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On Four-Dimensional Absolute Valued Algebras Containing a Nonzero Central Element

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Abstract:- **An absolute valued algebra is a nonzero real algebra that is equipped with a multiplicative norm ||ab|| = ||a|| ||b||. We classify, by an algebraic method, all four-dimensional absolute valued algebras containing a nonzero central element and commutative sub-algebra of dimension two. Moreover, we give some conditions implying that these new algebras having sub-algebras of dimension two.**

Keywords:- *Absolute Valued Algebra; Pre-Hilbert Algebra; Commutative Algebra; Central Element.*

I. INTRODUCTION

Let A be a non-necessarily associative real algebra which is normed as real vector space. We say that a real algebra is a pre-Hilbert algebra, if it's norm $\|.\|$ come from an inner product $(./.)$, and it's said to be absolute valued algebras, if it's norm satisfy the equality $\|ab\| = \|a\|\|b\|$, for all $a, b \in A$. Note that, the norm of any absolute valued algebras containing a nonzero central idempotent (or finite dimensional) comes from an inner product [3] and [4]. In 1947 Albert proved that the finite dimensional unital absolute valued algebras are classified by $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, and that every finite dimensional absolute valued algebra has dimension 1, 2, 4 or 8 [1]. Urbanik and Wright proved in 1960 that all unital absolute valued algebras are classified by $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ [9]. It is easily seen that the one-dimensional absolute valued algebras are classified by $\mathbb{R}$, and it is well-known that the two-dimensional absolute valued algebras are classified by $\mathbb{C}, \mathbb{C}^*, {}^*\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (the real algebras obtained by endowing the space $\mathbb{C}$ with the product $x * y = \bar{x}y$, $x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively) [7]. The four-dimensional absolute valued algebras have been described by M.I. Ramirez Alvarez in 1997 [5]. The problem of classifying all four (eight)-dimensional absolute valued algebras seems still to be open.

In 2016 [5], we classified all four-dimensional absolute valued algebras containing a nonzero central idempotent and we also proved such an algebra contains a commutative sub-algebra of dimension two. Here (**theorems 3.1 and 3.2**) we extend this result to more general situation. Indeed, Let A be a four-dimensional absolute valued algebra containing a nonzero central element a. If A has a commutative sub-algebra of dimension two, then A is isomorphic to a new absolute valued algebras of dimension four. We also show, in **proposition 2.9**, that A contains a sub-algebra of dimension two if and only if $(a^2a)a = (a^2)^2$. We denote that a central idempotent is central element, the reciprocal does not hold in general, and the counter example is given (**theorems 3.1 and 3.2**).

In section 2 we introduce the basic tools for the study of four-dimensional absolute valued algebras. We also give some properties related to central element satisfying some restrictions on commutativity (**lemmas 2.6, 2.7, 2.8 and proposition 2.9**). Moreover, the section 3 is devoted to construct, by algebraic method, some new class of the four-dimensional absolute valued algebras having commutative sub-algebras of dimension two, namely A_1, A_2, A_3, A_4, B_1, B_2, B_3 and B_4. The paper ends with the following main results:

- *Theorem 3.1 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra $B = A(e, i)$, where $i^2 = -e$, $ie = ei = \pm i$, then A is isomorphic to A_1, A_2, A_3, A_4, B_1, B_2, B_3 or B_4 .*

- *Theorem 3.2 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a such that $(a^2a)a = (a^2)^2$, then A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2 .*

II. NOTATION AND PRELIMINARIES RESULTS

In this paper all the algebras are considered over the real numbers field $\mathbb{R}$.

- *Definition 2.1 Let B be an arbitrary algebra.*

- B is called a normed algebra (respectively, absolute valued algebra) if it is endowed with a space norm: $\|\,.\,\|$ such that $\|xy\| \leq \|x\|\|y\|$ (respectively, $\|xy\| = \|x\|\|y\|$, for all $x, y \in B$).
- B is called a division algebra if the operators L_x and R_x of left and right multiplication by x are bijective for all $x \in B \setminus \{0\}$.
- B is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product (./.) such that

$$(./.) : B \times B \longrightarrow \mathbb{R}$$

$$(x, y) \mapsto \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

The most natural examples of absolute valued algebras are $\mathbb{R}, \mathbb{C}, H$ (the algebra of Hamilton quaternion) and O (the algebra of Cayley numbers) with norms equal to their usual absolute values [2] and [8]. The algebras by $\mathbb{C}^*$, $^*\mathbb{C}$ and $\overset{*}{\mathbb{C}}$ (obtained by endowing the space by $\mathbb{C}$ with the products defined by $x * y = \bar{x}y$, $x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively) where $x \to \bar{x}$ is the standard conjugation of $\mathbb{C}$. Note that the algebras $\mathbb{C}$ and $\overset{*}{\mathbb{C}}$ are the only two-dimensional commutative absolute valued algebras.

We need the following relevant results:

- *Theorem 2.2 [3] The norm of any finite dimensional absolute valued algebra comes from an inner product.*

- *Theorem 2.3 [4] The norm of any absolute valued algebra containing a nonzero central idempotent comes from an inner product.*

- *Theorem 2.4 [5] Let A be a four-dimensional absolute valued algebra containing a nonzero central idempotent e, then A is isomorphic to A_1, A_2, A_3 or A_4 defined by: $(\alpha^2+\beta^2=1)$*

A_1	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$-\alpha j - \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	$-e$	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	$-e$

A_2	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$-\alpha j - \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	$-e$	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	$-e$

A_3	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$-\alpha j - \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	$-e$	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	$-e$

A_4	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$-\alpha j - \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	$-e$	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	$-e$

- *Lemma 2.5 [5] Let A be a finite dimensional absolute valued algebra containing a nonzero central idempotent e, then A contains a commutative sub-algebra of dimension two.*

- *Lemma 2.6 Let A be a finite dimensional absolute valued algebra containing a nonzero central element a, then*

- $x^2 = -\|x\|^2 a^2$, for all $x \in \{a\}^{\perp} = \{x \in A, (x/a) = 0\}$
- $xy + yx = -2(x/y)a^2$ for all $x, y \in \{a\}^{\perp}$
- $(xy/yx) = -(x^2/y^2)$ for all $x, y \in \{a\}^{\perp}$ such that $(x/y) = 0$.

Proof. 1) By theorem 2.2, the norm of A comes from an inner product and we assume that $\|x\| = 1$ (where $x \in \{a\}^{\perp}$), we have :

$$\|x^2 - a^2\|^2 = \|x - a\|^2\|x + a\|^2 = 2$$

That is $(x^2/a^2) = -1$, which imply that $\|x^2 + a^2\|^2 = 0$ therefore $x^2 = -a^2$

2) It's clear.

3) We get this identity by simple linearization of the identity $\|x^2\| = \|x\|^2$.

- *Lemma 2.7 Let A be a four-dimensional absolute valued algebra containing a commutative sub-algebra B of dimension two. If $x, y \in B^{\perp}$, then $xy \in B$.*

Proof. According to Rodriguez's theorem [7], B is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. Then there exist an idempotent e and an element i such that $B = A(e, i)$, where $i^2 = -e$ and $ei = ie = \pm i$. Let $F = \{e, i, j, k\}$ be an orthonormal basis of A, as A is a division algebra then L_j is bijective, so there exist j' such that $i = jj'$. We have $(j'/e) = (jj'/je) = (i/je) = \pm (ie/je) = \pm (i/j) = 0$ and $(j'/i) = (jj'/ji) = (i/ji) = \pm (ei/ji) = \pm (e/j) = 0$ so $j' = \alpha j + \beta k$, with $\alpha, \beta \in \mathbb{R}$. Consequently we have $i = jj' = \alpha e + \beta\, jk$, which mean that $jk \in B$. Finally if we pose $x = p\,j + q\,k$ and $y = p'\,j + q'\,k$ with $p, q, p', q' \in \mathbb{R}$. We have $x\,y = (pp' + qq')e + (pq' - qp')\,jk \in B$ $(jk = -kj)$.

- *Lemma 2.8 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B, then $a \in B$.*

Proof. By Rodriguez's theorem [7], B is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. That is, there exist an idempotent e and an element i such that $B = A(e, i)$, where $i^2 = -e$ and $ie = ei = \pm i$. We distinguish the two following cases:

- If $(a/e) = 0$, by lemma 2.6, we have $a^2 = -e$, so $a = \pm i \in B$ $(ai = ia$ and $i^2 = -e$).
- If $(a/e) \neq 0$, we put $c = a - (a/e)\,e$, this imply that $(c/e) = 0$.

Since $ce = ec$, then $c^2 = -\|c\|^2 e = \|c\|^2 i^2$ which means that $c = \pm\|c\|\, i$ $(ci = ic)$, thus $a = c + (a/e)e \in B$. We can put $a = \lambda\, e + \mu\, i$, with $\lambda, \mu \in \mathbb{R}$ $(\lambda^2 + \mu^2 = 1)$ and let $b = \mu\, e - \lambda\, i \in B$ and $j \in A$ two elements orthogonal to a.

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As aj = ja, we get λ ej + μ ij = λ je +μ ji (1)

Using lemma 2.6, we have bj + jb = 0. This imply

μ ej + λ ij = $-\mu$ je $-$ λ ji (2)

From the equalities (1) and (2), we obtain
$2\lambda \mu$ ej + ij = $(\mu^2 - \lambda^2)$ ji $(\lambda^2 + \mu^2 = 1)$

Therefore $(2\lambda \mu \text{ ej} + \text{ij/ij}) = ((\mu^2 - \lambda^2) \text{ ji /ij})$

Applying lemma 3, we get

$1 = (\mu^2 - \lambda^2)(\text{ji/ ij}) = -(\mu^2 - \lambda^2)(j^2/ i^2)$ (3)

Since $a^2 = (\lambda^2 - \mu^2)e \pm 2\lambda \mu i$, $i^2 = -e$ and $j^2 = -a^2$

then the equality (3) gives $(\lambda^2 - \mu^2)^2 = 1$. Hence $\lambda^2 - \mu^2 = 1$ or $\lambda^2 - \mu^2 = -1$, as $\lambda^2 + \mu^2 = 1$, then $\lambda^2 = 1$ (because, $\lambda = (a/e) \neq 0$). That is, $a = \pm e \in B$.

We give some conditions implying the existence of two-dimensional sub-algebras.

➢ *Proposition 2.9 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a, then the following assertions are equivalent:*

$(a^2a)a = (a^2)^2$

A contains a commutative sub-algebra B of dimension two.

Proof. 1) $\Rightarrow$ 2) If $(a/a^2) = 0$, by lemma 2.6, we have $(a^2)^2 = -a^2$, so $a^2a = -a$ which means that B := A(a, a^2) is two dimensional commutative sub-algebra of A.

- If a and a^2 linearly dependent, then a is a nonzero central idempotent. Therefore A contains a commutative sub-algebra of dimension two (lemma 2.5),
- we assume that $(a/a^2) = m \neq 0$ such that $m^2 \neq 1$. We pose $d = a^2 - ma$, this imply that $(d/a) = 0$, using lemma 2.6, we have $d^2 = -\|d\|^2a^2 = -(1 - m^2)a^2$ which means that

$$- (1 - m^2)a^2 = (a^2 - ma)^2 = (a^2)^2 - 2ma^2a + m^2a^2$$
$$\text{So} \quad - (1 - m^2)a^2 = (a^2a)a - 2ma^2a + m^2a^2$$
$$\text{Hence} \quad - (1 - m^2)a = a^2a - 2ma^2 + m^2a = da - md$$

Therefore $ad = da = -(1 - m^2)a + md$ this means that A(a, d) is a two-dimensional commutative sub-algebra of A.
2) $\Rightarrow$ 1) Let B be a two-dimensional commutative sub-algebra of A, according to Rodriguez's theorem [7] B is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. That is B:= A(e, i) such that ie = ei = $\pm$i and $i^2 = -e$, where e is an idempotent of A. The lemma 2.8 proves that $a = \pm e$ or $a = \pm i$ and we have the two following cases:

- B is isomorphic to $\mathbb{C}$, hence a verifies the equality (1)
- B is isomorphic to $\overset{*}{\mathbb{C}}$, then $a = \pm e$ satisfies the identity (1).
- But since ie = ei = $-$i and $i^2 = -e$, thus $(a^2a)a \neq (a^2)^2$.

➢ *Remark 2.10 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B isomorphic to $\overset{*}{\mathbb{C}}$. If $(a^2a)\, a = (a^2)^2$, then a is a central idempotent of A*

III. MAIN RESULTS

➢ *Theorem 3.1 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B = A(e, i) , where $i^2 = -e$, ie = ei = $\pm$ i, then A is isomorphic to $A_1, A_2, A_3, A_4, B_1, B_2, B_3$ or B_4.*

Proof. According to lemma 1, we have the following cases:

- *a = $\pm$ e is a central idempotent of A, by theorem 2.3, A is isomorphic to A_1, A_2, A_3 or A_4.*
- 2) a = $\pm$ i is a central element of A. Let F = {e, i, j, k} be an orthonormal basis of A, since $j^2 = k^2 = -i^2 = e$ and jk = $-$kj $\in$ B (lemma 2.6), then $(jk/e) = (jk/j^2) = (k/j) = 0$ hence jk = $\pm$ i.
- The set {e, i, ij, ik} is an orthonormal basis of A, then $(ej/e) = (ej/e^2) = (e/j) = 0$, $(ej/i) = \pm (ej/ei) = \pm (j/i) = 0$ and $(ej/ij) = (e/j) = 0$.
- Which imply that ej = $\pm$jk, similarly $(ek/e) = (ek/e^2) = (e/k) = 0$, $(ek/i) = \pm (ek/ei) = \pm(k/i) = 0$ and $(ek/ik) = (e/k) = 0$ Then ek = $\pm$ ij. According to lemma 3, $(ej/je) = -(e^2/j^2) = (e/i^2) = -1$ hence ej = $-$je. Also $(ek/ke) = -(e^2/k^2) = (e/i^2) = -1$, thus ek = $-$ke. We assume that j k = i and we distinguish the following cases:

B isomorphic to $\mathbb{C}$, we have ei = ie = i and $i^2 = -e$. So ej = ik and ek = $-$ij. Indeed, if ej = $-$ik then

$$(e + k)j = ej + kj = -ik - jk = -ik - i = - ki - ei = - (e + k)i$$

Which gives i = $-$ j (A has no zero divisors), contradiction. Moreover, if ek = ij then

$$(e + j)k = ek + jk = ij + i = ji + ei = (e + j)i$$

The last gives k = i, which is absurd. We pose ej = αj + βk $(\alpha^2 + \beta^2 = 1)$, then ik = ki = αe + βj. And ek = λj + μk, where $\lambda^2 + \mu^2 = 1$. Since (ek/ej) = 0, we get $\alpha\lambda + \beta\mu = 0$. So

$$\begin{aligned}(\alpha \mu - \beta \lambda)^2 &= \alpha^2 \mu^2 - 2\alpha \mu \beta \lambda + \beta^2 \lambda^2 \\ &= \alpha^2 \mu^2 + 2\alpha^2 \lambda^2 + \beta^2 \lambda^2 \\ &= \alpha^2 (\mu^2 + \lambda^2) + \lambda^2 (\alpha^2 + \beta^2) \\ &= \alpha^2 + \lambda^2\end{aligned}$$

On the other hand we have.

$$
\begin{aligned}
(\alpha\,\mu - \beta\,\lambda)^2 &= \alpha^2\,\mu^2 - 2\alpha\,\mu\,\beta\,\lambda + \beta^2\,\lambda^2 \\
&= \alpha^2\,\mu^2 + 2\,\beta^2\,\mu^2 + \beta^2\,\lambda^2 \\
&= \mu^2\,(\alpha^2 + \beta^2) + \beta^2\,(\mu^2 + \lambda^2) \\
&= \mu^2 + \beta^2
\end{aligned}
$$

So $\alpha^2+\lambda^2 = \mu^2+\beta^2 = 2 - (\alpha^2+\lambda^2)$, which means that $\alpha^2+\lambda^2 = 1$ and consequently $\alpha\,\mu - \beta\,\lambda = \pm 1$.

*) If $\alpha\,\mu - \beta\,\lambda = 1$, then

$$\mu = \mu\,(\alpha\,\mu - \beta\,\lambda) = \alpha\,\mu^2 - \beta\,\lambda\,\mu = \alpha\,\mu^2 + \alpha\,\lambda^2 = \alpha$$

And $\lambda = \lambda\,(\alpha\,\mu - \beta\,\lambda) = \alpha\,\mu\,\lambda - \beta\,\lambda^2 = -\,\beta\,\mu^2 - \beta\,\lambda^2 = -\,\beta$

Therefore the multiplication table of A is given by:

B_1	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$\alpha j + \beta k$
j	$-\alpha j - \beta k$	$-\beta j + \alpha k$	e	i
k	$\beta j - \alpha k$	$\alpha j + \beta k$	$-i$	e

**) If $\alpha\,\mu - \beta\,\lambda = -1$, then

$$\mu = -\mu\,(\alpha\,\mu - \beta\,\lambda) = -\alpha\,\mu^2 + \beta\,\lambda\,\mu = -\alpha\,\mu^2 - \alpha\,\lambda^2 = -\alpha$$

And $\lambda = -\lambda\,(\alpha\,\mu - \beta\,\lambda) = -\alpha\,\mu\,\lambda + \beta\,\lambda^2 = \beta\,\mu^2 + \beta\,\lambda^2 = \beta$

Therefore the multiplication table of A is given by:

B_2	e	i	j	k
e	e	i	$\alpha j + \beta k$	$\beta j - \alpha k$
i	i	$-e$	$\beta j - \alpha k$	$\alpha j + \beta k$
j	$-\alpha j - \beta k$	$\beta j - \alpha k$	e	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	e

ii) B isomorphic to $\overset{*}{\mathbb{C}}$, we have $ei = ie = -\,i$, $i^2 = -e$ and $jk = i$. If we define a new multiplication on A by $x * y = \bar{x}\,\bar{y}$, we obtain an algebra $\overset{*}{A}$ which contains a sub-algebra isomorphic to $\mathbb{C}$. Therefore $\overset{*}{A}$ has an orthonormal basis which the multiplication tables are given previously. Consequently, the multiplication tables of the elements of the base F of A are given by :

B_3	e	i	J	k
e	e	$-i$	$-\alpha j - \beta k$	$\beta j - \alpha k$
i	$-i$	$-e$	$-\beta j + \alpha k$	$\alpha j + \beta k$
j	$\alpha j + \beta k$	$-\beta j + \alpha k$	e	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-\,i$	e

and

B_4	e	i	j	k
e	e	$-i$	$-\alpha j - \beta k$	$-\beta j + \alpha k$
i	$-i$	$-e$	$\beta j - \alpha k$	$\alpha j + \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	e	i
k	$\beta j - \alpha k$	$\alpha j + \beta k$	$-i$	e

➢ *Theorem 3.2 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a such that $(a^2a)a = (a^2)^2$, then A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2*

Proof. According to proposition 2.9, A contains a commutative sub-algebra B isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. By remark 2.10 and theorem 3.1, A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2.

IV. CONCLUSION

We have the following two classical results:

- Every four-dimensional real absolute valued algebra containing a nonzero central idempotent is isomorphic to A_1, A_2, A_3 or A_4
- If A is a four-dimensional real absolute valued algebra containing a nonzero central idempotent, then A contains a sub-algebra of dimension two. Based on the findings of this article, the following conclusions can be drawn:
- In general, if A is a four-dimensional real absolute valued algebra containing a nonzero central element, then, A is isomorphic to A_1, A_2, A_3, A_4, B_1, B_2, B_3 or B_4 .
- Note that, central idempotent is a central element. The reciprocal case does not hold in general, and the counter example is given (B_1 and B_2).
- We give some conditions implying that these new algebras having sub-algebras of dimension two. We show, that a four-dimensional real absolute valued algebra containing a nonzero central element a and having sub-algebra of dimension two if and only if $(a^2a)a = (a^2)^2$.

In future work, it is intended to classify all four dimensional real absolute valued algebra containing a nonzero central element.

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REFERENCES

[1]. A. A. Albert, Absolute valued real algebras, Ann. Math, 48 (1947), 405-501.

[2]. F. Hirzebruch, M. Koecher and R. Remmert, Numbers. Springer Verlag (1991).

[3]. M. L. El-Mallah, On finite dimensional absolute valued algebras satisfying (x, x, x) =0. Arch. Math. 49 (1987), 16-22.

[4]. EL-Mallah, Absolute valued algebras containing a central idempotent. J. Algebra. 128 (1990), 180-187. 108.
[5]. A. Moutassim and M. Benslimane, Four dimensional absolute valued algebras containing a nonzero central idempotent or with left unit. int. J. Algebra, Vol 10, 2016, no. 11, 513-524.
[6]. M. I. Ramirez, On four-dimensional absolute valued algebras, Proceedings of the International Conference on Jordan Structures (Malaga, 1997), univ. Malaga, Malaga, (1999), 169-173.
[7]. A. Rodriguez, Absolute valued algebras of degree two. In Non-associative Algebra and its applications (Ed. S. Gonzalez), Kluwer Academic Publishers, Dordrecht-Boston-London (1994), 350-356.
[8]. A. Rodriguez, Absolute-valued algebras, and absolute-valuable Banach spaces, Advanced Courses of Mathematical Analysis I, World Sci. Publ. Hackensack, NJ, 2004, 99-155.
[9]. K. Urbanik and F. B. Wright, Absolute valued algebras. Proc. Amer. Math. Soc. 11 (1960), 861-866.