ISSN No:-2456-2165

A Study on Cauchy-Schwarz Inequality and its Various Applications in Solving Inequalities with Exploring its Geometric Interpretation and Related Inequalities

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Abstract:- The Cauchy-Schwarz inequality is one of the most fundamental inequalities in the world of mathematics with its powerful ability for solving complex inequality problems. This research article illustrates the deep study of Cauchy-Schwarz inequality with its various proofs (algebraic and geometric) including its applications in the mathematical world of inequalities. The paper will to deepen our knowledge and understanding of this inequality exploring its implications. The article includes other related inequalities like Triangle Inequality, *Minkowski Inequality, and* Hö*lder's Inequality.*

I. INTRODUCTION

French mathematician, physicist, and engineer Augustin-Louis Cauchy, a pioneer in mathematical analysis and the theory of substitution groups [1], proposed a powerful inequality that was rediscovered by Hermann Amandus Schwarz. The Cauchy-Schwarz inequality is used in complex mathematical proofs, but also in a broad range of implications and applications in other fields such as probability theory, linear algebra, mathematical analysis, and optimization

A. *Objective*

The objective of this research article includes:

- To make a comprehensive study and analysis of the Cauchy-Schwarz inequality, exploring its fundamental principles,
- To dive into its application in solving inequalities,
- To illustrate the geometrical interpretation of the Cauchy-Schwarz inequality,
- To study briefly other related inequalities.

In algebraic form, Cauchy-Schwarz Inequality states that for any reals, $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$, is given by,

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2$$

with equality if and only if $\frac{x_i}{y_i} = c$ *for some constant* $c \in \mathbb{R}^+ \forall\, 1 \leq i \leq$ n and $x_iy_i \neq 0$.

In summation form, $\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \geq (\sum_{i=1}^{n} x_iy_i)^2$[2]

II. PROOF OF CAUCHY-SCHWARZ INEQUALITY

Before proving Cauchy-Schwarz inequality by algebraic method, it is good to learn about Arithmetic Mean-Geometric Mean (AM-GM) inequality.For any two non-negative reals x_1 and x_2, AM-GM inequality states,

$$\frac{x_1+x_2}{2} \geq \sqrt{x_1x_2}$$

For n reals,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1x_2 \dots x_n}$$

A. *Algebraic Proof*

Let $X = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \quad Y = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$.

By the arithmetic-geometric means inequality, we have

$$\sum_{i=1}^{n} \frac{x_iy_i}{XY} \leq \sum_{i=1}^{n} \frac{1}{2}\left(\frac{x_i^2}{X^2} + \frac{y_i^2}{Y^2}\right) = 1,$$

Or, $\sum_{i=1}^{n} x_iy_i \leq XY = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$

$$Hence, \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

This completes the proof. [3]

B. Geometrical Proof

Let us construct the vectors $X = (x_1, x_2, \cdots, x_n), \quad Y = (y_1, y_2, \cdots, y_n)$. By the formula of scalar product:

$$X \cdot Y = |X||Y| \cos(X, Y)$$

$$X \cdot Y \leq |X||Y|$$

From the expressions, $X \cdot Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$,

$|X|^2 = \sum_{i=1}^{n} x_i^2, \quad |Y|^2 = \sum_{i=1}^{n} y_i^2$,we get the required inequality. [4]

III. APPLICATION OF CAUCHY SCHWARZ INEQUALITY IN SOLVING OTHER INEQUALITIES

A. Linear Inequalities

Any statement which contains linear algebraic structure of the form $ax + b = 0$with $a \neq 0$and having inequality symbols ($>, <, \geq, or \ \leq$) is called linear inequality.[5] Cauchy-Schwarz inequality can be used to solve various linear inequalities by providing upper bounds on the products. Let's consider a simple example.

Problem: *Maximize:* $3x + 4y$ *from the constraint* $x^2 + y^2 = 25$, *with* $x, y \geq 0$.

Using Cauchy-Schwarz inequality in numbers $3, x$ and $4, y$.

$$(3x + 4y)^2 \leq (3^2 + 4^2)(x^2 + y^2)$$

Or, $|3x + 4y| \leq 25$

Since $x, y \geq 0$,

$|3x + 4y| = 3x + 4y$

Thus, the maximum value of $3x + 4y$ is 25.

B. Polynomial Inequalities

Any statement which contains polynomial expression $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$with $a_n \neq 0$and having inequality symbols ($>, <, \geq, or \ \leq$) is called polynomial inequality. For example, $8x^3 + 81x^4 \leq 144$. The Cauchy-Schwarz inequality helps solve polynomial inequalities by providing upper bounds on the products of polynomial expressions. It will always be clear if we consider an example.

Problem: Let there exist positive reals x, y *and* z such that $xyz = 1$. Then, prove that:

$x + y + z \leq x^2 + y^2 + z^2$ [6]

Using Cauchy-Schwarz inequality,

$$(x.1 + y.1 + z.1)^2 \leq (x^2 + y^2 + z^2)(1^2 + 1^2 + 1^2)$$

$$\text{Or, } x + y + z \leq \sqrt{x^2 + y^2 + z^2}\sqrt{3} \qquad (1)$$

From AM-GM inequality,

$$\sqrt[3]{x^2 y^2 z^2} \leq \frac{x^2 + y^2 + z^2}{3}$$

As $xyz = 1$ we get, $\sqrt{3} \leq \sqrt{x^2 + y^2 + z^2}$ (2)

From (1) and (2), we deduce that

$x + y + z \leq \sqrt{x^2 + y^2 + z^2}\sqrt{3} \leq \sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2 + z^2} = x^2 + y^2 + z^2$

which completes the proof.

IV. GEOMETRICAL INTERPRETATION OF CAUCHY-SCHWARZ INEQUALITY

Let's consider two vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ in n-dimensional space with real components.[7]

The dot product of these vectors is given as,

$$X \cdot Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

The magnitudes or lengths of the vectors from origin are defined as,

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \text{and} \quad \|Y\| = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$$

Geometrically, the Cauchy-Schwarz inequality states that the absolute value of the dot product of two vectors is less than or equal to the product of their lengths.

$$|X \cdot Y| \leq \|X\| \|Y\|$$

For geometrical interpretation, let α be the angle between the vectors X and Y.Their dot product is:

$$X \cdot Y = \|X\| \|Y\| \cos(\alpha)$$

From Cauchy-Schwarz inequality, we have

$$|X \cdot Y| = \|X\| \|Y\| \, |\cos(\alpha)| \leq \|X\| \|Y\|$$

It means the absolute value of the cosine of the angle between the vectors is less than or equal to 1 i.e. $-1 \leq \cos(\alpha) \leq 1$ and the maximum value of the $\cos(\alpha)$ is 1, which occurs if and only if the vectors are parallel or anti-parallel. In this case, the angle between the vectors becomes 0 or π and cos(α)=1.

V. RELATED INEQUALITIES

As mention earlier, Cauchy-Schwarz inequality is the fundamental inequality in mathematics, so it has many related inequalities. Let's briefly discuss three important ones — Triangle inequality, Minkowski's inequality, and Hölder's inequality.

A. *The Triangle Inequality*

The Triangle inequality states that for any two vectors X and Y, the length of their sum is less than or equal to the sum of their lengths. $\|X + Y\| \leq \|X\| + \|Y\|$

This inequality reflects the geometrical intuition that the shortest path between two points is a straight line.

B. *Minkowski's Inequality*

Minkowski's inequality is an extensive version of the Triangle inequality in vector space. Let's consider two n-dimensional vectors $X = (x_1, x_2, \cdots, x_n)$, $Y = (y_1, y_2, \cdots, y_n)$. The Minkowski's inequality states that for the vectors X and Y and a real number $p \geq 1$, the following result holds true:

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$ [8]

C. *Hölder's Inequality*

Just like in Minkowski's inequality, consider two n-dimensional vectors $X = (x_1, x_2, \cdots, x_n)$, $Y = (y_1, y_2, \cdots, y_n)$. The Hölder's inequality states that if there exists some reals $p > 1$ and $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

then the following condition holds true:

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$ [9]

VI. CONCLUSIONS

Cauchy-Schwarz inequality is a fundamental inequality in mathematics with its wide applications in other areas. This research paper aimed to provide a deep study of Cauchy-Schwarz inequality, including its basic fundamentals, proofs, applications in solving other forms of inequalities and finally its geometric interpretation. This research articles also touches other related inequalities — Triangle inequality, Minkowski's inequality, and Hölder's inequality. With the clear understanding and comprehensive study of the Cauchy-Schwarz inequality, this work aims to inspire further progress in the Inequality World of Mathematics.

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