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Finite and Infinite Generalized Back Ward q-Derivative Operator on its Application

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Abstract:- In this paper, the author define the generalized q-derivative oprator and obtain its relation with shift operator.Also, we present the discrete version of Leibtz theorem according to the generalized q-derivative operator.By defining its inverse,and using Stirling numbers of first kind, we establish formula for the sum of higher power of geometric progression in the field of first of Number Analysis.

***Keywords**:- Generalized q-Derivative Operator, Polynomial Factorial, Geometric Progression.*

I. INTRODUCTION

The theory of q- derivative equation is based on the definition of the q- derivative operator is definied as

$$\nabla_q y_k = y_{kq} - y_k$$

where y_k is a sequence of positive integers. The definition of $\nabla_q y_k$ is simply the derivative between two successive operator on two variable and turns to be suitable for dealing with the Cauchy polynomials. Also,derivative a binomial identity which unifies the two identities of Rota and Godman, as well as the q-Vandermond identity.

With this background, in this paper, we develop the basic theory for the generalized q-derivative operator $\nabla_{q(\alpha)} y_k$ and obtain relation connecting $\nabla_q u(k)$, and $\nabla_{q(\alpha)} u(k)$ and,$\nabla_{q(\alpha)} u(k)$ and E^q and the basic properties of $\nabla_{q(\alpha)} u(k)$ and also obtain a formula for finding the sum of the higher powers of geometric progressions using generalized inverse q-derivative operator.

II. PRELIMINARY

In this section, the author defined the generalized q-derivative operator and obtaining the relation between the shift operator and generalized q-derivative operator and polynomials.

2.1. Definition Let u(k) be a real valued *fuction defined on* $[0,\infty)$, *Then the generalized q-derivative oprator is defined as*

$$\nabla_{q(\alpha)} u(k) = u(qk) - \alpha u(k) \quad (1)$$

2.2. Lemma *The Relation between generalized q-derivative operator and q-shift oprator is*

$$\nabla_{q(\alpha)} = (E^q - \alpha) \quad (2)$$

$$\nabla_{q_1,q_2(\alpha)} = \prod_{t=1}^{2} (E^{q_t} - \alpha) \quad (3)$$

$$\nabla_{q_1,q_2,q_3(\alpha)} = \prod_{t=1}^{3} (E^{q_t} - \alpha) \quad (4)$$

2.3. Lemma *If* c_1 *and* c_2 *are non-zero sclars and* $u(k)$ *and* $v(k)$ *are real valued fuction on* $[0,\infty)$, *then*

$$\nabla_{q(\alpha)}[c_1 u(k) + c_2 v(k)] = c_1 \nabla_{q(\alpha)} u(k) + c_2 \nabla_{q(\alpha)} v(k)$$

$$\nabla_{q_1,q_2(\alpha)}[c_1 u(k) + c_2 v(k)] = c_1 \nabla_{q_1,q_2(\alpha)} u(k) + c_2 \nabla_{q_1,q_2(\alpha)} v(k)$$

$$\nabla_{q_1,q_2,q_3(\alpha)}[c_1 u(k) + c_2 v(k)] = c_1 \nabla_{q_1,q_2,q_3(\alpha)} u(k) + c_2 \nabla_{q_1,q_2,q_3(\alpha)} v(k)$$

2.4.Theorem *[9]If* k *is a positive integer,then*

$$\prod_{i=1}^{n} \nabla_{q_i(\alpha_i)}^{-1} k^n = \frac{k^n}{\prod_{i=1}^{n}(q_i - \alpha_i)}, q_i \neq \alpha_i \quad (5)$$

Proof: From (1) and Definition 4.1, and proof shoud end with a square □

III. FINITE Q-DERIVATIVE SERIERS.

3.1. Theorem *Let* $k \in (-\infty, \infty)$, *then*

$$\nabla_{q(\alpha)}^{-1} u(k) - \frac{1}{\alpha^t} \nabla_{q(\alpha)}^{-1} u(kq^t) = \sum_{r=1}^{t} \frac{-1}{\alpha^r} u(kq^{r-1}), k > 0, q \neq 0, \alpha > 0 \tag{6}$$

Proof: From (1) and Definition 4.1, we have

$$v(k) = -\frac{1}{\alpha} u(k) + \frac{1}{\alpha} v(kq) \tag{7}$$

Replacing k by kq in (16) and substituting in (16), we obtained

$$v(k) = -\frac{1}{\alpha} u(k) - \frac{1}{\alpha^2} u(kq) + \frac{1}{\alpha^2} v(kq^2)$$

Proceeding like this we get$v(k) = -\frac{1}{\alpha} u(k) - \frac{1}{\alpha^2} u(kq) - \frac{1}{\alpha^3} u(kq^2) - \cdots - \frac{1}{\alpha^t} u(kq^{t-1}) + \frac{1}{\alpha^t} v(kq^t)$

which gives (6)

3.2. Theorem *Let* $k \in (-\infty, \infty)$, *then*

$$[\tfrac{1}{\alpha^t} \nabla_{q(\alpha)}^{-1} u(kq^t)]_{t=n}^{m} = \sum_{r=m+1}^{n} \tfrac{-1}{\alpha^r} u(kq^{r-1}), k > 0, q \neq 0, \alpha > 0 \tag{8}$$

Proof: From (1) and Definition 4.1, we have

$$v(k) = -\frac{1}{\alpha} u(k) + \frac{1}{\alpha} v(kq) \tag{9}$$

Replacing k by kq in (1) and substituting in (9), we obtained

$$v(k) = -\frac{1}{\alpha} u(k) - \frac{1}{\alpha^2} u(kq) + \frac{1}{\alpha^2} v(kq^2)$$

Replacing k by kq repeately we find

$$v(k) - \frac{1}{\alpha^m} v(kq^m) = \sum_{r=1}^{m} -\frac{1}{\alpha^r} u(kq^{r-1}) \tag{10}$$

Repacing m by n in (19),we get

$$v(k) - \frac{1}{\alpha^n} v(kq^n) = \sum_{r=1}^{n} -\frac{1}{\alpha^r} u(kq^{r-1}) \tag{11}$$

Assume that $m < n$. Now $(8) - (11)$,gives,

$$\frac{1}{\alpha^m} v(kq^m) - \frac{1}{\alpha^n} v(kq^n) = \sum_{r=m+1}^{n} -\frac{1}{\alpha^r} u(kq^r) \tag{12}$$

The following examples illustrate (8)

3.3. Example *By taking* $p = 5$, *in Theorem3.6 and*

$k = 12, q = 6, \alpha = 5, m = 1$ *and* $n = 2$ *in (17), we get,*
$\frac{1244160}{60427296} - \frac{6220800}{469882653700} = \sum_{r=2}^{2} 5^{r-1} (\frac{12}{36})^5 = 0.020576131$

3.4. Corollary *If* $\alpha < q, u(k)$ *is bounded and*

$\lim_{n\to\infty} \alpha^n v(\frac{k}{q^n}) = 0$,*then*

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$\alpha^m \nabla_{q(\alpha)}^{-1} u(\frac{k}{q^m}) = \sum_{r=m+1}^{\infty} \alpha^{r-1} u(\frac{k}{q^r}$ (13)

In particular,

$\nabla_{q(\alpha)}^{-1} u(k) = \sum_{r=1}^{\infty} \alpha^{r-1} u(\frac{k}{q^r})$ (14)

Proof: Proof follows by taking $n \to \infty$ in (8) and (14) follows by putting $m = 0$ in (13)

3.5. Theorem *Let* $k \in (-\infty, \infty)$, *then*

$$\nabla_{q(\alpha)}^{-1} u(k) - \alpha^t \nabla_{q(\alpha)}^{-1} u\left(\frac{k}{q^t}\right) = \sum_{r=1}^{t} \alpha^{r-1} u\left(\frac{k}{q^r}\right), k > 0, q \neq 0, \alpha > 0 \quad (15)$$

Proof: From (1) and Definition 4.1, we have

$$v(k) = u(\frac{k}{q}) + \alpha v(\frac{k}{q}) \quad (16)$$

Replacing k by kq in (16) and substituting in (16), we obtained

$$v(k) = u(\frac{k}{q}) + \alpha u(\frac{k}{q^2}) + \alpha^2 v(\frac{k}{q^2})$$

Procuding like this we get

$$v(k) = u(\frac{k}{q}) + \alpha u(\frac{k}{q^2}) + \alpha^2 v(\frac{k}{q^2})$$

which gives (15)

3.6. Theorem *Let* $k \in (-\infty, \infty)$, *then*

$[\alpha^t \nabla_{q(\alpha)}^{-1} u(kq^t)]_{t=n}^{m} = \sum_{r=m+1}^{n} \frac{-1}{\alpha^r} u(\frac{k}{q^r}), k > 0, q \neq 0, \alpha > 0$ (17)

Proof: From (1) and Definition 4.1, we have

$v(k) = u(\frac{k}{q}) + \alpha v(\frac{k}{q})$ (18)

Replacing k by kq in (17) and substituting in (32), we obtained

$$v(k) = u(\frac{k}{q}) + \alpha u(\frac{k}{q^2}) + \alpha^2 v(\frac{k}{q^2})$$

Replacing k by kq repeately we find

$v(k) - \alpha^m v(\frac{k}{q^m}) = \sum_{r=1}^{m} \alpha^{m-1} u(\frac{k}{q^m})$ (19)

Repacing m by n in (19),we get

$v(k) - \alpha^n v(\frac{k}{q^n}) = \sum_{r=1}^{n} \alpha^{n-1} u(\frac{k}{q^n})$ (20)

Assume that $m < n$. Now $(20) - (19)$,gives,

$$\alpha^m v(\frac{k}{q^m}) - \alpha^n v(\frac{k}{q^n}) = \sum_{r=m+1}^{n} \alpha^{r-1} u(\frac{k}{q^r}) \quad (21)$$

The following examples illustrate (17)

IV. INFINITE Q-DERIVATIVE SERIES

In this section, the author derived the sum of higher power of geometric progressions using the inverse of generalized q-derivative operator.

4.1. Definition *The inverse of genegalized q-derivative operator denoted by* $\nabla_{q(\alpha)}^{-1}$ *is definied as if* $\nabla_{q(\alpha)}v(k) = u(k)$ *then* $v(k) = \nabla_{q(\alpha)}^{-1}u(k)$ *and the* n^{th} *order inverse operaor denoted by* $\nabla_{q(\alpha)}^{-n}$ *is definied as* $\nabla_{q(\alpha)}^{n}v(k) = u(k)$ *then* $v(k) = \nabla_{q(\alpha)}^{-n}u(k)$

4.2. Theorem *Let* $k \in (\infty, -\infty), \lim_{r\to\infty} \frac{1}{\alpha^r}u(kq^r) = 0$ *then*

$$\nabla_{q(\alpha)}^{-1}u(k) = -\frac{1}{\alpha}\sum_{r=0}^{\infty}\frac{1}{\alpha^r}u(kq^{r-1}), k > 0, q \neq 0 \tag{22}$$

Proof: From (1) and Definition 4.1, we have

$$v(k) = -\frac{1}{\alpha}u(k) + \frac{1}{\alpha}v(kq) \tag{23}$$

Replacing k by kq in (29) and substituting in (29), we obtained

$$v(k) = -\frac{1}{\alpha}u(k) - \frac{1}{\alpha^2}u(kq) + \frac{1}{\alpha^2}v(kq^2)$$

Continuing this process, we get (22).

4.3. Theorem *Let* $k \in (\infty, -\infty), \lim_{r\to\infty} \frac{1}{\alpha^r}u(kq^r) = 0$ *then*

$$\frac{1}{\alpha_1\alpha_2}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{1}{\alpha_1^{r_1}\alpha_2^{r_2}}u\left(kq_1^{r_1-1}q_2^{r_2-1}\right) = \nabla_{q_1(\alpha_1)}^{-1}\nabla_{q_2(\alpha_2)}^{-1}u(k) \tag{24}$$

Proof: Proof followed by From (1) and Definition 4.1 and Theorems 4.2.,

4.4. Theorem *Let* $k \in (\infty, -\infty), \lim_{r\to\infty} \frac{1}{\alpha^r}u(kq^r) = 0$, *then*

$$-\frac{1}{\alpha_1\alpha_2\alpha_3}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{1}{\alpha_1^{r_1}\alpha_2^{r_2}\alpha_3^{r_3}}u(kq_1^{r_1-1}q_2^{r_2-1}q_3^{r_3-1}) = \nabla_{q_1(\alpha_1)}^{-1}\nabla_{q_2(\alpha_2)}^{-1}\nabla_{q_3(\alpha_3)}^{-1}u(k) \tag{25}$$

Proof: Proof followed by From (1) and Definition 4.1 and Theorems 4.2., 4.3.

4.5.Theorem *Let* $k \in (\infty, -\infty)$,*then* $\sum_{(r)_{[1\to t]}}^{\infty}\prod_{t=1}^{p}\alpha_t^{-r_t}u\left(\prod_{t=1}^{n}q_t^{r_t}k\right) = (-1)^p\prod_{t=1}^{i}\alpha_t^{r_t}\nabla_{q(\alpha)_{[1\to t]}}^{-1}u(k)$ (26)

Proof: Proof followed by From (1) and Definition 4.1 and Theorems 4.2., 4.3.,4.4.

and proof shoud end with a square ?

4.6. Example *By taking* $p = 10$, *in Theorem2.4 and* $k = 15, q = 9, \alpha = 0.25, m = 4$ *and* $n = 2$ *in (17), we get,*
4.7.

$$\frac{15^{10}}{9^{10} - 0.25} = \sum_{r=1}^{\infty}(0.25)^{r-1}(\frac{15}{9^r})^{10} = 165.3817169$$

The following Corollary illustraes Theorem 4.8

4.7. Corollary *If* n *is positive integer, then*

$$\sum_{r=1}^{\infty}\frac{\alpha^{r-1}}{q^{rn}} = \frac{1}{q^n - \alpha}, \alpha > 0, q^n \neq \alpha \tag{27}$$

Proof: The proof follows by substituting $u(k) = k^n$ in (31)

4.8. Theorem *Let $k \in (-\infty, \infty)$, then*

$$\nabla_{q(\alpha)}^{-1} u(k) = \sum_{r=0}^{\infty} \alpha^r u(\frac{k}{q^{r+1}}), q \neq 0$$

(28)

Proof: From (1) and Definition 4.1, we have

$v(k) = u(\frac{k}{q}) + \alpha v(\frac{k}{q})$ (29)

Replacing k by $\frac{k}{q}$ in (29) and substituting in (29), we obtained

$$v(k) = u(\frac{k}{q}) + \alpha u(\frac{k}{q^2}) + \alpha^2 u(\frac{k}{q^2})$$

Continuing this process, we get (28).

4.9. Theorem *Let $k \in (\infty, -\infty)$, $\lim_{r\to\infty} \frac{1}{\alpha^r} u(kq^r) = 0$ then*

$$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \alpha_1^{r_1} \alpha_2^{r_2} u(\frac{k}{q_1^{r_1+1} q_2^{r_2+1}}) = \nabla_{q_1(\alpha_1)}^{-1} \nabla_{q_2(\alpha_2)}^{-1} u(k), q \neq 0 \quad (30)$$

Proof: From (1) and Definition 4.1

4.10. Theorem
Let $k \in (\infty, -\infty)$, $\lim_{r\to\infty} \frac{1}{\alpha^r} u(kq^r) = 0$

$$then \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{r_3=0}^{\infty} \alpha_1^{r_1} \alpha_2^{r_2} \alpha_3^{r_3} u(\frac{k}{q_1^{r_1+1} q_2^{r_2+1} q_3^{r_3+1}}) = \nabla_{q_1(\alpha_1)}^{-1} \nabla_{q_2(\alpha_2)}^{-1} \nabla_{q_3(\alpha_3)}^{-1} u(k), q \neq 0 \quad (31)$$

Proof: From (1) and Definition 4.1

4.11. Theorem *Let $k \in (\infty, -\infty)$, then*

$$\sum_{(r)_{[1\to t]}}^{\infty} \prod_{t=1}^{p} \alpha_t^{r_t} u(\prod_{t=1}^{n} q_t^{-r_t+1} k) = \prod_{t=1}^{p} \nabla_{q(\alpha)_{[1\to t]}}^{-1} u(k) (32)$$

Proof: Proof followed by From (1) and Definition 4.1 and Theorems 4.8, 4.9,4.10 and

Proof shoud end with a square □

V. CONCLUSION

The author derived several results and theorem using q-derivative and its inverse and they were verified with example. By taking different fuctions $u(k)$ and $v(k)$ one can obtain corresponding finite and infinite series formulas.

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