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A Fixed-Point Approach to Stability of Quadratic Functional Equation in Non-Archimedean 2-Normed Spaces

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Abstract:- In this paper, we use the fixed point method to investigate the Hyers Ulam-Rassias stability for the following quadratic functional equation:$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(x+z)$ in a non-Archimedean2-Normed spaces. Some applications of our results are illustrated.

Keywords:- *Hyers-Ulam Stability, A Non-Archimedean 2-Normed Spaces, Quadratic Mapping, Stability, Fixed Point Theory.*

I. INTRODUCTION

In 1940, Ulam [1] suggested the stability problem of functional equations concerning the stability of group homeomorphism as follows:

Let (G,o) be a group and let $(H,\star,d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist $a\ \delta = \delta(\epsilon) > 0$ such that if a mapping $F: G \to H$ satisfies the inequality:

$$d(f(x \circ y), f(x) * f(y)) < \delta$$

for all $x, y \in G$. Then a homomorphism $F: G \to H$ exists $d(f(x), F(x))$ for all $x \in G$?

In 1941, Hyers [2] give the first (partial) affirmative answer to the question of Ulam for Banach spaces. Thereafter, we call type the Hyers – Ulam stability.

The result of Hyers was generalized by Aoki [6] for approximate additive mappings and by Th.m. Rassias [7] for an approximate linear mapping following the difference Cauchy equation

$\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$

The functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y) \qquad (1)$$

Is related to symmetric bi-additive function and called a quadratic functional and every solution of the quadratic (1) is said to be quadratic mapping. Skof [20] proved the Hyers– Ulam stability of the quadratic functional equation (1).

The theory of linear 2-normed spaces was first developed by Gahler[8] in the mid 1960's, while That of 2-Banach spaces was studied later by Gahler [9] and White [10].

The functional equation

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(x+z) \qquad (2)$$

Be another quadratic mapping, by Skof [20] proved the Hyers– Ulam stability of quadratic functional equation (2). Jung [5] proved the Hyers– Ulam stability of quadratic functional equation (2) in the Banach space with respect to some conditions.

In 1978, Rassias[7] extended the Hyers- Ulam stability by considering variable. In 1994, it also has been generalized to function case by Gavruta [18].

Hensel[21] has introduced a normed space which does not have the Archimedean property, Rassias[22] proved the generalized Hyers- Ulam stability of the additive functional equation and the quadratic functional equation in non-Archimedean spaces.

Hyers's method used in [2], which is often called the direct method, has been applied for studying the stability of various functional equations but this method sometimes does not work [30]. Nevertheless, there are also other approaches proving the Hyers-Ulam stability, for example: the method of invariant means [32], the method of based on sandwich theorems [31], the method using the concept of shadowing [33] and the fixed-point method.

In this paper, we will use the fixed-point method which is the second most popular technique of proving the stability of quadratic functional equation in (2) in a non-Archimedean 2-Banach spaces under the approximately even (or odd) conditions and some asymptotic behaviors of quadratic and additive mappings shall be investigated and generalized the stability of the same functional equation in a non-Archimedean 2-Banach spaces.

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II. PRELIMINARIES

In this section we introduce some notions which we be used in the sequel.

- *Definition 2.1. [24] Let $\mathbb{K}$ be a field. A non-Archimedean absolute value (or valuation) on a filed $\mathbb{K}$ is a function $|.|: \mathbb{K} \longrightarrow \mathbb{R}$ such that for any $a.b \in \mathbb{K}$, we have,*
  (i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
  (ii) $|ab| = |a||b|$
  (iii) $|a+b| \leq max\{|a|, |b|\}$

Condition (iii) is called the strong triangle inequality. By (ii), we have $|-1| = |1| = 1$.

Thus, by induction, it follows from (iii) that $|n| \leq 1$, for each integer n. we always assume in addition that $|.|$ is non-trivial, that is, there is an $a_0 \in \mathbb{K}$ such tat $|a_0| \neq 0,1$

- *Definition 2.2. [24] Let X be a vector space over a non-Archimedean field K. A function $\|.\| : X \to R$ is called a non-Archimedean norm if it satisfies the following properties:*

(a) $\|x\| = 0$ *if and only if x = 0,*

(b) $\|rx\| = |r| \, \|x\|$

(c) $\|x+y\| = max\{\|x\|, \|y\|\}$

for all $x,y \in X$ and $r \in K$.
If $\|x\|$ is called a non-Archimedean norm on X and the pair $(X, \|.\|)$ is called a non-Archimedean normed space

- *Definition 2.3. [22] Let $(X, \|.\|)$ a non-Archimedean normed space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be convergent in $(X, \|.\|)$ if there exists an $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. In case, x is called the limit of the sequence $\{x_n\}$, and one denotes it by*

$\lim_{n\to\infty} x_n = x$. *A sequence $\{x_n\}$ is said to a Cauchy in $(X, \|.\|)$ if for all $p \in N$.*

- *Remark 2.4. [22] By (c) in Definition (2.2),*

$$\|x_n - x_m\| \leq max\{\|x_{j+1} - x_j\| \mid m \leq j \leq n-1\} \qquad (n > m) \tag{3}$$

A sequence $\{x_n\}$ is Cauchy in $(X, \|.\|)$ if and only if *$\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $(X, \|.\|)$. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.*

- *Definition 2.5. [26] Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following:*

(i) $d(x, y) = 0$ if only and only $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, y) + d(y,x)$ for all $x, y,z \in X$.

Then (X,d) is called a generalized metric space, (X,d) is called complete if every d- Cauchy sequence in X is d-convergent.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

- *Example 2.6. [26] Let $X := C(\mathbb{R})$ "the space of the continuous functions on $\mathbb{R}$" and let $d: X^2 \to [0, \infty]$ given by $d(x, y) = \sup_{t\in\mathbb{R}}|x(t) - y(t)|$*

Then the pair (X,d) is a generalized complete metric space.

- *Definition 2.7. [27] Let (X,d) be a generalized complete metric space, a mapping $J : X \longrightarrow X$ satisfies a Lipschitz condition with a constant $L > 0$ "Lipschitz constant" if*

$$d(J(x),J(y)) \leq Ld(x, y) \tag{4}$$

for all $x,y \in X$. If $L < 1$, then J is called a strictly contractive operator.

We remark that only difference between the generalized metric and the usual metric is that the range of former is permitted to include the infinity.

By these notions, B. Margolis and J. Diaz gave one of the fundamental results of the fixed-point theory. For the proof, we refer to [26].

- *Theorem 2.8. [26] Let (X, d) be a generalized complete metric space and $J : X \to X$ be strictly contractive mapping with the Lipschitz constant L. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all non-negative integers n or there exists a positive integer n_0 such that;

(i) $d(J^n x, J^{n+1}x) < \infty$,

for all $n \geq n_0$

(ii) the sequence $J^n x$ converges to a fixed point y^* of J,
(iii) y^* is the unique fixed point of J in the set
$Y = \{y \in X : d(J^{n_0}(x), y) < \infty\}$
(iv) $d(y, y^*) \leq \frac{1}{1-L} d(J(y), y)$

for all $y \in Y$.

- *Definition 2.9. [22] Let X be a real linear space over non-Archimedean field K with dim X > 1 and let $\|x, y\|: X \times X \to [0, \infty[$ be a function satisfying the following properties:*

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(N A1) $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*

(N A2) $\|x, y\| = \|y, x\|$

(N A3) $\|\propto x, y\| = |\propto|\|x, y\|$
(A A4) $\|x, (y + z)\| \leq max\{\|x, y\|, \|x, z\|\}$

for all $x, y, z \in X$ *and* $\alpha \in R$*. Then the function* $(\|.,.\|)$ *is called a non-Archimedean 2- norm on X and the pair*
$(X, \|.,.\|)$ *is called a non-Archimedean 2-normed space.*

- *Lemma 2.10. [22] Let* $(X, \|.,.\|)$ *be a non-Archimedean 2-normed space. If* $x \in X$ *and* $\| x, y \| = 0$ *for all* $y \in X$*, then* $x = 0$*.*

- *Definition 2.11. [22] A sequence* $\{x_n\}$ *in a non-Archimedean 2-normed space* $(X, \|.,.\|)$ *is called a Cauchy sequence if*

$\| x_n - x_m, y \| = 0$

for all y ∈X

- *Definition 2.12. [23] A sequence* $\{x_n\}$ *in a non-Archimedean 2-normed space* $(X, \|.,.\|)$ *is called a Cauchy sequence if*

$\| x_n - x, y \| = 0$

for all y ∈X If $\{x_n\}$ *converges to x, write* $x_n \rightarrow x$ *as* $n \rightarrow \infty$ *and call x the limit of* $\{x_n\}$*. In this case, we also write*

$$\lim_{n\to\infty} x_n = x$$

Let $\{x_n\}$ be a sequence in a non-Archimedean 2-normed space $(X, \|.,.\|)$ It follows from (N A4) that

$$\|x_n - x_{m,y}\| \leq max\{\|x_{j+1} - x_{j,y}\| \mid m \leq j \leq n - 1\}\ (n > m)$$

for all $y \in X$ and so a sequence $\{x_n\}$ is a Cauchy sequence in $(X, \|.,.\|)$ if and only if $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|.,.\|)$

A non-Archimedean 2-normed space $(X, \|.,.\|)$ is called a non-Archimedean 2-Banach space if every Cauchy sequence in $(X, \|.,.\|)$ is convergent.

- *Lemma 2.13. [23] For a convergent sequence* $\{x_n\}$ *a non-Archimedean 2-normed space* $(X, \|.,.\|)$

$$\lim_{n\to\infty} \|x_n, y\| = \left\|\lim_{n\to\infty} x_n, y\right\|$$

for all y ∈X.

- *Lemma 2.14. [23] Let* $(X, \|.,.\|)$ *be a non-Archimedean 2-normed space. Then*

$$\|x, z\| - \|y, z\| \leq \|x - y, z\|$$

for all x, y,z ∈X.

- *Definition 2.15. [22] A non-Archimedean 2-Banach space X is called a normed non-Archimedean 2- Banach space if X is a 2-Banach space with norm.*

III. MAIN RESULTS

Hereafter, we will assume that X be a linear vector space over a non-Archimedean field K with a valuation |.| and Y a non-Archimedean 2- Banach spaces with $dimY > 1$. For convenience we use following abbreviation for a given mapping $f : X \rightarrow Y$

$$Df(x, y,z) := f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(x + z) - f(y + z) \quad (5)$$

for all x, y,z ∈X.

If for some $\varphi: X^3 \times Y \rightarrow [0, \infty]$, a mapping $f : X \rightarrow Y$ satisfies $\| Df(x, y, z), u \| \leq \varphi(x, y, z, u)$ for all $x, y, z \in X$ and all $u \in Y$, then f is called a φ -approximately quadratic function.

- *Theorem 3.1. [28] Let* $f : X \rightarrow Y$ *be mapping satisfies the following inequality:*

$$\|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(z + x), u\| \leq \varphi(x, y, z, u)$$

for all x,y,z ∈X , u ∈Y , with f(0) = 0, where $\varphi : X^3 \times Y \rightarrow [0,\infty)$ *is arbitrary mapping. Then,*

$$\left\|f(x) - \frac{2^n + 1}{2^{2n} + 1} f(2^n x) + \frac{2^n + 1}{2^{2n} + 1} f(-2^n x), u\right\| \leq$$
$$\max\{\{\tfrac{|2^{i+1}-1|}{|2|^{2i+3}}\varphi(2^i x, -2^i x, 2^i x, u)\}, \{\tfrac{|2^{i+1}+1|}{|2|^{2i+3}}\varphi(2^i x, 2^i x, -2^i x, u)\} : 0 \leq i \leq n - 1\}$$

for all x ∈X, all u ∈Y and n ∈N.

- *Lemma 3.2. Let* $\psi: X \rightarrow [0, \infty)$ *be a function, the set* $\mathcal{M} = \{g: X \rightarrow Y \mid g(0) = 0\}$ *and define*

$$d(g, h) = \inf\{\alpha > 0 : \|g(x) - h(x), u\| \leq \propto \psi(x)\} \qquad (g, h \in \mathcal{M})$$

∀x ∈ X, ∀u ∈ Y. Then d is generalized metric on $\mathcal{M}$*.*

Proof. Let $g, h, k \in \mathcal{M}$ and $\propto_1, \propto_2 > 0$ such that $d(g, h) < \propto_1$ and $d(h, k) < \propto_2$. Then by the definition
$\|g(x) - h(x), u\| \leq \propto_1 \psi(x)$ and $\|h(x) - k(x), u\| \leq \propto_2 \psi(x)$ for each $x \in X$, and all $u \in Y$, it follows that

$$\|g(x) - k(x), u\| \leq \|g(x) - h(x), u\| + \|h(x) - k(x), u\| \leq \propto_1 \psi(x) \leq \propto_2 \psi(x) = (\propto_1 + \propto_2)\psi(x)$$

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Therefore $d(g,h) \le (\propto_1 + \propto_2)$. This proves the triangle inequality for d. The rest of proof is similar to the proof of main result of [34].

In the following theorem, Hyers-Ulam stability of equation (2) is proved under approximately even condition in a non-Archimedean 2-Banach spaces.

- *Theorem 3.3. Let $f: X \to Y$ be a φ-approximately quadratic function with f(0) = 0 such that for all x, y,z $\in$ X and all u $\in$ Y,*

$$\|f(x) - f(-x), u\| \le \delta(x) \tag{6}$$

where $\delta : X \to [0,\infty)$. *Let* 0 < L < 1 be a constant such that,

$$|2|^2 \varphi(2^{-1}x, 2^{-1}y, 2^{-1}z, u) \le L\varphi(x,y,z,u) \tag{7}$$

for each x, y,z $\in$ X, and all u $\in$ Y. Then there exists a unique quadratic mapping Q: X → Y such that

$$\|f(x) - Q(x), u\| \le \frac{L}{|2|^2 - |2|^2.L}\psi(x,x,x,u) \tag{8}$$

for all x $\in$X and all u $\in$Y, where

$$\psi(x,x,x,u) = \max\{\max\left\{\frac{1}{|2|^3}\varphi(-x,-x,x,u), \frac{|3|}{|2|^3}\varphi(x,x,-x,u)\right\} \tag{9}$$

Proof. *Let $f: X \to Y$ be a φ* -approximately quadratic function satisfies the inequality (6) and f(0)=0, then for all *x* $\in$X, and all *u* $\in$Y , we have for *n* $\in$N

$$\left\|\frac{2^n-1}{2^{2n+1}}f(2^nx) - \frac{2^n-1}{2^{2n+1}}f(-2^nx), u\right\| = \frac{|2^n-1|}{|2|^{2n+1}}\|f(2^nx) - f(-2^nx), u\| \le \frac{|2^n-1|}{|2|^{2n+1}}\varphi(2^nx) \tag{10}$$

for all *x* $\in$X, *u* $\in$Y and *n* $\in$N,

$$\left\|f(x) - \frac{1}{2^{2n}}f(2^nx), u\right\| = \left\|f(x) - \frac{1}{2^{2n}}f(2^nx) + \frac{2^n+1}{2^{2n+1}}f(2^nx) - \frac{2^n+1}{2^{2n+1}}f(2^nx) + \frac{2^n-1}{2^{2n+1}}f(-2^nx) - \frac{2^n-1}{2^{2n+1}}f(-2^nx), u\right\|$$

$$= \left\|f(x) - \frac{2^n+1}{2^{2n+1}}f(2^nx) + \frac{2^n-1}{2^{2n+1}}f(-2^nx) - \frac{1}{2^{2n}}f(2^nx) + \frac{2^n+1}{2^{2n+1}}f(2^nx) - \frac{2^n-1}{2^{2n+1}}f(-2^nx), u\right\|$$

$$= \left\|f(x) - \frac{2^n+1}{2^{2n+1}}f(2^nx) + \frac{2^n-1}{2^{2n+1}}f(-2^nx) + \left(-\frac{1}{2^{2n}}f(2^nx) + \frac{2^n+1}{2^{2n+1}}f(2^nx) - \frac{2^n-1}{2^{2n+1}}f(-2^nx), u\right)\right\|$$

$$\le max\left\{\left\|f(x) - \frac{2^n+1}{2^{2n+1}}f(2^nx) + \frac{2^n-1}{2^{2n+1}}f(-2^nx)\right\|, \left\|\frac{2^n-1}{2^{2n+1}}f(2^nx) - \frac{2^n-1}{2^{2n+1}}f(-2^nx), u\right\|\right\}$$

by(10) and Theorem (3.1), the right side satisfies

$$\le max\left\{max\left\{\frac{|2^{i+1}-1|}{|2|^{2i+3}}\varphi(-2^ix, -2^ix, 2^ix, u), \frac{|2^{i+}}{|2} : 0 \le i \le n-1\right\}, \frac{|2^n-1|}{|2|^{2n+1}}\delta(2^nx)\right\} \tag{11}$$

for all $n \in N$. In particular,

$$\|f(x) - 2^{-2}f(2x), u\| \le \psi(x,x,x,u) \qquad \forall x \in X, \forall u \in Y, \tag{12}$$

ψ is defined by (9)

$$\text{So } \|4f(x) - f(2x), u\| \le |2|^2\psi(x,x,x,u) \le \psi(x,x,x,u) \qquad \forall x \in X, \forall u \in Y \tag{13}$$

Replacing x by 2^1x in (13), it follows that for each $x \in X$ and all $\in Y$,

$$\|4f(x) - 4f(2^{-1}x), u\| \le \psi(2^{-1}x, 2^{-1}x, 2^{-1}x, u) \tag{14}$$

Let us consider the set

$$\mathcal{F} := \{g: X \to Y | \, g(0) = 0\} \tag{15}$$

and introduce a generalized metric on *F* as following:

$$d(g,h) = inf\{\propto > 0: \|g(x) - h(x), u\| \le \propto \psi(x,x,x,u)\} \qquad \forall x \in X \text{ and all } u \in Y \tag{16}$$

where, as usual *inf* $\phi = +\infty$. Its easy to show that $(\mathcal{F}, d)$ is complete (see for example ([29]).

Now, we consider the linear mapping $J : \mathcal{F} \to \mathcal{F}$such that $J(h) = 4h(2^1x)$. we assert that *J* is strictly contractive on $\mathcal{F}$.

Given $g,h \in \mathcal{F}$, and let $\propto \in [0,\infty)$ be an arbitrary constant with $d(g,h) \le \propto \psi(x,x,x,u)$ that

$\|g(x) - h(x), u\| \le \propto \psi(x,x,x,u)$ $(x \in X, u \in Y)$, then by (7)

$$\|J(g)(x) - J(h)(x), u\| = \|4g(2^{-1}x) - 4h(2^{-1}x), u\| \le \propto |2|^2\psi(2^{-1}x), 2^{-1}x, 2^{-1}x) \le L \propto \psi(x,x,x,u)$$

for all *x* $\in$X. It follows that

$$d(J(g), J(h)) \le Ld(g,h) \qquad (g,h \in \mathcal{F})$$

Hence d is strictly contractive mapping with Lipschitz constant, L. By (14) we have, for each $x \in X$ and $u \in Y$

$$\begin{aligned}\|(Jf)(x) - f(x), u\| &= \|4f(2^{-1}x) - f(x), u\| \\ &\leq \psi(2^{-1}x, 2^{-1}x, 2^{-1}x, u) \\ &\leq \frac{L}{|2|^2}\psi(x, x, x, u)\end{aligned}$$

This means that $d(J(f), f) \leq \frac{L}{|2|^2}$.

By Theorem (2.8) there exists a mapping $Q\colon X \to Y$ satisfies the following:

(1) Q is fixed point of J, and Q is a unique fixed point of J in the set

$$\mathcal{J} = \{g \in \mathcal{F} : d(g, J(f)) < \infty\}$$

That Q is a unique mapping such that there exists $\alpha \in [0, \infty)$ satisfying

$$\|f(x) - Q(x), u\| \leq \alpha\psi(x, x, x, u)$$

for all $x \in X$ and all $u \in Y$.

(2) $\lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} 2^{2n} f(2^{-n}x) = Q(x)$ (for all $x \in X$ and all $u \in Y$).

Therefore for all $x,y,z \in X$ and $u \in Y$,

$$\begin{aligned}&\|Q(x+y+z) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(x+z) - Q(y+z), u\| \\ &= \lim_{n\to\infty} |2|^{2n} \|f(2^{-n}(x+y+z)) + f(2^{-n}(x)) + f(2^{-n}(y)) + f(2^{-n}(z)) \\ &\quad - f(2^{-n}(x+y)) - f(2^{-n}(x+z)) - f(2^{-n}(y+z), u\| \\ &\leq \lim_{n\to\infty} |2|^{2n}\, \varphi(2^{-n}x, 2^{-n}y, 2^{-n}z, u) \qquad (17)\end{aligned}$$

it easy to show that by induction $|2|^{2n}\varphi(2^{-n}x, 2^{-n}y, 2^{-n}z, u) \leq L^n\varphi(x, y, z, u)$,

so the right side of $(17) \leq \lim_{n\to\infty} L^n\, \varphi(x, y, z, u) = 0$ $(since\ L < 1)$

By lemma (2.10) this show that Q is quadratic .

(3) $d(f, Q) \leq \frac{1}{1-L} d\big(f, J(f)\big)$. This implies $d(f, Q) \leq \frac{1}{1-L} \cdot \frac{L}{|2|^2} = \frac{L}{|2|^2 - |2|^2.L}$

Thus $\|f(x) - Q(x), u\| \leq \frac{L}{|2|^2 - |2|^2.L}\psi(x, x, x, u)$. Then we have the inequality (8).

Hence the proof of the theorem is end.

Similarly, as the proof theorem (3.3), the Hyers-Ulam stability of equation (2) is proved under approximately odd condition in a non-Archimedean 2-Banach spaces.

- *Theorem 3.4. Let $f : X \to Y$ be a φ-approximately quadratic function with $f(0) = 0$ such that for all x,y,z $\in$ X and all* $u \in Y$,

$$\|f(x) + f(-x), u\| \leq \delta(x) \qquad (18)$$

where $\delta : X \to [0, \infty)$. Let $0 < L < 1$ be a constant such that,

$$|2|^{2n}\varphi(2^{-1}x, 2^{-1}y, 2^{-1}z, u) \leq L\varphi(x, y, z, u) \qquad (19)$$

for each x,y $\in X$. Then there exists a unique quadratic mapping Q: $X \to Y$ such that

$$\|f(x) - Q(x), u\| \leq \frac{L}{|2| - |2|.L}\psi(x, x, x, u) \qquad (20)$$

for all $x \in X$ and all $u \in Y$, where

$$\psi(x, x, x, u) = \max\{\max\{\frac{1}{|2|^3}\varphi(-x, -x, x, u), \frac{|3|}{|2|^3}\varphi(x, x, x, u)\}, \qquad (21)$$

Proof. Let $f : X \to Y$ *be a* φ-approximately quadratic function with $f(0) = 0$ satisfies the inequality (18), So for all $x \in X$, and $u \in Y$, we have for $n \in N$

$$\left\|\frac{2^n - 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x)\right\| \leq \frac{|2^n - 1|}{|2|^{2n+1}}\psi(2^n x) \qquad (22)$$

for all $x \in X$, $u \in Y$ and $n \in N$,

$$\begin{aligned}\left\|f(x) - \frac{1}{2^n} f(2^n x), u\right\| &= \left\|f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x) + \frac{-2^n - 1}{2^{2n+1}} f(2^n x) - \frac{2^n - 1}{2^{2n+1}} f(2^n x), u\right\| \\ &\leq max\left\{\left\|f(x) - \frac{2^n + 1}{2^{2n+1}}\, f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x)\right\|,\ \left\|\frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u\right\|\right\}\end{aligned}$$

By (22) and Theorem (3.1), the right side satisfies

$$\leq max\left\{max\left\{\frac{|2^{i+1} - 1|}{|2|^{2i+3}}\varphi(-2^i x, -2^i x, 2^i x, u), \frac{|2}{|} : 0 \leq i \leq n-1\right\}, \frac{|2^n - 1|}{|2|^{2n+1}}\psi(2^n x)\right\} \qquad (23)$$

for all $n \in N$. In particular,

$$\|f(x) - 2^{-1}f(2x), u\| \leq \psi(x, x, x, u) \qquad (x \in X, u \in Y), \quad \psi \text{ is defined by (21)} \qquad (24)$$

this follows that

$$\|2f(x)-f(2x),u\| \le |2|\psi(x,x,x,u) \le \psi(x,x,x,u) \qquad (x \in X, u \in Y) \tag{25}$$

Replacing x by $2^{-1}x$ in (25), it follows that for each $x \in X$ and all $u \in Y$,

$$\|f(x)-2f(2^{-1}x),u\| \le \psi(2^{-1}x,2^{-1}x,2^{-1}x,u) \tag{26}$$

Similarly, as proof of the theorem (3.3) we can consider the set $\mathcal{F}$ is defined by (15) and introduce d is a generalized metric on $\mathcal{F}$ which defined by (16). It clearly $(\mathcal{F},d)$ is complete.

Now we consider the linear mapping $J:\mathcal{F}\to\mathcal{F}$ such that $J(h)=2h(2^{-1}x)$ Similarly, as proof of the theorem (3.3) of J is strictly contractive on $\mathcal{F}$. By (26) we have, for all $x \in X$ and all $u \in Y$

$$\begin{aligned}\|(Jf)(x)-f(x),u\| &= \|2f(2^{-1}x)-f(x),u\| \\ &\le \psi(2^{-1}x,2^{-1}x,2^{-1}x,u) \\ &\le \frac{L}{|2|}\psi(x,x,x,u)\end{aligned}$$

This means that $d(J(f),f) \le \frac{L}{|2|}$. By Theorem (2.8) there exists a mapping $Q:X\to Y$ satisfies the following:

(1) Q is fixed point of J, and Q is a unique fixed point of J in the set

$$\mathcal{J}=\{g\in\mathcal{F}: d(g,J(f))<\infty\}$$

That Q is a unique mapping such that there exists $\alpha \in [0,\infty)$ satisfying

$$\|f(x)-Q(x),u\| \le \alpha\psi(x,x,x,u)$$

for all $x \in X$ and all $u \in Y$.

(2) $\lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} 2^{2n} f(2^{-n}x) = Q(x)$ (for all $x \in X$ and all $u \in Y$).

Therefore for all $x,y,z \in X$ and $u \in Y$,

$$\begin{aligned}&\|Q(x+y+z)+Q(x)+Q(y)+Q(z)-Q(x+y)-Q(x+z)-Q(y+z),u\| \\ &= \lim_{n\to\infty}|2|^n\,\|f(2^{-n}(x+y+z))+f(2^{-n}(x))+f(2^{-n}(y)) + f(2^{-n}(z))-f(2^{-n}(x+y)) - f(2^{-n}(x+z))-f(2^{-n}(y+z)),u\| \\ &\le \lim_{n\to\infty}|2|^n\,\varphi(2^{-n}x,2^{-n}y,2^{-n}z,u)\end{aligned} \tag{27}$$

it easy to show that by induction $|2|^n\varphi(2^{-n}x,2^{-n}y,2^{-n}z,u) \le L^n\varphi(x,y,z,u)$

so the right side of (27)

$$\le \lim_{n\to\infty} L^n\,\varphi(x,y,z,u) = 0 \quad (since\ L<1)$$

By lemma (2.10) this show that Q is quadratic.

(3) $d(f,Q) \le \frac{1}{1-L} d(f,J(f))$. This implies $d(f,Q) \le \frac{1}{1-L}\cdot\frac{L}{|2|} = \frac{L}{|2|-|2|.L}$

Thus $\|f(x)-Q(x),u\| \le \frac{L}{|2|-|2|.L}\psi(x,x,x,u)$. Then we have the inequality (20).

Hence the proof of the theorem is end.

- *Theorem 3.5. Let $f:X\to Y$ be a φ-approximately quadratic function with $f(0)=0$ such that for all x, y, z $\in X$ and all u $\in Y$,*

$$\|f(x)-f(-x),u\| \le \delta(x) \tag{28}$$

where $\delta: X \to [0,\infty)$. Let 0 < L < 1 be a constant such that,

$$|2|^{-2}\varphi(2x,2y,2z,u) \le L\varphi(x,y,z,u) \tag{29}$$

for each x,y,z $\in X$. Then there exists a unique quadratic mapping Q : X → Y such that

$$\|f(x)-Q(x),u\| \le \frac{1}{1-L}\psi(x,x,x,u) \tag{30}$$

for all x $\in X$ and all u $\in Y$, where

$$\psi(x,x,x,u) = \max\{\max\{\frac{1}{|2|^3}\varphi(-x,-x,x,u),\frac{|3|}{|2|^3}\varphi(x,x,-x,u)\}, \tag{31}$$

Proof. Let $f: X\to Y$ *be a* φ-approximately quadratic function satisfies the inequality (28) and $f(0)=0$, then for all

$x \in X$, and $u \in Y$. As theorem (3.3) we have for all $n \in N$,

$$\|f(x)-2^{-2}f(2x),u\| \le \psi(x,x,x,u) \qquad (x\in X, u \in Y), \tag{32}$$

and we consider the set $\mathcal{F}$ is defined by (15) and introduce a generalized metric d is defined by (16) on $\mathcal{F}$. Its clearly $(\mathcal{F},d)$ is complete.

Now, we consider the linear mapping $J:\mathcal{F}\to\mathcal{F}$ such that $J(h)=2^{-2}h(2x)$. we assert that J is strictly contractive on $\mathcal{F}$.

Given $g,h\in\mathcal{F}$, and let $\alpha\in[0,\infty)$be an arbitrary constant with $d(g,h) \le \alpha\psi(x,x,x,u)$that

$$\|g(x)-h(x),u\| \le \alpha\psi(x,x,x,u) \quad (x\in X, u\in Y),$$

then by (29) we have

$$\|(Jf)(x) - f(x), u\| = \left\|\frac{1}{4}g(2x) - \frac{1}{4}h(2x), u\right\|$$
$$\leq \alpha|2|^{-2}\psi(2x, 2x, 2x, u)$$
$$\leq L\alpha\psi(x, x, x, u)$$

for all $x \in X$ and all $u \in Y$. It follows that

$$d\big(J(g), J(h)\big) \leq Ld(g, h) \qquad (g, h \in \mathcal{F})$$

Hence d is strictly contractive mapping with Lipschitz constant, L. By (32) we have, for each $x \in X$ and $u \in Y$

$$\|(Jf)(x) - f(x), u\| = \left\|\frac{1}{4}f(2x) - f(x), u\right\|$$
$$\leq \psi(x, x, x, u)$$

This means that $d(J(f), f) \leq 1 < \infty$. By Theorem (2.8) there exists a mapping $Q : X \to Y$ satisfies the following:

(1) Q is fixed point of J, and Q is a unique fixed point of J in the set

$$\mathcal{J} = \{g \in \mathcal{F} : d(g, J(f)) < \infty\}$$

That Q is a unique mapping such that there exists $\alpha \in [0, \infty)$ satisfying

$$\|f(x) - Q(x), u\| \leq \alpha\psi(x, x, x, u)$$

for all $x \in X$ and all $u \in Y$.

(2) $\lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x) = Q(x)$ (for all $x \in X$ and all $u \in Y$).

Therefore for all $x, y, z \in X$ and $u \in Y$,

$$\|Q(x + y + z) + Q(x) + Q(y) + Q(z) - Q(x + y) - Q(x + z) - Q(y + z), u\|$$

$$= \lim_{n\to\infty} |2|^{-2n} \|f(2^n(x + y + z)) + f(2^n x) + f(2^n y) + f(2^n z) - f(2^n(x + y)) - f(2^n(x + z)) - f(2^n(y + z), u\|$$

$$\leq \lim_{n\to\infty} |2|^{-2n} \varphi(2^n x, 2^n y, 2^n z, u) \qquad (33)$$

t easy to show that by induction

$$|2|^{-2n}\varphi(2^n x, 2^n y, 2^n z, u) \leq L^n\varphi(x, y, z, u)$$

so the right side of (33)

$$\leq \lim_{n\to\infty} L^n \varphi(x, y, z, u) = 0 \qquad (since\ L < 1)$$

By lemma (2.10) this show that Q is quadratic.

(3) $d(f, Q) \leq \frac{1}{1-L} d\big(f, J(f)\big)$.

This implies

$$d(f, Q) \leq \frac{1}{1 - L}$$

Thus $\|f(x) - Q(x), u\| \leq \frac{1}{1-L}\psi(x, x, x, u)$ for all $x \in X$ and all $u \in Y$. Then we have the inequality (30).

Hence the proof of the theorem is end.

➢ *Theorem 3.6. Let $f : X \to Y$ be a φ-approximately quadratic function with $f(0) = 0$ such that for all x,y,z $\in X$ and all u $\in Y$,*

$$\|f(x) + f(-x), u\| \leq \delta(x) \qquad (34)$$

where $\delta : X \to [0, \infty)$. Let $0 < L < 1$ be a constant such that,

$$|2|^{-2}\varphi(2x, 2y, 2z, u) \leq L\varphi(x, y, z, u) \qquad (35)$$

for each x,y,z $\in X$, and all u $\in Y$. Then there exists a unique quadratic mapping Q : X ⟶ Y such that

$$\|f(x) - Q(x), u\| \leq \frac{1}{1 - L}\psi(x, x, x, u) \qquad (36)$$

for all x $\in X$ and all u $\in Y$, where

$$\psi(x, x, x, u) = \max\{\max\{\frac{1}{|2|^3}\varphi(-x, -x, x, u), \frac{|3|}{|2|^3}\varphi(x, x, -x, u)\}, \qquad (37)$$

Proof. Similarly, as proof the last theorem we can proof this theorem with consider the linear mapping $J : \mathcal{F} \to \mathcal{F}$ which is strictly contractive on $\mathcal{F}$ such that $J(h) = \frac{1}{2}h(2x)$, by (24)

$$\|(Jf)(x) - f(x), u\| = \left\|\frac{1}{2}f(2x) - f(x), u\right\|$$
$$\leq \psi(x, x, x, u)$$

This means that $d(J(f), f) \leq 1$. The rest of proof as proof of theorem (3.5).

IV. APPLICATIONS

As example o $\varphi(x, y, z, u)$f in Theorems (3.3),(3.5), we can take $\varphi(x, y, z, u) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)\|u\|$ for all x, y, z and all $u \in Y$ and some positive real number θ. Then we have the following corollaries:

➢ *Corollary 4.1. Let θ, δ, p be positive real numbers such that p < 2, and let f : X ⟶ Y is mapping satisfying*

$$\|f(x + y + z) + f(x) + f(y) - f(x + y) - f(x + z) - f(y + z), u\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)\|u\|, \qquad (38)$$

$$\|f(x) - f(-x), u\| < \delta \qquad (39)$$

for all x,y,z ∈ X and all u ∈ Y . Then there exists a unique quadratic mapping Q: X ⟶ Y satisfying

$$\|f(x)-Q(x),u\| \leq \frac{1}{|2|^p-|2|^2}\psi(x,x,x,u) \qquad (40)$$

where,

$$\psi(x,x,x,u) = max\left\{\frac{|3|}{|2|^3}\theta\|x\|^p\|u\|, \frac{1}{|2|^3}\delta\right\} \qquad (41)$$

Proof. Let $f : X \to Y$ be mapping satisfying (38), (39). Put $x = y = z = 0$ in (38). This implies $f(0) = 0$.
Now let $\varphi(x,y,z,u) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)\|u\|$ for all $u \in Y$ and all $x,y,z \in X$, so

$$|2|^2\varphi(2^{-1}x, 2^{-1}y, 2^{-1}z, u) = |2|^2\theta(\|2^{-1}x\|^p + \|2^{-1}y\|^p + \|2^{-1}z\|^p)\|u\| = |2|^{2-p}(x,x,x,u)$$

since $p < 2$ we have $L = |2|^{2-p} < 1$ and by Theorem (3.3) there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x),u\| \leq \frac{1}{|2|^2(1-|2|^{2-p})}\psi(x,x,x,u) = \frac{1}{|2|^p-|2|^2}\psi(x,x,x,u)$$

for and all $u \in Y$ and all $x,y,z \in X$ where ψ is defined by (41).

➢ *Corollary 4.2. Let θ, δ, p be positive real numbers such that $p > 2$, and let $f : X \to Y$ is mapping satisfying*

$$\|f(x+y+z)+f(x)+f(y)-f(x+y) - f(x+z)-f(y+z),u\| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p)\|u\|, \qquad (42)$$

$$\|f(x)-f(-x),u\| < \delta \qquad (43)$$

for all x,y,z ∈ X and all u ∈ Y . Then there exists a unique quadratic mapping Q: X ⟶ Y satisfying

$$\|f(x)-Q(x),u\| \leq \frac{1}{1-|2|^{p-2}}\psi(x,x,x,u) \qquad (44)$$

where,

$$\psi(x,x,x,u) = max\left\{\frac{|3|}{|2|^3}\theta\|x\|^p\|u\|, \frac{1}{|2|^3}\delta\right\} \qquad (45)$$

Proof. Let $f : X \to Y$ be mapping satisfying (42),(43). its clearly $f(0) = 0$.

Now let $\varphi(x,y,z,u) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)\|u\|$ for all $u \in Y$ and all $x,y,z \in X$, so

$$|2|^{-2}\varphi(2x,2y,2z,u) = |2|^{-2}\theta(\|2x\|^p+\|2y\|^p+\|2z\|^p)\|u\| = |2|^{p-2}\varphi(x,x,x,u)$$

since $p > 2$ we have $L = |2|^{p-2} < 1$ and by Theorem (3.5) there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x),u\| \leq \frac{1}{1-|2|^{p-2}}\psi(x,x,x,u) \qquad (45).$$

for all $u \in Y$ and all $x,y,z \in X$, where ψ is defined by

➢ *Corollary 4.3. Let θ,δ,p be positive real numbers such that p < 2, and let f : X ⟶ Y is mapping satisfying*

$$\|f(x+y+z)+f(x)+f(y)-f(x+y) - f(x+z)-f(y+z),u\| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p)\|u\|, \qquad (46)$$

$$\|f(x)-f(-x),u\| < \delta \qquad (47)$$

for all x,y,z ∈ X and all u ∈ Y . Then there exists a unique quadratic mapping Q: X ⟶ Y satisfying

$$\|f(x)-Q(x),u\| \leq \frac{|2|^{2-p}}{|2|\,(1-|2|^{2-p})}\psi(x,x,x,u) \qquad (48)$$

where,

$$\psi(x,x,x,u) = max\left\{\frac{|3|}{|2|^3}\theta\|x\|^p\|u\|, \frac{1}{|2|^3}\delta\right\} \qquad (49)$$

Proof. Similarly as proof of corollary (4.1) Let $f : X \to Y$ be mapping satisfying (46), (47). and by theorem (3.4) with p < 2 and a constant $L = |2|^{2-p} < 1$ we have (48), where ψ is defined by (49).

➢ *Corollary 4.4. Let θ,δ,p be positive real numbers such that p > 2, and let f : X ⟶ Y is mapping satisfying*

$$\|f(x+y+z)+f(x)+f(y)-f(x+y) - f(x+z)-f(y+z),u\| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p)\|u\|, \qquad (50)$$

$$\|f(x)+f(-x),u\| < \delta \qquad (51)$$

for all x,y,z ∈ X and all u ∈ Y . Then there exists a unique quadratic mapping Q: X ⟶ Y satisfying

$$\|f(x)-Q(x),u\| \leq \frac{1}{1-|2|^{p-2}}\psi(x,x,x,u) \qquad (52)$$

where,

$$\psi(x,x,x,u) = max\left\{\frac{|3|}{|2|^3}\theta\|x\|^p\|u\|, \frac{1}{|2|^3}\delta\right\} \qquad (53)$$

Proof. Similarly as proof of the previous corollaries and by theorem (3.6) with p > 2 and constant Lipschitz $L = |2|^{p-2} < 1$ we have (52), ψ is defined by (53).

As another example of $\varphi(x,y,z,u)$ in theorems (3.3),(3.4), (3.5) and theorem(3.6), we can take

$\varphi(x,y,z,u) = \theta(\|x\|^p\|y\|^p\|z\|^p)\|u\|$ for all $x,y,z \in X$ and all $u \in Y$ and some positive real numbers p,q,r,θ. Then we have the following corollaries:

- *Corollary 4.5. Let p,q,r,δ and θ be positive real numbers such that p + q + r = 2, f : X → Y is a mapping satisfying*

$$\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y) -f(x+z)-f(y+z),u\| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p)\|u\|, \quad (54)$$

$$\|f(x)-f(-x),u\| < \delta \quad (55)$$

for all u ∈ Y and all x,y,z ∈ X. Then there exists a unique quadratic mapping Q: X ⟶ Y satisfying

$$\|f(x)-Q(x),u\| \leq \begin{cases} \frac{1}{1-|2|^{p+q+r-2}}\psi(x,x,x,u), & p+q+r>2 \\ \frac{1}{|2|^2-|2|^2.|2|^{2-p+q+r}}\psi(x,x,x,u), & p+q+r<2 \end{cases}$$

where,

$$\psi(x,x,x,u) = max\left\{\frac{1}{|2|^3}\theta\|x\|^{p+q+r}, \frac{1}{|2|^3}\delta\right\} \quad \text{for all } x \in X \quad (56)$$

and all u ∈Y

Proof. Let $f: X \to Y$ satisfies (54) and (55). $\varphi(x,y,z,u) = \theta(\|x\|^p\|y\|^p\|z\|^r)\|u\|$ for all $x,y,z \in X$, all $u \in Y$ and $p+q+r \neq 2$, (*p,q,r* and θ be positive real numbers).
Then we have,

$$\varphi(2x,2y,2z,u) = |2|^{p+q+r}\theta(\|x\|^p\|y\|^p\|z\|^r)\|u\| = |2|^{p+q+r}\varphi(x,y,z,u) = |2|^2|2|^{p+q+r-2}\varphi(x,y,z,u).$$

Hence $if\ p+q+r>2$, by theorem(3.5) and let $L = |2|^{p+q+r-2} < 1$ there exists a unique quadratic mapping $Q: X \to Y$

satisfying $\|f(x)-Q(x),u\| \leq \left\{\frac{1}{1-|2|^{p+q+r-2}}\psi(x,x,x,u)\right.$ for all $x,y,z \in X$ and all $u \in Y$.

Where $\psi(x,x,x,u) = max = \left\{\frac{1}{|2|^3}\|x\|^{p+q+r}, \frac{1}{|2|^3}\delta\right\}$.

Now $if\ p+q+r<2$, for all $x,y,z \in X$ and $u \in Y$

$$\varphi(x,y,z,u) = |2|^{-(p+q+r)}\varphi(2x,2y,2z,u) = |2|^{-2}|2|^{2-p-q-r}\varphi(2x,2y,2z,u).$$

Thus by Theorem(3.3) there exists a unique quadratic mapping $Q: X \to Y$ such that

$$\|f(x)-Q(x),u\| \leq \frac{1}{|2|^2-|2|^2.|2|^{2-p-q-r}}\psi(x,x,x,u),$$

for all $x \in X$ and all $u \in Y$, where ψ is defined by (56).

- *Corollary 4.6. Let p, q, r, δ and θ be positive real numbers with p+q+r≠2. Suppose that f :X⟶Y is a mapping satisfying*

$$\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y) -f(x+z)+f(y+z),u\| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p)\|u\|, \quad (57)$$

$$\|f(x)+f(-x),u\| < \delta \quad (58)$$

for all u ∈ Y and all x,y,z ∈ X. Then there exists a unique quadratic mapping Q: X ⟶ Y satisfying

$$\|f(x)-Q(x),u\| \leq \begin{cases} \frac{1}{1-|2|^{p+q+r-2}}\psi(x,x,x,u), & p+q+r>2 \\ \frac{1}{|2|-|2|.|2|^{2-p-q-r}}\psi(x,x,x,u), & p+q+r<2 \end{cases}$$

where,

$$\psi(x,x,x,u) = max\left\{\frac{1}{|2|^3}\theta\|x\|^{p+q+r}, \frac{1}{|2|^3}\delta\right\} \quad (59)$$

for all x ∈X and all u ∈Y

Proof. Similarly as proof of corollary (4.5). $If\ p+q+r>2$, by Theorem (3.6), we have

$$\|f(x)-Q(x),u\| \leq \frac{1}{1-|2|^{p+q+r-2}}\psi(x,x,x,u),$$

for all $x \in X$ and all $u \in Y$, where ψ is defined by (59).

Thus $if\ p+q+r<2$, by Theorem (3.4) we have

$$\|f(x)-Q(x),u\| \leq \frac{|2|^{2-p-q-r}}{|2|-|2|.|2|^{2-p-q-r}}\psi(x,x,x,u),$$

for all $x \in X$ and all $u \in Y$, where ψ is defined by (59).

Now, we will study the stability of the following functional equation with several variable on a non-Archimedean 2-Banach spaces as following:

$$D_f(x_1,\dots,x_k) = f\left(\sum_{i=1}^{k}x_i\right) + (k-2)\sum_{i=1}^{k}f(x_i) - \sum_{i=1}^{k}\sum_{j=1,j>i}^{k}f(x_i+x_j) \quad (60)$$

for any $k \geq 3$.

By Janfada [14] we can see that the quadratic function f :

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$X \to Y$ defined by $f(x)= x^2$ and any additive mapping not only satisfies the following equation functional:

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z) + f(x+z) \tag{61}$$

but also,

$$D_f(x_1,\ldots,x_k) = 0 \tag{62}$$

For all $x_i \in X, i = 1,2,\ldots,k$.
In following we prove the generalized Hyers-Ulam-Rassias stability of (62) will be proved in non-Archimedean 2-normed spaces by using Theorems (3.3),(3.4),(3.5) and (3.6).

➢ *Theorem 4.7. Let X and Y be common domain and range of the f 's in the functional equations (61) and (62). Then the functional equation (62) is equivalent to (61).*

Proof. See [14]

➢ *Corollary 4.8. Let $k \in N$ and $k \geq 3$. Assume that mapping $f : X^k \times Y \to Y$ such that $f(0) = 0$ and f satisfies the following inequalities:*

$$\left\| f\left(\sum_{i=1}^{k} x_i\right) + (k-2)\sum_{i=1}^{k} f(x_i) - \sum_{i=1}^{k}\sum_{j=1,j>i}^{k} f(x_i+x_j), u \right\| \leq \emptyset(x_1,x_2,\ldots,x_k,u) \tag{63}$$

$$\|f(x)-f(-x),u\| \leq \delta(x) \tag{64}$$

Where $\phi : X^k \times Y \to [0,\infty)$ and $\delta : X \to [0,\infty)$ are mapping such that for all $x_1,x_2,x_3,\ldots,x_k \in X$ and $\forall u \in Y$. For $k \in N, 0 < L < 1$

$$|2|^2\emptyset(2^{-1}x_1,2^{-1}x_2,2^{-1}x_3,\ldots,2^{-1}x_k,u) \leq L\emptyset(x_1,x_2,x_3,\ldots,x_k,u) \tag{65}$$

for all $x,x_1,x_2,x_3,\ldots,x_k \in X$ and $\forall u \in Y$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x),u\| \leq \frac{1}{|2|^2-|2|^2L}\psi(x,x,x,u) \tag{66}$$

for all $x \in X$ and all $u \in Y$, where

$$\psi(x,x,x,u) = \max\left\{ \begin{matrix} \{max \frac{1}{|2|^3}\emptyset(-x,-x,x,0,\ldots,0,u), \frac{3}{|2|^3}\emptyset \\ (x,x,-x,0,\ldots,0,u)\}, \frac{1}{|2|^3}\delta(2x) \end{matrix} \right\} \tag{67}$$

for all $x \in X$ and all $u \in Y$

Proof. Let $f : X^k \times Y \to Y$ be mapping satisfies (63), $f(0) = 0$ and by setting $(x_1,x_2,x_3,\ldots,x_n,u) = (x_1,x_2,x_3,0,0,\ldots,0,u)$ in (63), we obtain,

$$\begin{aligned}\|f(x_1+x_2+x_3)+f(x_1)+f(x_2)+f(x_3) &- f(x_1+x_2)-f(x_2+x_3) \\ &- f(x_1+x_3),u\| \\ &\leq \emptyset(x_1,x_2,x_3,0,0,\ldots,0,u)\end{aligned} \tag{68}$$

for all $x_1,x_2,x_3 \in X$, and all $u \in Y$.

Now by Considering

$$\varphi(x_1,x_2,x_3,u) = \emptyset(x_1,x_2,x_3,0,0,\ldots,0,u),$$

we obtain

$$\begin{aligned}\varphi(2^{-1}x_1,2^{-1}x_2,2^{-1}x_3,u) &= \emptyset(2^{-1}x_1,2^{-1}x_2,2^{-1}x_3,0,0,\ldots,0,u) \\ &\leq (x_1,x_2,x_3,0,0,\ldots,0,u) \\ &= L\emptyset(x_1,x_2,x_3,u)\end{aligned}$$

for all $x_1,x_2,x_3 \in X$, and all $u \in Y$.

By theorem (3.3) we have (66) with ψ is defined by (67).

➢ *Corollary 4.9. Let $k \in N$ and $k \geq 3$. Assume that mapping $f : X^k \times Y \to Y$ such that $f(0) = 0$ and f satisfies the following inequalities:*

$$\left\| f\left(\sum_{i=1}^{k} x_i\right) + (k-2)\sum_{i=1}^{k} f(x_i) - \sum_{i=1}^{k}\sum_{j=1,j>i}^{k} f(x_i+x_j), u \right\| \leq \emptyset(x_1,x_2,\ldots,x_k,u) \tag{69}$$

$$\|f(x)+f(-x),u\| \leq \delta(x) \tag{70}$$

Where $\phi : X^k \times Y \to [0,\infty)$ and $\delta : X \to [0,\infty)$ are mapping such that for all $x_1,x_2,x_3,\ldots,x_k \in X$ and $\forall u \in Y$. For $k \in N,\ 0 < L < 1$

$$|2|^2\emptyset(2^{-1}x_1,2^{-1}x_2,2^{-1}x_3,\ldots,2^{-1}x_k,u) \leq L\emptyset(x_1,x_2,x_3,\ldots,x_k,u) \tag{71}$$

for all $x_1,x_2,x_3,\ldots,x_k \in X$ and $\forall u \in Y$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x),u\| \leq \frac{L}{|2|-|2|L}\psi(x,x,x,u) \tag{72}$$

for all $x \in X$ and all $u \in Y$, where

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$$\psi(x,x,x,u) = \max\left\{ \begin{array}{c} \{max\frac{1}{|2|^3}\emptyset(-x,-x,x,0,..,0,u), \\ \frac{3}{|2|^3}\emptyset(x,x,-x,0,..,0,u)\},\frac{1}{|2|^3}\delta(2x) \end{array} \right\} \quad (73)$$

for all $x \in X$ and all $u \in Y$.
Proof. Similarly as proof corollary(4.8) and by Theorem(3.4) we have (72).

➢ *Corollary 4.10. Let $k \in N$ and $k \geq 3$. Assume that mapping $f : X^k \times Y \to Y$ such that $f(0) = 0$ and f satisfies the following inequalities:*

$$\left\| f\left(\sum_{i=1}^{k} x_i\right) + (k-2)\sum_{i=1}^{k} f(x_i) - \sum_{i=1}^{k}\sum_{j=1,j>i}^{k} f(x_i + x_j), u \right\| \leq \emptyset(x_1, x_2, \dots, x_k, u) \quad (74)$$

$$\|f(x) - f(-x), u\| \leq \delta(x) \quad (75)$$

Where $\phi : X^k \times Y \to [0,\infty)$ and $\delta : X \to [0,\infty)$ are mapping such that for all $x_1, x_2, x_3, \dots, x_k \in X$ and $\forall u \in Y$.
For $k \in N, 0 < L < 1$

$$|2|^{-2}\emptyset(2x_1, 2x_2, 2x_3, \dots, 2x_k, u) \leq L\emptyset(x_1, x_2, x_3, \dots, x_k, u) \quad (76)$$

for all $x_1, x_2, x_3, \dots, x_k \in X$ and $\forall u \in Y$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x), u\| \leq \frac{1}{1-L}\psi(x,x,x,u) \quad (77)$$

for all $x \in X$ and all $u \in Y$, where

$$\psi(x,x,x,u) = \max\left\{ \begin{array}{c} \{max\frac{1}{|2|^3}\emptyset(-x,-x,x,0,..,0,u) \\ ,\frac{3}{|2|^3}\emptyset(x,x,-x,0,..,0,u)\},\frac{1}{|2|^3}\delta(2x) \end{array} \right\} \quad (78)$$

for all $x \in X$ and all $u \in Y$
Proof. Similarly as proof corollary (4.8) and by Theorem (3.5) we have (77).

➢ *Corollary 4.11. Let $k \in N$ and $k \geq 3$. Assume that mapping $f : X^k \times Y \to Y$ such that $f(0) = 0$ and f satisfies the following inequalities:*

$$\left\| f\left(\sum_{i=1}^{k} x_i\right) + (k-2)\sum_{i=1}^{k} f(x_i) - \sum_{i=1}^{k}\sum_{j=1,j>i}^{k} f(x_i + x_j), u \right\| \leq \emptyset(x_1, x_2, \dots, x_k, u) \quad (79)$$

$$\|f(x) + f(-x), u\| \leq \delta(x) \quad (80)$$

Where $\phi : X^k \times Y \to [0,\infty)$ and $\delta : X \to [0,\infty)$ are mapping such that for all $x, x_1, x_2, x_3, \dots, x_k \in X$ and $\forall u \in Y$.

For $k \in N, 0 < L < 1$

$$|2|^{-2}\emptyset(2x_1, 2x_2, 2x_3, \dots, 2x_k, u) \leq L\emptyset(x_1, x_2, x_3, \dots, x_k, u)$$

for all $x_1, x_2, x_3, \dots, x_k \in X$ and $\forall u \in Y$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x), u\| \leq \frac{1}{1-L}\psi(x,x,x,u) \quad (82)$$

for all $x \in X$ and all $u \in Y$, where

$$\psi(x,x,x,u) = \max\left\{ \begin{array}{c} \{max\frac{1}{|2|^3}\emptyset(-x,-x,x,0,..,0,u) \\ ,\frac{3}{|2|^3}\emptyset(x,x,-x,0,..,0,u)\},\frac{1}{|2|^3}\delta(2x) \end{array} \right\} \quad (83)$$

for all $x \in X$ and all $u \in Y$
Proof. Similarly as proof corollary (4.8) and by Theorem(3.6) we have (82).

V. CONCLUSION

A study of the stability properties of a type of quadratic equation in non-Archimedean 2-Banach spaces by fixed point method has been done. The stability quadratic functional equation with n-variable has been proved on the same space. It would be interesting also to study slimier properties for n-normed spaces.

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