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Alpha Difference Operator on its Finite and Infinite Series for Positive Variable K

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Abstract:- In this paper, the author extend the theory on finite and infinite positive variable *k* of the generalized *α*-difference equation and also from real line *K*(*R*) obtained the two solutions that is closed form solution and inverse form solutions of *α*-difference equation.

Keywords:- *Generalized α-difference equation, Inverse solution, Closed form solution.*

I. INTRODUCTION

The theory of difference equation is developed with the definition of the difference operator $\Delta_{(-1)}u(k) = u(k-1) - u(k)$, $k \in N$, where N is the set of natural numbers. Many authors suggested the possible study by redefining the operator as $\Delta_{-l}u(k) = u(k-l) - u(k)$, $k \in R$. The theory developed already with the difference operator Δ agrees when `$=-1$.

In 2011, M.MariaSusai Manuel, et.al, [6], have extended the definition of Δ_α to $\Delta_{\alpha(-1`)}$ which is defined as $\Delta_{\alpha(-1`)}v(k) = v(k-l) - \alpha v(k)$ for the real valued function $v(k)$, $k \in (0,\infty)$. In [7], the authors have used the generalized α-difference equation;

$$v(k-l) - \alpha v(k) = u(k),\ k \in [0,-\infty),\ 0 < l < k \quad (1)$$

and obtained a summation solution of the above equation in the form

$$v(k) = \Delta^{-1}_{\alpha(-\ell)}u(k) - \alpha^{[\frac{k}{\ell}]}\Delta^{-1}_{\alpha(-\ell)}u(\check{\ell}(k)) = \sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}u(k+r\ell \quad (2)$$

$\check{\ell}(k) = (k + [\frac{k}{\ell}]\ell)$ where, $[\frac{k}{\ell}]$ denotes the integer part of $\frac{k}{\ell}$.

II. PRELIMINARIES

In this section, the authorpresent some basic definition and some results on generalized α-difference operator and polynomial factorials, which will be useful for subsequent discussion.

- **Definition:** *The inverse of the generalized α−difference operator denoted by* $\Delta^{-1}_{\alpha(-\ell)}$ *on* $u(k)$ *is defined as,if* $\Delta_{\alpha(-`)}v(k) = u(k)$ *and* ˇ(k) *is defined, then*

$$\Delta^{-1}_{\alpha(-\ell)}u(k) = v(k) - \alpha^{[\frac{k}{\ell}]}c_j, \quad (3)$$

where c_j *is a constant for all* $k \in R_{-`}(j)$, $j = (l\,ˇ(k))$.

III. FINITE SERIES

In this section, we present some significant results, and applications on finite sums of k^npowers of α using the inverse of $\Delta_{\alpha(-1`)}$.

- **Lemma 3.1** *If* $k > 0, 0 < l < k, \alpha > 1$ *,then*

$$\Delta^{-1}_{\alpha(-\ell)}u(k) - \frac{1}{\alpha^{[\frac{k}{\ell}]+1}}\Delta^{-1}_{\alpha(-\ell)}u(\tilde{\ell}(k)) = \sum_{r=0}^{[\frac{k}{\ell}]}(\frac{-1}{\alpha^{r+1}})u(k-r\ell \quad (4)$$

Proof: By taking $\Delta^{-1}_{\alpha(-\ell)}u(k) = v(k)$,

we have $\Delta_{\alpha(-1`)}v(k) = u(k)$, *which gives*

$$v(k) = \frac{-1}{\alpha}u(k) + \frac{1}{\alpha}v(k-\ell \quad (5)$$

Replacing k by k −l in (5), we get

$$v(k-\ell) = \frac{-1}{\alpha}u(k-\ell) + \frac{1}{\alpha}v(k-2\ell \qquad (6)$$

Substituting (6) in (5), we get

$$v(k) = \frac{-1}{\alpha}u(k) - \frac{1}{\alpha^2}u(k-\ell) + \frac{1}{\alpha^2}v(k-2\ell \qquad (7)$$

Proceeding like this we get

$$v(k) = \tfrac{-1}{\alpha}u(k) - \tfrac{1}{\alpha^2}u(k-\ell) - \tfrac{1}{\alpha^3}u(k-2\ell) - \ldots$$

$$-\tfrac{1}{\alpha^{[\frac{k}{\ell}]+1}}u(k-[\tfrac{k}{\ell}]\ell) + \tfrac{1}{\alpha^{[\frac{k}{\ell}]+1}}v(k-([\tfrac{k}{\ell}]+1)\ell),$$

which gives (4)

- **Example 3.2** *Let* $u(k) = k, \alpha = 2, l = 4, k = 11$ *in (4),(17) we obtain*

$$\sum_{r=0}^{[\frac{11}{4}]}(\frac{-1}{2^{r+1}})(11-4r) = -7.625 = \frac{11}{1-2} + \frac{4}{(1-2)^2}$$

$$-\frac{1}{2^{[\frac{11}{4}]+1}}\left\{\frac{11-([\frac{11}{4}]+1)4}{1-2} + \frac{4}{(1-2)^2}\right\}$$

- **Example 3.3** *Let* $u(k) = k^2, \alpha = 3, l = 5, k = 13$ *in (4) and (18) we obtain*

$$\sum_{r=0}^{[\frac{13}{5}]}(\frac{-1}{3^{r+1}})(13-5r)^2 = -63.77777778 = \frac{169}{1-3} + \frac{105}{(1-3)^2} + \frac{50}{(1-3)^3}$$

$$-\frac{1}{3^{[\frac{13}{5}]+1}}\left\{\frac{(13-([\frac{13}{5}]+1)5)^2}{1-3} + \frac{[10(13-([\frac{13}{5}]+1)5)-25]}{(1-3)^2} + \frac{50}{(1-3)^3}\right\}$$

- ***Theorem 3.4***
  For $k \in [0,\infty), 0 < l < k$ *and* $n \in N(1)$,

$$\Delta^{-1}_{\alpha(-\ell)}k^n - \frac{1}{\alpha^{[\frac{k}{\ell}]+1}}\Delta^{-1}_{\alpha(-\ell)}(\tilde{\ell}(k))^n = \sum_{r=0}^{[\frac{k}{\ell}]}(\frac{-1}{\alpha^{r+1}})(k-r\ell)^n \qquad (8)$$

Where

$$\Delta^{-1}_{\alpha(-\ell)}k^n = \frac{1}{1-\alpha}\left\{k^n - \sum_{r=1}^{n}(-1)^r\binom{n}{r}\ell^r\Delta^{-1}_{\alpha(-\ell)}k^{n-r}\right\}$$

In particular when $n = 4$,

$$\Delta^{-1}_{\alpha(-\ell)}k^4 - \frac{1}{\alpha^{[\frac{k}{\ell}]+1}}\Delta^{-1}_{\alpha(-\ell)}(\tilde{\ell}(k))^4 = \sum_{r=0}^{[\frac{k}{\ell}]}(\frac{-1}{\alpha^{r+1}})(k-r\ell)^4$$

$$\Delta^{-1}_{\alpha(-\ell)}(k^4) = \tfrac{k^4}{1-\alpha} - \tfrac{4\ell}{1-\alpha} \times \left[\tfrac{k^3}{1-\alpha} - \tfrac{\ell}{(1-\alpha)^2}\left(3k(k+\ell) - \tfrac{6\ell}{1-\alpha}\left(k - \tfrac{\ell}{1-\alpha} + \ell\right) + \ell^2\right)\right]$$
$$-\tfrac{6\ell^2}{1-\alpha}\left[\tfrac{k^2}{1-\alpha} - \tfrac{\ell}{(1-\alpha)^2}\left[2\left(k - \tfrac{\ell}{1-\alpha} + \ell\right)\right]\right]$$
$$-\tfrac{4\ell^3}{(1-\alpha)^2}\left(k + \tfrac{\ell}{1-\alpha}\right) - \tfrac{\ell^4}{(1-\alpha)^2}.$$

where

*Proof: As in the proof of Lemma 3.1, by applying (**8**) to the expression of Δα(−l`), we get*

$$\Delta^{-1}_{\alpha(-\ell)} k^n = \frac{1}{1-\alpha}\left\{k^n - \frac{1}{\alpha^{[\frac{k}{\ell}]+1}}(\tilde{\ell}(k))^n \sum_{r=1}^{n} (-1)^r \binom{n}{r} \ell^r \Delta^{-1}_{\alpha(-\ell)} k^{n-r}\right\} \tag{9}$$

Now the proof follows by substituting $\alpha_2 = \alpha$ and ` $l_2 = l$ ` in (16),(17),(18),etc,we get $\Delta^{-1}_{\alpha(-1)}(1), \Delta^{-1}_{\alpha(-\ell)} k \cdots, \Delta^{-1}_{\alpha(-\ell)} k^{n-1}$ in (9) and (4)

IV. HIGHER ORDER SERIERS

In this section, the authorobtain the sum of higher order alpha series by equating the closed and summation form solutions of the generalized higher kind alpha difference equation. The higher order generalized α-difference equation is defined as

- **Theorem 4.1**
  For $k \in [0,\infty)$, $0 < \ell < k$, $k > 0$ *and* $\alpha > 1$.

$$\sum_{i=1}^{n} \sum_{(r\ell)_{[1\to i]}}^{[k]} \prod_{i=1}^{n} \left(\frac{-1}{\alpha_i^{r_i+1}}\right) u\left(k - \sum_{i=1}^{n} r_i \ell_i\right) = \prod_{i=1}^{n} \Delta^{-1}_{\alpha_i(-\ell_i)} u(k)$$
$$- \left\{\frac{1}{\alpha_1^{\left[\frac{k}{\ell_1}\right]}}\right\} \prod_{i=1}^{n} \Delta^{-1}_{\alpha_i(-\ell_i)} (\tilde{\ell}_1(k))$$
$$- \sum_{i=1}^{n-1} \sum_{(r\ell)_{[1\to i]}}^{[k]} \left\{\frac{1}{\alpha_{i+1}^{\left[\frac{k - \sum_{i=1}^{n-1} r_i \ell_i}{\ell_i + 1}\right]} \alpha_i^{r_i - 1} \alpha_{i-1}^{r_i - 2} \alpha_{i-3}^{r_i - 3} \cdots \alpha_{i-(1-n)}^{r_i - n}}\right\}$$

Proof. Replacing ` by l_1 and α by α_1 in (4), we have

$$\tfrac{-1}{\alpha_1} u(k) - \tfrac{1}{\alpha_1^2} u(k - \ell_1) - \tfrac{1}{\alpha_1^3} u(k - 2\ell_1) - \cdots - \tfrac{1}{\alpha_1^{[\frac{k}{\ell_1}]+1}} u(k - [\tfrac{k}{\ell_1}]\ell_1)$$
$$= \Delta^{-1}_{\alpha_1(-\ell_1)} u(k) - \left\{\frac{1}{\alpha_1^{[\frac{k}{\ell_1}]+1}}\right\} \Delta^{-1}_{\alpha_1(-\ell_1)} u(k - [\frac{k}{\ell_2}]\ell_2). \tag{10}$$

Replacing l_1 *by* l_2 and α_1 *by* α_2 in (10), we get

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$$\frac{-1}{\alpha_2}u(k)-\frac{1}{\alpha_2^2}u(k-\ell_2)-\frac{1}{\alpha_2^3}u(k-2\ell_2)-\ldots-\frac{1}{\alpha_2^{[\frac{k}{\ell_2}]+1}}u(k-[\tfrac{k}{\ell_2}]\ell_2)$$

$$=\Delta^{-1}_{\alpha_2(-\ell_2)}u(k)-\left\{\frac{1}{\alpha_2^{[\frac{k}{\ell_2}]+1}}\right\}\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{\ell}_2(k)). \quad (11)$$

For $r=1,2,3...[\frac{k}{\ell_1}]$ replacing k by $k-r`_1$ in (11) and multiplying both sides by $\left\{\frac{-1}{\alpha_1^r}\right\}$, we find that

$$\left\{\frac{-1}{\alpha_2}\right\}u(k-r\ell_1)+\left\{\frac{-1}{\alpha_2^2}\right\}u(k-r\ell_1-\ell_2)+\left\{\frac{-1}{\alpha_2^3}\right\}u(k-r\ell_1-2\ell_2)+\cdots+\left\{\frac{-1}{\alpha_2^{[\frac{k-r\ell_1}{\ell_2}]}}\right\}u\left(k-r\ell_1-[\tfrac{k-r\ell_1}{\ell_1}]\ell_1\right)$$

$$=\Delta^{-1}_{\alpha_2(-\ell_2)}u(k-r\ell_1-\ell_2)-\left\{\frac{1}{\alpha_2^{[\frac{k-r\ell_1}{\ell_2}]+1}}\right\}\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{\ell}_2(k-r\ell_1-\ell_2)). \quad (12)$$

Adding (11) and (12) for $r=1,2,3...[\frac{k}{\ell_1}]$ and applying (10), we derive

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\left(\frac{-1}{\alpha_1^{r_1+1}}\right)\left(\frac{-1}{\alpha_2^{r_2+1}}\right)u(k-r_1\ell_1-r_2\ell_2)=\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k)$$
$$-\left(\frac{1}{\alpha_1^{[\frac{k}{\ell_1}]+1}}\right)\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(\tilde{\ell}_1(k))$$

Replacing r_1,r_2 *by* r_2,r_3 and l_1,l_2 *by* l_2,l_3 in (13), we find

$$\sum_{r_2=0}^{[\frac{k}{\ell_2}]}\sum_{r_3=0}^{[\frac{k-r_2\ell_2}{\ell_3}]}\left(\frac{-1}{\alpha_1^{r_2+1}}\right)\left(\frac{-1}{\alpha_3^{r_3+1}}\right)u(k-r_2\ell_2-r_3\ell_3)=$$
$$\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k)-\left(\frac{1}{\alpha_2^{[\frac{k}{\ell_2}]+1}}\right)\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{\ell}_2(k))$$
$$+\sum_{r_2=0}^{[\frac{k}{\ell_2}]}\left(\frac{1}{\alpha_2^{r_2+1}}\right)\left(\frac{1}{\alpha_3^{[\frac{(k-r_2\ell_2)}{\ell_3}]+1}}\right)\Delta^{-1}_{\alpha_3(-\ell_3)}u(\tilde{\ell}_3(k-r_2\ell_2)) \quad (14)$$

Replacing k by $k-rl_1$ in (14) and multiplying both sides by $\left\{\frac{-1}{\alpha_1^r}\right\}$ and adding the corresponding expressions for $r=0,1,2,...[\frac{k}{\ell_1}]$, we derive

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\sum_{r_3=0}^{[\frac{k-r_1\ell_1-r_2\ell_2}{\ell_3})]}\left(\frac{-1}{\alpha_1^{r_1+1}}\right)\left(\frac{-1}{\alpha_2^{r_2+1}}\right)\left(\frac{-1}{\alpha_3^{r_3+1}}\right)u(k-r_1\ell_1-r_2\ell_2-r_3\ell_3)$$
$$=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k)-\left(\frac{1}{\alpha_1^{[\frac{-k}{\ell_1}]+1}}\right)\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(\tilde{\ell}_1(k))$$
$$+\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\left(\frac{1}{\alpha_1^{r_1+1}}\right)\left(\frac{1}{\alpha_2^{[\frac{(k-r_1\ell_1)}{\ell_2}]+1}}\right)\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{\ell}_2(k-r_1\ell_1))$$
$$+\sum_{r_1=0}^{[\frac{k}{\ell}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\left\{\frac{1}{\alpha_3^{[\frac{k-r_1\ell_1-r_2\ell_2}{-\ell_3}]}}\right\}\left\{\frac{1}{\alpha_2^{r_2}}\right\}\left\{\frac{1}{\alpha_1^{r_1}}\right\}\Delta^{-1}_{\alpha_3(-\ell_3)}u(\tilde{\ell}_3(k-r_1\ell_1-r_2\ell_2))$$

Continuing this process we get the proof of the theorem.

- **Corollary 4.2** *Taking* $n = 2$, $u(k) = k^2$ *in the Theorem 4.1, we have*

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{-r_1 1}{\ell_2}]}(\frac{-1}{\alpha_1^{r_1+1}})(\frac{-1}{\alpha_2^{r_2+1}})u(k-r_1\ell_1-r_2\ell_2)^2 = \Delta^{-1}_{\alpha_1(-\ell_1)}\Delta^{-1}_{\alpha_2(-\ell_2)}k^2_{\;k}$$

$$-(\frac{1}{\alpha_1^{[\frac{k}{\ell_1}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(\tilde{\ell}_1(k))^2$$

$$+\sum_{r_1=0}^{[\frac{k}{\ell_1}]}(\frac{1}{\alpha_1^{r_1+1}})(\frac{1}{\alpha_2^{[\frac{(k-r_1\ell_1)}{\ell_2}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{\ell}_2(k-r_1\ell_1))^2 \quad (15)$$

Proof. Since $\Delta_{\alpha 2(l_2)}k^0 = (k - l_2)^0 - \alpha_2 k^0 = (1 - \alpha_2)(1)$, we have

$$\Delta^{-1}_{\alpha_2(-\ell_2)}(1) = \frac{1}{1-\alpha_2}. \quad (16)$$

From $\Delta_{\alpha 2(l_2)}k = (k - l_2) - \alpha_2 k = k(1 - \alpha_2) - l_2(1)$ and (16), we get

$$\Delta^{-1}_{\alpha_2(-\ell_2)}k = \frac{k}{1-\alpha_2} + \frac{\ell_2}{(1-\alpha_2)^2}. \quad (17)$$

Now $\Delta_{\alpha_2(-\ell_2)}k^2 = (k-\ell_2)^2 - \alpha_2 k^2 = k^2(1-\alpha_2) - 2\ell_2 k + \ell_2^2$ yields

$$\Delta^{-1}_{\alpha_2(-\ell_2)}k^2 = \frac{k^2}{(1-\alpha_2)} + \frac{2\ell_2}{(1-\alpha_2)}\Delta^{-1}_{\alpha_2(-\ell_2)}(k) - \frac{\ell_2^2}{(1-\alpha_2)}\Delta^{-1}_{\alpha_2(-\ell_2)}(1)$$

and hence by (16) and (17), we find

$$\Delta^{-1}_{\alpha_2(-\ell_2)}k^2 = \frac{k^2}{(1-\alpha_2)} + \frac{[2\ell_2 k - \ell_2^2]}{(1-\alpha_2)^2} + \frac{2\ell_2^2}{(1-\alpha_2)^3}. \quad (18)$$

Taking $\Delta^{-1}_{\alpha_1(-\ell_1)}$ on both sides of (18) and applying (16) and(17) for $`l_1$, we arrive

$$\Delta^{-1}_{\alpha_1(-\ell_1)}\Delta^{-1}_{\alpha_2(-\ell_2)}k^2 = \frac{k^2}{(1-\alpha_1)(1-\alpha_2)} + \frac{2\ell_2 k - \ell_2^2}{(1-\alpha_1)(1-\alpha_2)^2} \quad (19)$$

$$+\frac{2\ell_1 k - \ell_1^2}{(1-\alpha_1)^2(1-\alpha_2)} + \frac{2\ell_2\ell_1}{(1-\alpha_1)^2(1-\alpha_2)^2} + \frac{2\ell_2^2}{(1-\alpha_1)(1-\alpha_2)^3} + \frac{2\ell_1^2}{(1-\alpha_1)^3(1-\alpha_2)}$$

Now the proof follows by applying (18) and(19) in the Theorem (4.1). Following example is an verification of corollary (4.2).

- **Example 4.3** *Let* $k = 10, l_1 = 3, l_2 = 4$ *and* $\alpha_1 = 2$, $\alpha_2 = 3$ *in Corollary 4.2,(18) and (19)*

$$\sum_{r_1=0}^{[\frac{10}{3}]}\sum_{r_2=0}^{[\frac{10-3r_1}{4}]}(\frac{-1}{2^{r_1+1}})(\frac{-1}{3^{r_2+1}})u(10-3r_1-4r_2)^2 = 23.7962963 = \Delta^{-1}_{\alpha_1(-\ell_1)}\Delta^{-1}_{\alpha_2(-\ell_2)}10^2$$

$$-(\frac{1}{2^{[\frac{10}{3}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(\tilde{\ell}_1(10))^2$$

$$+\sum_{r_1=0}^{[\frac{10}{3}]}(\frac{1}{2^{r_1+1}})(\frac{1}{3^{[\frac{(10-3r_1)}{4}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{4}(10-3r_1))^2 \quad (20)$$

Corollary 4.4 Taking $n = 2$, $u(k) = k$ *in the Theorem 4.1*

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$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{-r_1}{\ell_2}\frac{1}{}]}(\frac{-1}{\alpha_1^{r_1+1}})(\frac{-1}{\alpha_2^{r_2+1}})(k-r_1\ell_1-r_2\ell_2)=\Delta^{-1}_{\alpha_1(-\ell_1)}\Delta^{-1}_{\alpha_2(-\ell_2)}k_k$$

$$-(\frac{1}{\alpha_1^{[\frac{k}{\ell_1}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(\tilde{\ell}_1(k))$$

$$+\sum_{r_1=0}^{[\frac{k}{\ell_1}]}(\frac{1}{\alpha_1^{r_1+1}})(\frac{1}{\alpha_2^{[\frac{(k-r_1\ell_1)}{\ell_2}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}u(\tilde{\ell}_2(k-r_1\ell_1)) \quad (21)$$

- **Example 4.5**
  Let $k = 7, l_1 = 2 = \alpha_1, \alpha_2 = 3 = l_2$ in (21)and using (19)

$$\sum_{r_1=0}^{[\frac{7}{2}]}\sum_{r_2=0}^{[\frac{7-2r_1}{3}]}(\frac{-1}{2^{r_1+1}})(\frac{-1}{3^{r_2+1}})(7-2r_1-3r_2)2=2.025462963=\Delta^{-1}_{\alpha_1(-\ell_1)}\Delta^{-1}_{\alpha_2(-\ell_2)}7$$

$$-(\frac{1}{2^{[\frac{7}{2}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}(\tilde{2}(7))$$

$$+\sum_{r_1=0}^{[\frac{7}{2}]}(\frac{1}{2^{r_1+1}})(\frac{1}{3^{[\frac{(7-2r_1)}{3}]+1}})\Delta^{-1}_{\alpha_2(-\ell_2)}(\tilde{3}(7-2r_1)) \quad (22)$$

- ***Corollary 4.6***
  Taking $n = 2$, $u(k) = k^2, \alpha_1 = \alpha_2 = \alpha$ in the Theorem 4.1, we have

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}(\frac{1}{\alpha^{r_1+r_2+2}})u(k-r_1\ell_1-r_2\ell_2)^2=\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}k^2$$

$$-(\frac{1}{\alpha^{[\frac{k}{\ell_1}]+1}})\Delta^{-1}_{\alpha(-\ell_2)}\Delta^{-1}_{\alpha(-\ell_1)}u(\tilde{\ell}_1(k))^2$$

$$+\sum_{r_1=0}^{[\frac{k}{\ell_1}]}(\frac{1}{\alpha^{r_1+1}})(\frac{1}{\alpha^{[\frac{(k-r_1\ell_1)}{\ell_2}]+1}})\Delta^{-1}_{\alpha(-\ell_2)}u(\tilde{\ell}_2(k-r_1\ell_1))^2 \quad (23)$$

- **Example 4.7**
  Let $k = 32 l_1 = 6, l_2 = 7$ and $\alpha = 5$ in equation (23),(18) and (19)

$$\sum_{r_1=0}^{[\frac{32}{6}]}\sum_{r_2=0}^{[\frac{32-6r_1}{7}]}(\frac{-1}{5^{r_1+r_2+2}})(32-6r_1-7r_2)^2=\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}32^2$$

$$-(\frac{1}{5^{[\frac{32}{6}]+1}})\Delta^{-1}_{\alpha(-\ell_2)}\Delta^{-1}_{\alpha(-\ell_1)}(\tilde{6}(32))^2$$

$$+\sum_{r_1=0}^{[\frac{32}{6}]}(\frac{1}{5^{r_1+1}})(\frac{1}{5^{[\frac{(32-6r_1)}{7}]+1}})\Delta^{-1}_{\alpha(-\ell_2)}(\tilde{7}(32-6r_1))^2 \quad (24)$$

V. INFINITE SERIES

- **Lemma 5.1**
  If

$$k>0,\ \lim_{r\to\infty}\alpha^r\Delta^{-1}_{\alpha(-\ell)}u(k+r\ell)=0,\ 0<\alpha<1$$ *, then*

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$$\sum_{r=1}^{\infty} \alpha^{r-1} u(k+r\ell) = \Delta^{-1}_{\alpha(-\ell)} u(k \quad (25)$$

Proof: By taking $\Delta^{-1}_{\alpha(-\ell)} u(k) = v(k)$,
we have $\Delta_{\alpha(-1^{`})} v(k) = u(k)$*, which gives*

$$v(k-l) - \alpha v(k) = u(k)$$

$$v(k-l) = u(k) + \alpha v(k) \quad (26)$$

Replacing k by k +l in (26), we get
$\Delta^{-1}_{\alpha(-\ell)} u(k) = u(k+\ell) + \alpha u(k+2\ell) + \alpha^2 u(k+3\ell) + \alpha^3 u(k+3\ell) + \ldots$

- **Example 5.2**
  Let
  $u(k) = k, \alpha = \frac{1}{4}, \ell = 7, k = 21$ *in (25), we obtain*
  $$\sum_{r=1}^{\infty} \left(\frac{1}{4}\right)^{r-1} (21+7r) = \frac{21}{0.75} + \frac{7}{0.75^2} = 40.444$$

- **Example 5.3**
  Let
  $u(k) = k^2, \alpha = \frac{1}{2}, \ell = 3, k = 7$ *in (25), we obtain*
  $$\sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^{r-1} (7+3r)^2 = \frac{49}{0.5} + \frac{33}{0.25} + \frac{12}{0.125} = 326$$

➢ ***Theorem 5.4***
If
$k > 0, \Delta^{-1}_{\alpha(-\ell)} \lim_{r\to\infty} \alpha^r (k+r\ell)^n = 0, 0 < \alpha < 1$,
$0 < l < k, k \in [0,\infty)$, *then*

$$\sum_{r=1}^{\infty} \alpha^{r-1} (k+r\ell)^n = \Delta^{-1}_{\alpha(-\ell)} k^n \quad (27)$$

where

$$\Delta^{-1}_{\alpha(-\ell)} k^n = \frac{1}{1-\alpha} \left\{ k^n - \sum_{r=1}^{n} (-1)^r \binom{n}{r} \ell^r \Delta^{-1}_{\alpha(-\ell)} k^{n-r} \right\}$$

VI. HIGHER ORDER SERIES

➢ ***Theorem 6.1***
If
$k > 0, \Delta^{-1}_{\alpha(-\ell)} \lim_{r\to\infty} \alpha^r (k+r\ell)^n = 0, 0 < \alpha < 1, 0 < \ell_i < k,$

Then

$$\prod_{i=1}^{n} \Delta^{-1}_{\alpha_i(-\ell_i)} u(k) = \sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \sum_{r_3=1}^{\infty} \sum_{r_4=1}^{\infty} \sum_{r_5=1}^{\infty} \cdots \sum_{r_i=1}^{\infty} \left(\prod_{i=1}^{n} \alpha_i^{r_i-1}\right) u\left(k + \sum_{i=1}^{n} r_i \ell_i\right) \quad (28)$$

Proof. Replacing ` by l_1 *and* α *by* α_1 *in (25), we have*

$$u(k+\ell_1) + \alpha_1 u(k+2\ell_1) + \alpha_1^2 u(k+3\ell_1) + \cdots = \Delta^{-1}_{\alpha_1(-\ell_1)} u(k). \quad (29)$$

Replacing l_1 by l_2 and α_1 by α_2 in (29), we get:

$$u(k+\ell_2)+\alpha_2 u(k+2\ell_2)+\alpha_2^2 u(k+3\ell_2)+\cdots=\Delta^{-1}_{\alpha_2(-\ell_2)}u(k). \quad (30)$$

For r = 1,2,3⋯ replacing k by k + rl_1 in (30) and multiplying both sides by α_1^{r-1} $u(k+rl_1+l_2)+\alpha_2 u(k+rl_1+2l_2)+\alpha_2^2 u(k+rl_1+3l_2)+\cdots$

$$=\alpha_1^{r-1}\Delta^{-1}_{\alpha_2(-\ell_2)}u(k+r\ell_1). \quad (31)$$

Adding (30) and (31) for r = 1,2,3⋯ and applying (29), we derive

) (32)

Replacing r_1,r_2 by r_2,r_3 and l_1,l_2 by l_2,l_3 in (32), we find

u(k + r2`2 + r3`3) = Δ−α1(−1`)Δ−α1(−1)u(k) α2

3 3 2 2 (33)

Replacing k by k + rl_1 in (33) and multiplying both sides by α_1^{r-1} and adding the corresponding expressions for r = 0,1,2,⋯ we derive;

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\alpha_1^{r_1-1}\alpha_2^{r_1-1}\alpha_3^{r_3-1}u(k+r_1\ell_1+r_2\ell_2+r_3\ell_3)=\Delta^{-1}_{\alpha_3(-\ell_3)}\Delta^{-1}_{\alpha_2(-\ell_2)}\Delta^{-1}_{\alpha_1(-\ell_1)}u(k) \quad (34)$$

Continuing this process we get the proof of the theorem.

- **Corollary 6.2:** *Taking n = 2, u(k) = k in the Theorem 6.1, we have*
  $\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\alpha_1^{-1}\alpha_2^{-1}(k+r1l_1+r2l_2)=\Delta^{-1}\alpha1(l1)\Delta^{-1}\alpha2(l2)k$ (35)

- ***Example 6.3*** *Put k = 12, l_1 = 4, l_2 = 6 and*
  $\alpha_1=\frac{1}{100}$, $\alpha_2=\frac{1}{10}$, *inequation (35) and (19)*

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}(0.01)^{r_1-1}(0.1)^{r_2-1}(12+4r_1+6r_2)=25.4849=$$

$$\frac{12}{(1-0.01)(1-0.1)}+\frac{6}{(1-0.01)(1-0.1)^2}+\frac{4}{(1-0.01)^2(1-0.1)}$$

Corollary 6.4 *If*
$k>0, \Delta^{-1}_{\alpha(-\ell)} \lim_{r\to\infty}\alpha^r(k+r\ell)^n=0, 0<\alpha<1,$
0 < l_i < k, *then*

$$\prod_{i=1}^{n}\Delta^{-1}_{\alpha(-\ell_i)}u(k)=\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty}\sum_{r_4=1}^{\infty}\sum_{r_5=1}^{\infty}\cdots\sum_{r_i=1}^{\infty}(\prod_{i=1}^{n}\alpha^{r_i-1})u(k+\sum_{i=1}^{n}r_i\ell_i) \quad (36)$$

Proof.The proof follows by taking $\alpha_1=\alpha_2=\ldots=\alpha$ in Theorem 6.1

- **Corollary 6.5**
  Taking n = 2,u(k) = k^2 in the Corollary 6.4, we have

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\alpha^{r_1+r_2-1}(k+r_1\ell_1+r_2\ell_2)^2=\Delta^{-1}_{\alpha(-\ell_1)}\Delta^{-1}_{\alpha(-\ell_2)}k^2 \quad (37)$$

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- **Example 6.6** *Put* $k = 13, l_1 = 3, l_2 = 4$ *and* $\alpha = \frac{1}{2}$ *in equation (37)*

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}(0.5)^{r_1+r_2-2}(13+3r_1+4r_2)^2 = 732 = \frac{13^2}{(1-0.5)^2}+\frac{4}{(1-0.5)^3}+\frac{3}{(1-0.5)^3}$$

VII. POLYNOMIAL FACTORIAL OF A SERIES

In this section, the author*present some significant results, and applications on positive variable k is finite and infinite sums of* $\frac{1}{k^{(n)}_{(-\ell)}}$ *and powers of α using the inverse of* $\Delta_{\alpha(-1)}$. *Suitable examples are given to illustrate our main results.*

- ***Theorem 7.1 If***

k >0,*α*>1,*l*∈ (0,∞), *then*

$$\frac{1}{k^{(n)}_{-\ell}} = \Delta^{-1}_{\alpha(-\ell)}\left\{\frac{k(1-\alpha)+\ell((n-1)+\alpha)}{(k-\ell)^{(n+1)}_{-\ell}}\right\} \tag{38}$$

Proof:From (2.1) and taking n ∈*R*(1) *in (38),we get (38)*

- ***Theorem 7.2***

Ifα **>1,*l*∈ (0,∞), *then***

$$\sum_{r=1}^{[\frac{k}{\ell}]}\alpha^{r-1}\left\{\frac{(k+r\ell)(1-\alpha)+\ell((n-1)+\alpha)}{(k+r\ell-\ell)^{(n+1)}_{-\ell}}\right\} = \frac{1}{k^{(n)}_{-\ell}} - \alpha^{[\frac{k}{\ell}]}\left\{\frac{1}{(\check{\ell}(k))^{(n)}_{-\ell}}\right\} \tag{39}$$

Proof:Now the proof follows by

$u(k) = \frac{1}{k^{(n)}_{(-\ell)}}$ *in (1)*

- **Example 7.3***k* = 5,*l*= 3,*α* = 2,*n* = 1 *in equation* (39)

$$\sum_{r=1}^{1}2^{r-1}\left(\frac{(5+3r)(1-2)+6}{(2+3r)^{(2)}_{-3}}\right) = \frac{1}{5^{(1)}_{-3}} - 2\left\{\frac{1}{8^{(1)}_{-3}}\right\} = -0.05$$

- **Example 7.4k = 5,*l*= 2 = α,n= 2 in equation (39)**

$$\sum_{r=1}^{[\frac{5}{2}]}2^{r-1}\left(\frac{(5+2r)(1-2)+6}{(3+2r)^{(3)}_{-2}}\right) = \frac{1}{5^{(2)}_{-2}} - 2^2\left\{\frac{1}{9^{(2)}_{-2}}\right\} = -0.011832611$$

- **Theorem 7.5*If***

$k>0, 0<\alpha<1, \lim_{r\to\infty}\left(\frac{-1}{\alpha^{r+1}}\right)\left\{\frac{1}{(\check{\ell}(k))^{(n)}_{-\ell}}\right\} = 0$ *, then*

$$\sum_{r=1}^{\infty}\alpha^{r-1}\left\{\frac{(k+r\ell)(1-\alpha)+\ell((n-1)+\alpha)}{(k+r\ell-\ell)^{(n+1)}_{-\ell}}\right\} = \frac{1}{k^{(n)}_{-\ell}} \tag{40}$$

Proof:Now the proof follows by $u(k) = \frac{1}{k^{(n)}_{(-\ell)}}$ *in (25)*

- **Example 7.6**If
$k=7, \ell=3, \alpha=\frac{1}{2}, n=1$ *in equation* (40)

$$\sum_{r=1}^{\infty}(0.5)^{r-1}\left\{\frac{(7+3r)(0.5)+1.5}{(7+3r-3)^{(2)}_{(-3)}}\right\}=\frac{1}{(7)^{(1)}_{(-3)}}=0.14285714286$$

$-0.05=-0.05$

- **Example 7.7**If
$k=13, \ell=2, \alpha=\frac{1}{4}, n=2$ *in equation* (40)

$$\sum_{r=1}^{\infty}(0.25)^{r-1}\left\{\frac{(13+2r)(0.75)+(2.5)}{(11+2r)^{(3)}_{(-2)}}\right\}=\frac{1}{13^{(2)}_{(-2)}}=0.00512820513$$

$-0.011832611=-0.011832611$

Proof.The proof follows by taking $\alpha_1=\alpha_2=\ldots\alpha_n=1$ *in Theorem 6.1*

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