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Economic Applications of Calculus: Computing Optimal Production Rates

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Abstract:- This paper investigates the use of the Cobb-Douglas production functionto reach optimum rates of production. It uses statistics of an Indian firm in the automotive industry as an example to explore the dynamics behind the function.

I. INTRODUCTION

The theory of calculus was invented by Isaac Newton and Gottfried Leibniz and is known as the mathematical study of change. This groundbreaking theory has been implemented for practical applications in various fields like electrical, civil, and mechanical engineering. However, one of its other important uses lies in economic theory, the basis for the transactions that make up any nation's economy. This research paper implements the tools of calculus to model economic development by analyzing the impact of various components such as labor, capital, and technology on economic output. This helps procure the most efficient ratio of the above-mentioned external factors for maximizing a firm's revenue, a metric which is used to measure profit.

II. COBB-DOUGLAS PRODUCTION FUNCTION

The Cobb-Douglas production function, founded by Charles Cobb and Paul Douglas between 1927-1947, is a form of the production function used to depict the relationship between total output and other input quantities like labor, capital, and total factor productivity. The function also takes into account two more inputs namely output elasticity of labor and output elasticity of capital. It states that: -

$$f(L,K) = Y = A.L^{\alpha}.K^{\beta} \tag{2.1}$$

Where,

- Y is the *Total Output* (the real value of all goods produced in a year);
- A is the *Total factor productivity* (ratio of the aggregate output to the aggregateinput);
- L is the *Labor Input* (a measure of the total wages used for human services);
- K is the *Capital Input* (a measure of all machinery, equipment, and buildingsused for production);
- α is the *Output elasticity of labor* (responsiveness of output to a change inlabor);
- β is the *Output elasticity of capital* (responsiveness of output to a change incapital).

➢ *Partial Derivation*

To determine the optimum ratio of labor and capital in the production of a good, first we must find the differential of the function with respect to the changing labor and capital (factor productivity remains constant in a firm). This will help us find the maxima of the function, the amount of capital and labor at which output is maximized.

Differentials are defined as the following under the fundamental theory of calculus.

$$\frac{\partial Y}{\partial L} = \lim_{h \to 0} \frac{f(L+h) - f(L)}{h} \tag{2.2}$$

$$\frac{\partial Y}{\partial K} = \lim_{h \to 0} \frac{f(K+h) - f(K)}{h}. \tag{2.3}$$

Application of power rule under partial differentiation in 2.1

- *Differentiating Change in Output w.r.t a Change in Labor to Define α in L and K*

$$\frac{\partial Y}{\partial L} = A.\alpha.K^{\beta}.L^{\alpha-1} \tag{2.4}$$

$$\frac{\partial Y}{\partial L} = \frac{A.\alpha.K^{\beta}.L^{\alpha}}{L} \tag{2.5}$$

Replacing the original function, we get

$$\frac{\partial Y}{\partial L} = \frac{\alpha.Y}{L} \tag{2.6}$$

$$\alpha = \frac{\partial Y.L}{\partial L.Y} \tag{2.7}$$

- *Similarly, Differentiating Change in Output W.R.T A Change in Capital to Define B in L and K*

$$\frac{\partial Y}{\partial K} = A.\beta.L^{\alpha}.K^{\beta-1} \tag{2.8}$$

$$\frac{\partial Y}{\partial K} = \frac{A.\beta.K^{\beta}.L^{\alpha}}{K} \quad (2.9)$$

$$\frac{\partial Y}{\partial K} = \frac{\beta.Y}{K} \quad (2.10)$$

$$\beta = \frac{\partial Y.K}{\partial K.Y} \quad (2.11)$$

- *The value of α and β factors can be used to predict the result of the returns to scale such that:*

- *If α + β = 1, there are constant returns to scale, meaning optimum quantity is produced.*
- *If α + β >1, there are increasing returns to scale, meaning more quantity of the good should be produced.*
- *If α + β <1, there are decreasing returns to scale, meaning lesser quantity of the good should be produced.*

For calculation, assuming $\alpha + \beta \approx 1$, and $0 \leq \alpha, \beta \leq 1$. This would help us get an appropriate formula for % change in output.

$$\alpha + \beta \approx 1 \quad (2.12)$$

$$\frac{\partial Y.L}{\partial L.Y} + \frac{\partial Y.K}{\partial K.Y} \approx 1 \quad (2.13)$$

$$\frac{\partial Y.L.\partial K}{\partial L.Y.\partial K} + \frac{\partial Y.K.\partial L}{\partial K.Y.\partial L} \approx 1 \quad (2.14)$$

$$\partial Y.L.\partial K + \partial Y.K.\partial L \approx \partial L.Y.\partial K \quad (2.15)$$

$$\partial Y.L.\partial K + \partial Y.K.\partial L - \partial L.Y.\partial K \approx 0 \quad (2.16)$$

$$\partial Y(L.\partial K + K.\partial L) \approx \partial L.Y.\partial K \quad (2.17)$$

$$\frac{\partial Y}{Y} \approx \frac{\partial L.\partial K}{L.\partial K + K.\partial L} \quad (2.18)$$

Hence, if:

$$\frac{\partial Y}{Y} = \frac{\partial L.\partial K}{L.\partial K + K.\partial L} \quad (2.19)$$

- *An X% Increase in Capital and Labor Causes an Equal X% Increase in Output.*

$$\frac{\partial Y}{Y} > \frac{\partial L.\partial K}{L.\partial K + K.\partial L} \quad (2.20)$$

- *An X% Increase in Capital and Labor Causes a Greater than X% Increase in Output.*

$$\frac{\partial Y}{Y} < \frac{\partial L.\partial K}{L.\partial K + K.\partial L} \quad (2.21)$$

- *An X % Increase in Capital and Labor Causes a Less than X% Increase in Output.*

III. CALCULATING INPUT QUANTITIES

I have been able to attain the cost sheet of an Indian firm "Superite Private Ltd." which specializes in manufacturing brake linings for automobiles. I will be using its data for the years 2011 and 2012 to empirically test the above-mentioned formulae and understand their accuracy.

- *Total Output*

After some calculations, I have been able to produce an estimated value of output for the two mentioned years -

Total output in 2011 = ₹22,014,318 Total output in 2012 = ₹21,463,005

- *Capital*

Table 1 Complete Breakdown of the Capital Investment (in INR) by the Firm in 2011and 2012

Capital (Gross block - Depreciation) in INR		
Fixed Assets	**2011**	**2012**
Land	282,000	282,000
Factory Building	560,900	504,810
Plant and Machinery	2,509,150	2,084,875
Electric fittings	197,540	170,062
Dyes and Tools	244,790	148,070
Computer	1,731	1,039
Generator	69,160	59,540
Handling equipment	25,477	21,933
Measuring instrument	6,334	5,453
Cost of materials consumed	15,363,390	15,129,301
TOTAL	₹19,260,472	₹18,407,083

➢ *Labor*

- *Amount of money spent on wages in 2011 = |1,811,374Amount of money spent on wages in 2012 = |1,764,164*
- *Average number of daily working hours = 10Estimated number of working days = 280*
- *Number of men employed in 2011 = 45 people*
- *Number of person-hours worked in 2011 = 45*10*280 = 126,000 hours*
- *Number of men employed in 2012 = 43 people*
- *Number of person-hours worked in 2011 = 43*10*280 = 120,400 hours*

➢ *Total Factor Productivity*

$$\text{Total Factory Productivity for the year 2011} = \frac{\text{Total Output in 2011}}{(\text{Capital + Labour input in 2011})}$$

$$= \frac{22,014,318}{19,260,472 + 1,811,374} = \frac{22,014,318}{21,071,846} = 1.045(\text{rounded to 3sf}) \quad (3.1)$$

$$\text{Total Factory Productivity for the year 2012} = \frac{\text{Total Output in 2012}}{(\text{Capital + Labour input in 2012})}$$

$$= \frac{21,463,005}{18,407,083 + 1,764,164} = \frac{21,463,005}{20,171,247} = 1.064(\text{rounded to 3sf}) \quad (3.2)$$

➢ *Calculating α and β*

- ∂ Y = 22,014,318 - 21,463,005 = ₹551313
- ∂ L = 126,000 - 120,400 = 5600 hours
- ∂ K = 19,260,472 - 18,407,083 = ₹853,389

As calculated in 2.7,As calculated in 2.11,

$$\alpha = \frac{551313 * 126000}{22014318 * 5600} = 0.563 \quad (3.3)$$

As calculated in 2.11,

$$\beta = \frac{551313 * 19260472}{22014318 * 853389} = 0.565 \quad (3.4)$$

$$\alpha + \beta = 1.128 \quad (3.5)$$

Thus, since $\alpha + \beta > 1$, one can assume that the firm is thriving as there are increasingreturns to scale.

This conclusion aligns with the statement made by the owner of the factory, whorecounts reasonable profits made during that year.

IV. EMPIRICAL TESTING

Let us test the efficiency of the firm by inputting the above values in 2.1 and comparing our actual output with this predicted output for the year 2011.

$$f(L, K) = Y = A.L^{\alpha}.K^{\beta} \quad (4.1)$$

$$Y = 1.045 * 1,811,374^{0.563} * 9,260,472^{0.565} = 30,096,956.7 \quad (4.2)$$

This value is significantly greater than the actual value of output in 2011, which is 22,014,318. This can possibly be due to inefficient production by a firm, discounting external variables like hourly breaks, broken pieces, and other unforeseen circum- stances.

Hence, the firm can maximize total output, without increasing investment, by ensuring greater efficiency of the different factors of production.

$$\text{Efficiency of firm} = \frac{22,014,318}{30,096,956.7} \times 100 = 73.14\% \quad (4.3)$$

V. GRAPHING

In order to make the original function linear, taking log on both sides of 2.1,

$$logY = log(A.L^{\alpha}.K^{\beta}) \quad (5.1)$$

$$logY = logA + logL^{\alpha} + logK^{\beta} \quad (5.2)$$

$$logY = logA + \alpha.logL + \beta.logK \quad (5.3)$$

After inputting the values for A, α, and β, I used 3-dimensional graphing to see the change in output with a change in L and K.

In the below graph, the *x*-axis represents Labor, the *y*-axis represents Capital, and the *z*-axis represents Output.

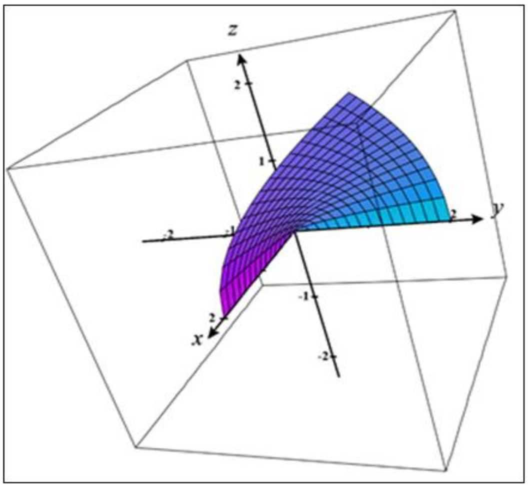

Fig 1 3-D Modelling of Change in Output of the Firm with Respect to Capital and Labor

The graph depicts how output increases with labor and capital, with each of the two variables making an almost equal contribution to this change.

VI. CONCLUSION

In conclusion, the Cobb-Douglas production effectively demonstrates the relation between total output and the other factors of labor and capital. It can be readily utilized to predict the theoretical output of a firm, depending upon total factor productivity and the elasticity of the two variables.

In this case, the function helped depict how the firm "Superite Private Ltd" was operating at an efficiency rate of 73.14%. As a result, the firm can introspect and increase output not by investment, but by ensuring greater efficiency of labor and capital. Hence, investment money could be saved. Even if the firm can reach an efficiency rate of 85%, it can increase output by |3568000 without spending a single penny.

This paper depicts how the Cobb-Douglas production function is a powerful tool, with endless applications in economic theory. Its uses are not just limited to a microeconomic level. With slight adjustment and change, the function can also be adapted for macroeconomic purposes like predicting the GDP (Gross Domestic Product) ofa nation.

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