{16}(a_2)K_{16}(b_2)]K_{15}(d_1)}{[a_0K_{17}(a_1)K_{16}(b_2)+b_0K_{16}(a_2)K_{17}(b_1)+K_{16}(a_2)K_{16}(b_2)]K_{16}(d_1)-K_{17}(a_1)K_{17}(b_1)K_{15}(d_2)}\right]=\left[\frac{283054812672}{137102857708}\right]=2$$

$$d_{18}=\left[\frac{[a_0K_{18}(a_1)K_{17}(b_2)+b_0K_{17}(a_2)K_{18}(b_1)+K_{17}(a_2)K_{17}(b_2)]K_{16}(d_1)-K_{18}(a_1)K_{18}(b_1)K_{15}(d_2)}{K_{18}(a_1)K_{18}(b_1)K_{16}(d_1)-[a_0K_{18}(a_1)K_{17}(b_2)+b_0K_{17}(a_2)K_{18}(b_1)+K_{17}(a_2)K_{17}(b_2)]K_{17}(d_1)}\right]=\left[\frac{742746713752}{68320249726}\right]=10$$

$$d_{19}=\left[\frac{K_{19}(a_1)K_{19}(b_1)K_{16}(d_2)-[a_0K_{19}(a_1)K_{18}(b_2)+b_0K_{18}(a_2)K_{19}(b_1)+K_{18}(a_2)K_{18}(b_2)]K_{17}(d_1)}{[a_0K_{19}(a_1)K_{18}(b_2)+b_0K_{18}(a_2)K_{19}(b_1)+K_{18}(a_2)K_{18}(b_2)]K_{18}(d_1)-K_{19}(a_1)K_{19}(b_1)K_{17}(d_2)}\right]=\left[\frac{13556049441}{366746024552}\right]=0$$

$$d_{20}=\left[\frac{[a_0K_{20}(a_1)K_{19}(b_2)+b_0K_{19}(a_2)K_{20}(b_1)+K_{19}(a_2)K_{19}(b_2)]K_{18}(d_1)-K_{20}(a_1)K_{20}(b_1)K_{17}(d_2)}{K_{20}(a_1)K_{20}(b_1)K_{18}(d_1)-[a_0K_{20}(a_1)K_{19}(b_2)+b_0K_{19}(a_2)K_{20}(b_1)+K_{19}(a_2)K_{19}(b_2)]K_{19}(d_1)}\right]=\left[\frac{21817281525737}{8387820345631}\right]=2$$

$$d_{21}=\left[\frac{K_{21}(a_1)K_{21}(b_1)K_{18}(d_2)-[a_0K_{21}(a_1)K_{20}(b_2)+b_0K_{20}(a_2)K_{21}(b_1)+K_{20}(a_2)K_{20}(b_2)]K_{19}(d_1)}{[a_0K_{21}(a_1)K_{20}(b_2)+b_0K_{20}(a_2)K_{21}(b_1)+K_{20}(a_2)K_{20}(b_2)]K_{20}(d_1)-K_{21}(a_1)K_{21}(b_1)K_{19}(d_2)}\right]=\left[\frac{354344404522903}{214649437154052}\right]=1$$

$$d_{22}=\left[\frac{[a_0K_{22}(a_1)K_{21}(b_2)+b_0K_{21}(a_2)K_{22}(b_1)+K_{21}(a_2)K_{21}(b_2)]K_{20}(d_1)-K_{22}(a_1)K_{22}(b_1)K_{19}(d_2)}{K_{22}(a_1)K_{22}(b_1)K_{20}(d_1)-[a_0K_{22}(a_1)K_{21}(b_2)+b_0K_{21}(a_2)K_{22}(b_1)+K_{21}(a_2)K_{21}(b_2)]K_{21}(d_1)}\right]=\left[\frac{332029152384101}{217463566431841}\right]=1$$

$$d_{23}=\left[\frac{K_{23}(a_1)K_{23}(b_1)K_{20}(d_2)-[a_0K_{23}(a_1)K_{22}(b_2)+b_0K_{22}(a_2)K_{23}(b_1)+K_{22}(a_2)K_{22}(b_2)]K_{21}(d_1)}{[a_0K_{23}(a_1)K_{22}(b_2)+b_0K_{22}(a_2)K_{23}(b_1)+K_{22}(a_2)K_{22}(b_2)]K_{22}(d_1)-K_{23}(a_1)K_{23}(b_1)K_{21}(d_2)}\right]=\left[\frac{9550392737833532}{5048455425835407}\right]=1$$

$$d_{24}=\left[\frac{[a_0K_{24}(a_1)K_{23}(b_2)+b_0K_{23}(a_2)K_{24}(b_1)+K_{23}(a_2)K_{23}(b_2)]K_{22}(d_1)-K_{24}(a_1)K_{24}(b_1)K_{21}(d_2)}{K_{24}(a_1)K_{24}(b_1)K_{22}(d_1)-[a_0K_{24}(a_1)K_{23}(b_2)+b_0K_{23}(a_2)K_{24}(b_1)+K_{23}(a_2)K_{23}(b_2)]K_{23}(d_1)}\right]=\left[\frac{6954763809533449}{6211394778067234}\right]=1$$

$$d_{25}=\left[\frac{K_{25}(a_1)K_{25}(b_1)K_{22}(d_2)-[a_0K_{25}(a_1)K_{24}(b_2)+b_0K_{24}(a_2)K_{25}(b_1)+K_{24}(a_2)K_{24}(b_2)]K_{23}(d_1)}{[a_0K_{25}(a_1)K_{24}(b_2)+b_0K_{24}(a_2)K_{25}(b_1)+K_{24}(a_2)K_{24}(b_2)]K_{24}(d_1)-K_{25}(a_1)K_{25}(b_1)K_{23}(d_2)}\right]=\left[\frac{33267013815497016}{3997453989998394}\right]=8$$

$$d_{26}=\left[\frac{[a_0K_{26}(a_1)K_{25}(b_2)+b_0K_{25}(a_2)K_{26}(b_1)+K_{25}(a_2)K_{25}(b_2)]K_{24}(d_1)-K_{26}(a_1)K_{26}(b_1)K_{23}(d_2)}{K_{26}(a_1)K_{26}(b_1)K_{24}(d_1)-[a_0K_{26}(a_1)K_{25}(b_2)+b_0K_{25}(a_2)K_{26}(b_1)+K_{25}(a_2)K_{25}(b_2)]K_{25}(d_1)}\right]=\left[\frac{14325582964057850}{4686747979327547}\right]=3$$

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$$d_{27} = \left[\frac{K_{27}(a_1)K_{27}(b_1)K_{24}(d_2) - [a_0K_{27}(a_1)K_{26}(b_2)+b_0K_{26}(a_2)K_{27}(b_1) + K_{26}(a_2)K_{26}(b_2)]K_{25}(d_1)}{[a_0K_{27}(a_1)K_{26}(b_2)+b_0K_{26}(a_2)K_{27}(b_1) + K_{26}(a_2)K_{26}(b_2)]K_{26}(d_1) - K_{27}(a_1)K_{27}(b_1)K_{25}(d_2)}\right]$$
$$= \left[\frac{17657947089174719}{1251900564178360}\right] = 14$$

$$d_{28} = \left[\frac{[a_0K_{27}(a_1)K_{27}(b_2)+b_0K_{26}(a_2)K_{28}(b_1) + K_{26}(a_2)K_{27}(b_2)]K_{26}(d_1) - K_{27}(a_1)K_{28}(b_1)K_{25}(d_2)}{K_{27}(a_1)K_{28}(b_1)K_{26}(d_1) - [a_0K_{27}(a_1)K_{27}(b_2)+b_0K_{26}(a_2)K_{28}(b_1) + K_{26}(a_2)K_{27}(b_2)]K_{27}(d_1)}\right]$$
$$= \left[\frac{3209035617509080}{2846124441830387}\right] = 1$$

$$d_{29} = \left[\frac{K_{27}(a_1)K_{29}(b_1)K_{26}(d_2) - [a_0K_{27}(a_1)K_{28}(b_2)+b_0K_{26}(a_2)K_{29}(b_1) + K_{26}(a_2)K_{28}(b_2)]K_{27}(d_1)}{[a_0K_{27}(a_1)K_{28}(b_2)+b_0K_{26}(a_2)K_{29}(b_1) + K_{26}(a_2)K_{28}(b_2)]K_{28}(d_1) - K_{27}(a_1)K_{29}(b_1)K_{27}(d_2)}\right]$$
$$= \left[\frac{2977463632508066}{1483472549179374}\right] = 2$$

$$d_{30} = \left[\frac{[a_0K_{27}(a_1)K_{29}(b_2)+b_0K_{26}(a_2)K_{30}(b_1) + K_{26}(a_2)K_{29}(b_2)]K_{28}(d_1) - K_{27}(a_1)K_{30}(b_1)K_{27}(d_2)}{K_{27}(a_1)K_{30}(b_1)K_{28}(d_1) - [a_0K_{27}(a_1)K_{29}(b_2)+b_0K_{26}(a_2)K_{30}(b_1) + K_{26}(a_2)K_{29}(b_2)]K_{29}(d_1)}\right]$$
$$= \left[\frac{1846383724858067}{2130820624622319}\right] = 0$$

$$d_{31} = \left[\frac{K_{27}(a_1)K_{31}(b_1)K_{28}(d_2) - [a_0K_{27}(a_1)K_{30}(b_2)+b_0K_{26}(a_2)K_{31}(b_1) + K_{26}(a_2)K_{30}(b_2)]K_{29}(d_1)}{[a_0K_{27}(a_1)K_{30}(b_2)+b_0K_{26}(a_2)K_{31}(b_1) + K_{26}(a_2)K_{30}(b_2)]K_{30}(d_1) - K_{27}(a_1)K_{31}(b_1)K_{29}(d_2)}\right]$$
$$= \left[\frac{4272159783393956}{5176239998895508}\right] = 0$$

$$d_{32} = \left[\frac{[a_0K_{27}(a_1)K_{31}(b_2)+b_0K_{26}(a_2)K_{32}(b_1) + K_{26}(a_2)K_{31}(b_2)]K_{30}(d_1) - K_{27}(a_1)K_{32}(b_1)K_{29}(d_2)}{K_{27}(a_1)K_{32}(b_1)K_{30}(d_1) - [a_0K_{27}(a_1)K_{31}(b_2)+b_0K_{26}(a_2)K_{32}(b_1) + K_{26}(a_2)K_{31}(b_2)]K_{31}(d_1)}\right]$$
$$= \left[\frac{7022623723753575}{6402980408016257}\right] = 1$$

$$d_{33} = \left[\frac{K_{27}(a_1)K_{33}(b_1)K_{30}(d_2) - [a_0K_{27}(a_1)K_{32}(b_2)+b_0K_{26}(a_2)K_{33}(b_1) + K_{26}(a_2)K_{32}(b_2)]K_{31}(d_1)}{[a_0K_{27}(a_1)K_{32}(b_2)+b_0K_{26}(a_2)K_{33}(b_1) + K_{26}(a_2)K_{32}(b_2)]K_{32}(d_1) - K_{27}(a_1)K_{33}(b_1)K_{31}(d_2)}\right]$$
$$= \left[\frac{10675140191410231}{1523723531238852}\right] = 7$$

$$d_{34} = \left[\frac{[a_0K_{27}(a_1)K_{33}(b_2)+b_0K_{26}(a_2)K_{34}(b_1) + K_{26}(a_2)K_{33}(b_2)]K_{32}(d_1) - K_{27}(a_1)K_{34}(b_1)K_{31}(d_2)}{K_{27}(a_1)K_{34}(b_1)K_{32}(d_1) - [a_0K_{27}(a_1)K_{33}(b_2)+b_0K_{26}(a_2)K_{34}(b_1) + K_{26}(a_2)K_{33}(b_2)]K_{33}(d_1)}\right]$$
$$= \left[\frac{11285708034409264}{2129005507023044}\right] = 5$$

$$d_{35} = \left[\frac{K_{27}(a_1)K_{35}(b_1)K_{32}(d_2) - [a_0K_{27}(a_1)K_{34}(b_2)+b_0K_{26}(a_2)K_{35}(b_1) + K_{26}(a_2)K_{34}(b_2)]K_{33}(d_1)}{[a_0K_{27}(a_1)K_{34}(b_2)+b_0K_{26}(a_2)K_{35}(b_1) + K_{26}(a_2)K_{34}(b_2)]K_{34}(d_1) - K_{27}(a_1)K_{35}(b_1)K_{33}(d_2)}\right]$$
$$= \left[\frac{8525097500830443}{4041068164723693}\right] = 2$$

$$d_{36} = \left[\frac{[a_0K_{27}(a_1)K_{35}(b_2)+b_0K_{26}(a_2)K_{36}(b_1) + K_{26}(a_2)K_{35}(b_2)]K_{34}(d_1) - K_{27}(a_1)K_{36}(b_1)K_{33}(d_2)}{K_{27}(a_1)K_{36}(b_1)K_{34}(d_1) - [a_0K_{27}(a_1)K_{35}(b_2)+b_0K_{26}(a_2)K_{36}(b_1) + K_{26}(a_2)K_{35}(b_2)]K_{35}(d_1)}\right]$$
$$= \left[\frac{4681748664017737}{1290605679818013}\right] = 3$$

$$d_{37} = \frac{K_{27}(a_1)K_{37}(b_1)K_{34}(d_2) - [a_0K_{27}(a_1)K_{36}(b_2)+b_0K_{26}(a_2)K_{37}(b_1) + K_{26}(a_2)K_{36}(b_2)]K_{35}(d_1)}{[a_0K_{27}(a_1)K_{36}(b_2)+b_0K_{26}(a_2)K_{37}(b_1) + K_{26}(a_2)K_{36}(b_2)]K_{36}(d_1) - K_{27}(a_1)K_{37}(b_1)K_{35}(d_2)}$$

$$= \frac{10767806609927161}{9191637647084106}$$

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d_{37} is expanded into simple continued fraction, $d_{37} = [1;\ 5, 1, 4, 1, 15, 2, 54, 1, 2, 2, 2, 1, 31, 1, 1, 1, 1, 3, 1, 3, 1, 1, 2, 40, 1, 1, 1, 1, 2, 1, 1, 2, 3, 2, 6]$, then $[d_0;\ d_1, d_2, \ldots, d_{37}] = [6;\ 0, 2, 1, 1, 5, 1, 3, 1, 3, 0, 1, 12, 3, 2, 1, 5, 2, 10, 0, 2, 1, 1, 1, 1, 8, 3, 14, 1, 2, 0, 0, 1, 7, 5, 2, 3, 1, 5, 1, 4, 1, 15, 2, 54, 1, 2, 2, 2, 1, 31, 1, 1, 1, 1, 3, 1, 3, 1, 1, 2, 40, 1, 1, 1, 1, 2, 1, 1, 2, 3, 2, 6] = [8;\ 1, 1, 5, 1, 3, 1, 4, 12, 3, 2, 1, 5, 2, 12, 1, 1, 1, 1, 8, 3, 14, 1, 2, 1, 7, 5, 2, 3, 1, 5, 1, 4, 1, 15, 2, 54, 1, 2, 2, 2, 1, 31, 1, 1, 1, 1, 3, 1, 3, 1, 1, 2, 40, 1, 1, 1, 1, 2, 1, 1, 2, 3, 2, 6]$.

- *Best Rational Approximation of πe:*

Best rational approximation can be a convergent or an intermediate fraction. To determine whether a convergent or an intermediate fraction is a best rational approximation or not, is by comparing their distance to its true value. We then find the absolute value of the difference of convergent or intermediate fraction and $\widehat{\pi e}$. $\widehat{\pi e}$ is true value πe limited to 15 decimal places, which is 8.539734222673567 with a round-off error of $6.546355086954657\ldots \times 10^{-17}$. Convergent or intermediate fraction that has smaller distance than the distance of all convergents or intermediate fractions with smaller denominators is the best rational approximation. The best rational approximations are then obtained in the following table.

Table 1 Best Rational Approximation

Convergents and Intermediate Fractions	Distance to $\widehat{\pi e}$
9	0.460265777326433
17/2	0.039734222673567
60/7	0.031694348755004…
77/9	0.015821332881988…
94/11	0.005720322780978…
111/13	0.001272684212028…
316/37	0.000806317866973…
427/50	0.000265777326433
538/63	0.000051682991027…
1503/176	0.000038504599160…
2041/239	0.000014731301328…
2579/302	$8.766641813443708\ldots \times 10^{-7}$
18591/2177	$6.443547796779972\ldots \times 10^{-7}$
21170/2479	$4.590592063707139\ldots \times 10^{-7}$
23749/2781	$3.140076194991010\ldots \times 10^{-7}$
26328/3083	$1.973735345640609\ldots \times 10^{-7}$
28907/3385	$1.015509672954209\ldots \times 10^{-7}$
31486/3687	$2.142594020314618\ldots \times 10^{-8}$
65551/7676	$1.390798589213131\ldots \times 10^{-8}$
97037/11363	$2.443039529965678\ldots \times 10^{-9}$
225560/26413	$8.888397823420285\ldots \times 10^{-10}$

322597/37776	1.133876802202456... × 10^{-10}
1193351/139741	7.604730284597934... × 10^{-11}
1515948/177517	3.573513037624565... × 10^{-11}
1838545/215293	9.569563947736340... × 10^{-12}
2161142/253069	8.784453556144766... × 10^{-12}
3999687/468362	3.476215961158249... × 10^{-13}
29836354/3493827	2.634849719233379... × 10^{-13}
33836041/3962189	1.912473541771985... × 10^{-13}
37835728/4430551	1.342824892434372... × 10^{-14}
41835415/4898913	8.820990921863686... × 10^{-14}
45835102/5367275	5.017814906074311... × 10^{-14}
49834789/5835637	1.825116589671358... × 10^{-14}
53834476/6303999	8.931722070387384... × 10^{-15}
103669265/12139636	4.135347386033650... × 10^{-15}
157503741/18443635	3.309518432781824... × 10^{-16}

*Convergents are Marked in Bold.

Another way of finding best rational approximation of πe is by approximating π and e themselves. Using simple continued fractions of π and e which are chopped to 15 decimal places, we get best rational approximation of π is $\frac{245850922}{78256779}$ with 15 decimal places precision, and best rational approximation of e is $\frac{325368125}{119696244}$ with 15 decimal places precision. Multiplication of the two best rational approximation is $\frac{39996026760330625}{4683521256919038}$, with its distance to $\widehat{\pi e}$ is 7.375599324045643... × 10^{-16}. Therefore $\frac{157503741}{18443635}$ is better approximation than $\frac{39996026760330625}{4683521256919038}$.

IV. CONCLUSION

Infinite simple continued fraction of π and e are $\pi = [3;\ 7, 15, 1, 292, 1, \dots]$ and $e = [2;\ 1, 2, 1, 1, 4, \dots]$. Using irrational numbers π and e chopped to 15 decimal places, the multiplication of their simple continued fractions is $[8;\ 1, 1, 5, 1, 3, 1, 4, 12, 3, 2, 1, 5, 2, 12, 1, 1, 1, 1, 8, 3, 14, 1, 2, 1, 7, 5, 2, 3, 1, 5, 1, 4, 1, 15, 2, 54, 1, 2, 2, 2, 1, 31, 1, 1, 1, 1, 3, 1, 3, 1, 1, 2, 40, 1, 1, 1, 1, 2, 1, 1, 2, 3, 2, 6]$. Best rational approximation of πe with 15 decimal places precision is $\mathbf{\frac{157503741}{18443635}}$.

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