ISSN No:-2456-2165

A Comparative Study of the Standard and Generalized Formalism Associated with the Mathematical Framework Based on the Spacer Component Matrices and the Set of Related Mathematical Elements

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Abstract:- The paper presents a study of the comparative and contrasting features of the standard formalism and the generalized formalism associated with the mathematical framework of the Spacer component matrices and related set of mathematical elements, the analytical results are presented and numerically demonstrated using an appropriate case study.

Keywords:- *Spacer Matrix Components, the Core Component Matrix Associated with the Spacer Component Matrices, Correlation Component Matrices and their Building-Block Matrices, Completely Positive Trace Preserving Transformations, the Standard and Generalized Formalism Associated with the Mathematical Framework of Spacer Component Matrices*

I. NOTATIONS

- N denotes the set of all Natural numbers
- C denotes the set of all Complex numbers
- $I_{w\times w}$ denotes the Identity matrix of order 'w'
- C^{w} denotes the complex coordinate space of order 'w'
- $c^{\bullet}$ denotes the complex conjugate of the complex number ' c '
- $M_{x\times y}(C)$ denotes the complex Matrix space of order 'x' by 'y'
- ' s ' denotes the Embedding dimension
- $M_{s\times s}(C)$ denotes the Embedded Matrix Space
-
- $$|\omega\rangle \in C^{d},\ |\omega\rangle = \begin{bmatrix} \omega_1 \\ \omega_2 \\ . \\ . \\ \omega_d \end{bmatrix}_{d\times 1},\ \langle\omega| = \begin{bmatrix} \omega^{\bullet}_1 & \omega^{\bullet}_2 & . & . & \omega^{\bullet}_d \end{bmatrix}_{1\times d},\ \omega_1,\omega_2,....,\omega_d \in C$$
- $$|\omega\rangle \in C^{d},\ |\mu\rangle \in C^{f},\ T_{d\times f} = \left[T_{ij}\right],\ T_{d\times f} \in M_{d\times f}(C),\ \text{therefore: } \langle\omega|T|\mu\rangle = \sum_{i=1}^{d}\sum_{j=1}^{f}\omega^{\bullet}_{i}T_{ij}\mu_{j}$$
- $$|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m\times 1},\ |n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1},\ |s\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{s\times 1},\ |s-m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{(s-m)\times 1},\ |s-n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{(s-n)\times 1}$$

- $|e_s\rangle = \frac{1}{\sqrt{s}}|s\rangle$, $|e_m\rangle = \frac{1}{\sqrt{m}}|m\rangle$, $|e_n\rangle = \frac{1}{\sqrt{n}}|n\rangle$
- A^H denotes the Hermitian conjugate of the matrix A
- $(X_{w\times w})^{-1}$ denote the Proper Inverse of the Invertible matrix $X_{w\times w}$, i.e. $(X_{w\times w})^{-1} X_{w\times w} = X_{w\times w}(X_{w\times w})^{-1} = I_{w\times w}$
- $X_{w\times w}$ is hermitian and Positive definite, we define the hermitian and Positive definite matrices $(X_{w\times w})^{+1/2}$ and $(X_{w\times w})^{-1/2}$ such that: $(X_{w\times w})^{-1/2} = ((X_{w\times w})^{+1/2})^{-1}$, $(X_{w\times w})^{+1/2}(X_{w\times w})^{+1/2} = X_{w\times w}$ and $(X_{w\times w})^{-1/2}(X_{w\times w})^{-1/2} = (X_{w\times w})^{-1}$
- $\max(a,b)$ denotes the maximum of the two inputs 'a' and 'b'
- $|a-b|$ denotes the absolute value of the difference between the two inputs 'a' and 'b'
- $E.S(X_{w\times w})$ denotes the Eigenvalue spectrum of the matrix $X_{w\times w}$
- $X \subset Y$ denotes that the set X is a proper subset of the set Y
- $Nsp(A_{x\times y})$ denotes the Null space of the matrix $A_{x\times y}$
- $\dim.[Nsp(A_{x\times y})]$ denotes the dimension of the Null space of the matrix $A_{x\times y}$
- $V o W = C^s$ denotes the Orthogonal decomposition of the space C^s into the subspaces V and W
- 'σ' denotes the number of distinct eigenvalues of the Standard Core component matrix $Z_{s\times s}$
- 'ν' denotes the number of distinct eigenvalues of the Generalized Core component matrix $\hat{Z}_{s\times s}$
- $\{H(j)_{m\times m}, T(j)_{n\times n}, E(j)_{m\times n} \mid j = 1,2,....,\sigma\}$ denotes the set of Building-Block matrices associated with the Standard formalism
- $\{\hat{H}(t)_{m\times m}, \hat{T}(t)_{n\times n}, \hat{E}(t)_{m\times n} \mid t = 1,2,....,\nu\}$ denotes the set of Building-Block matrices associated with the Generalized formalism
- $\{R(m,m\,|\,\mu)_{m\times m}, R(n,n\,|\,\mu)_{n\times n}, R(m,n\,|\,\mu)_{m\times n} \mid 0 < \mu \le 1\}$ denotes the set of Correlation Component matrices associated with the Standard formalism
- $\{\hat{R}(m,m\,|\,\mu)_{m\times m}, \hat{R}(n,n\,|\,\mu)_{n\times n}, \hat{R}(m,n\,|\,\mu)_{m\times n} \mid 0 < \mu \le 1\}$ denotes the set of Correlation Component matrices associated with the Generalized formalism

II. INTRODUCTION

The mathematical framework based on the Spacer matrix components associated with strictly rectangular complex matrix spaces, presented in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], provides a methodology that can be used to quantify correlation between vectors belonging to non-compatible dimensions [6,10], quantify intrinsic overlap in matrices belonging to strictly rectangular complex matrix spaces [7] and construction of several mathematical elements of utility in theoretical and numerical linear algebra [8,12], the results of theoretical studies on the properties of spacer matrix components and set of matrices and mathematical elements generated from them, has been previously presented [9, 14, 15 and 16].

ISSN No:-2456-2165

The paper presents a comparative study of the two formalisms associated with the mathematical framework based on the Spacer matrix components: the standard formalism and the generalized formalism. The Standard formalism corresponds to the situation where there is synchronicity; the embedding matrix components being $G_{s\times m}$ and $W_{s\times n}$ while the Core component matrix being $Z_{s\times s}$, both of these component units are generated only from the spacer matrix components $P_{n\times s}$ and $Q_{s\times m}$ which result in certain symmetric features being incorporated into the framework [9, 15 and 16]. The Generalized formalism attempts to remove this synchronicity between the embedding and the core matrix components by use of a particular form of completely positive trace preserving transformation [1, 19, 21, 23, 24, 26, 27, 29], abbreviated as CPTP transformation, resulting in the formation of the generalized Core component matrix $\hat{Z}_{s\times s}$ from the standard Core component matrix $Z_{s\times s}$, the CPTP transformation changes the eigenspaces and the associated eigenvalues thereby resulting in an alternate mathematical structure interrelating the embedding and the core component units.

Numerical demonstration of the two formalisms is presented using the case of $(m=2, n=4)$ Complex Matrix space. The numerical section presents a particular CPTP transformation to demonstrate the contrasting aspects of these two formalisms. The article concludes with a discussion of the observations, results of the numerical case study and insights obtained with respect to the properties of the two formalisms under consideration.

❖ *Mathematical Framework*

➢ $m \in N, n \in N$ and $m \neq n$, $s = \max(m,n) + |m-n|$, therefore $s \in N$, $s > m$ and $s > n$

The Spacer component matrices $P_{n\times s}$ and $Q_{s\times m}$, are defined as follows:

$$P_{n\times s} = \begin{bmatrix} I_{n\times n} & (\frac{1}{n})|n\rangle\langle s-n| \end{bmatrix}_{n\times s} = \left[\begin{array}{ccccc|ccccc} 1 & 0 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ 0 & 1 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 1/n & 1/n & . & . & 1/n \end{array}\right]_{n\times s}$$

$$Q_{s\times m} = \begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})|s-m\rangle\langle m| \end{bmatrix}_{s\times m} = \left[\begin{array}{ccccc} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \\ \hline 1/m & 1/m & . & . & 1/m \\ 1/m & 1/m & . & . & 1/m \\ . & . & . & . & . \\ . & . & . & . & . \\ 1/m & 1/m & . & . & 1/m \end{array}\right]_{s\times m}$$

We define the matrices $G_{s\times m}$ and $W_{s\times n}$ as follows:

$$W_{s\times n} = (P_{n\times s})^H[(PP^H)^{-1/2}]_{n\times n} \quad , \quad G_{s\times m} = (Q_{s\times m})[(Q^H Q)^{-1/2}]_{m\times m}$$

Therefore $(W_{s\times n})^H W_{s\times n} = I_{n\times n}$, $(G_{s\times m})^H G_{s\times m} = I_{m\times m}$

❖ *The Standard Formalism*

➢ $Z_{s\times s} = (\frac{1}{2})(P^H P)_{s\times s} + (\frac{1}{2})(QQ^H)_{s\times s}$, therefore $Z_{s\times s}$ is real, Hermitian and Positive semi-definite

➢ $E.S(Z_{s\times s}) = \{\lambda_1, d_1; \lambda_2, d_2; ....; \lambda_\sigma, d_\sigma\}$ where $\lambda_1 > \lambda_2 > .... > \lambda_{\sigma-1} > 0$ and $\lambda_\sigma = 0$

➢ $V(j) = Nsp(Z_{s\times s} - \lambda_j I_{s\times s})$, $\dim.[V(j)] = d_j$, where $j = 1, 2, ..., \sigma$

➢ The associated Orthogonal decomposition of the space C^s is given as follows:

$V(1)oV(2)o....oV(\sigma) = C^s$, therefore $\sum_{j=1}^{\sigma} d_j = s$

➢ $F(j)_{s\times s}$ denotes the Orthogonal Projector onto the subspace $V(j)$, therefore: $(F(j)_{s\times s})^H = F(j)_{s\times s}$ $\forall$ $j = 1, 2, ..., \sigma$, $\sum_{j=1}^{\sigma} F(j)_{s\times s} = I_{s\times s}$ and $Z_{s\times s} = \sum_{j=1}^{\sigma} \lambda_j F(j)_{s\times s}$

➢ $\Omega_{s\times s}(\mu) = \mu I_{s\times s} + (1-\mu) Z_{s\times s}$ where $0 < \mu \le 1$, therefore $\Omega_{s\times s}(\mu) \in M_{s\times s}(C)$, $\Omega_{s\times s}(\mu)$ is real, Hermitian and Positive definite $\forall \mu \in (0,1]$

➢ $\Omega_{s\times s}(\mu) = \sum_{j=1}^{\sigma} \hat{\lambda}_j(\mu) F(j)_{s\times s}$ where $\hat{\lambda}_j(\mu) = \mu + (1-\mu)\lambda_j$, $j = 1, 2, ..., \sigma$

➢ $H(j)_{m\times m} = (G_{s\times m})^H F(j)_{s\times s} G_{s\times m}$, $T(j)_{n\times n} = (W_{s\times n})^H F(j)_{s\times s} W_{s\times n}$, $E(j)_{m\times n} = (G_{s\times m})^H F(j)_{s\times s} W_{s\times n}$ where $j = 1, 2, ..., \sigma$

➢ $R(m, n \mid \mu)_{m\times n} = (G_{s\times m})^H \Omega(\mu)_{s\times s} W_{s\times n} = \sum_{j=1}^{\sigma} \hat{\lambda}_j(\mu) E(j)_{m\times n}$

➢ $R(m, m \mid \mu)_{m\times m} = (G_{s\times m})^H \Omega(\mu)_{s\times s} G_{s\times m} = \sum_{j=1}^{\sigma} \hat{\lambda}_j(\mu) H(j)_{m\times m}$

➢ $R(n, n \mid \mu)_{n\times n} = (W_{s\times n})^H \Omega(\mu)_{s\times s} W_{s\times n} = \sum_{j=1}^{\sigma} \hat{\lambda}_j(\mu) T(j)_{n\times n}$

❖ *The Generalized Formalism*

➢ $\hat{Z}_{s\times s} = \sum_{t=1}^{s} p_t [(U(t)_{s\times s})^H Z_{s\times s} U(t)_{s\times s}]$ where $p_t \ge 0$ $\forall t = 1, 2, ..., s$ and $\sum_{t=1}^{s} p_t = 1$

$U(t)_{s\times s} \in M_{s\times s}(C)$, $(U(t)_{s\times s})^H = (U(t)_{s\times s})^{-1}$ $\forall t = 1, 2, ..., s$

Therefore, $\hat{Z}_{s\times s}$ is Hermitian and Positive semi-definite or Positive definite

➢ $E.S(\hat{Z}_{s\times s}) = \{\varepsilon_1, \delta_1; \varepsilon_2, \delta_2; ....; \varepsilon_\nu, \delta_\nu\}$ where $\varepsilon_1 > \varepsilon_2 > .... > \varepsilon_\nu \ge 0$

- $\hat{V}(t) = Nsp(\hat{Z}_{s\times s} - \varepsilon_t I_{s\times s})$, $\dim.[\hat{V}(t)] = \delta_t$, where $t = 1,2,....,\nu$

- The associated Orthogonal decomposition of the space C^s is given as follows:

$\hat{V}(1) o \hat{V}(2) o....o \hat{V}(\nu) = C^s$, therefore $\sum_{t=1}^{\nu} \delta_t = s$

- $\hat{F}(t)_{s\times s}$ denotes the Orthogonal Projector onto the subspace $\hat{V}(t)$, therefore:

$(\hat{F}(t)_{s\times s})^H = \hat{F}(t)_{s\times s} \quad \forall t = 1,2,....,\nu$, $\sum_{t=1}^{\nu} \hat{F}(t)_{s\times s} = I_{s\times s}$ and $\hat{Z}_{s\times s} = \sum_{t=1}^{\nu} \varepsilon_t \hat{F}(t)_{s\times s}$

- $\hat{\Omega}_{s\times s}(\mu) = \mu I_{s\times s} + (1-\mu)\hat{Z}_{s\times s}$ where $0 < \mu \le 1$, therefore $\hat{\Omega}_{s\times s}(\mu) \in M_{s\times s}(C)$, $\hat{\Omega}_{s\times s}(\mu)$ is Hermitian and Positive definite $\forall \mu \in (0,1]$

- $\hat{\Omega}_{s\times s}(\mu) = \sum_{t=1}^{\nu} \hat{\varepsilon}_t(\mu) \hat{F}(t)_{s\times s}$ where $\hat{\varepsilon}_t(\mu) = \mu + (1-\mu)\varepsilon_t$, $t = 1,2,....,\nu$

- $\hat{H}(t)_{m\times m} = (G_{s\times m})^H \hat{F}(t)_{s\times s} G_{s\times m}$, $\hat{T}(t)_{n\times n} = (W_{s\times n})^H \hat{F}(t)_{s\times s} W_{s\times n}$, $\hat{E}(t)_{m\times n} = (G_{s\times m})^H \hat{F}(t)_{s\times s} W_{s\times n}$ where $t = 1,2,....,\nu$

- $\hat{R}(m,n\,|\,\mu)_{m\times n} = (G_{s\times m})^H \hat{\Omega}(\mu)_{s\times s} W_{s\times n} = \sum_{t=1}^{\nu} \hat{\varepsilon}_t(\mu) \hat{E}(t)_{m\times n}$

- $\hat{R}(m,m\,|\,\mu)_{m\times m} = (G_{s\times m})^H \hat{\Omega}(\mu)_{s\times s} G_{s\times m} = \sum_{t=1}^{\nu} \hat{\varepsilon}_t(\mu) \hat{H}(t)_{m\times m}$

- $\hat{R}(n,n\,|\,\mu)_{n\times n} = (W_{s\times n})^H \hat{\Omega}(\mu)_{s\times s} W_{s\times n} = \sum_{t=1}^{\nu} \hat{\varepsilon}_t(\mu) \hat{T}(t)_{n\times n}$

❖ *Results on Conservation and Invariance Relationships*

- $trace(Z_{s\times s}) = \sum_{j=1}^{\sigma} \lambda_j d_j = \sum_{t=1}^{\nu} \varepsilon_t \delta_t = trace(\hat{Z}_{s\times s})$

- $trace(\Omega(\mu)_{s\times s}) = \sum_{j=1}^{\sigma} \hat{\lambda}_j(\mu) d_j = \sum_{t=1}^{\nu} \hat{\varepsilon}_t(\mu) \delta_t = trace(\hat{\Omega}(\mu)_{s\times s})$, where $0 < \mu \le 1$

- $I_{m\times m} = (G_{s\times m})^H I_{s\times s} G_{s\times m} = \sum_{j=1}^{\sigma} H(j)_{m\times m} = \sum_{t=1}^{\nu} \hat{H}(t)_{m\times m}$

- $I_{n\times n} = (W_{s\times n})^H I_{s\times s} W_{s\times n} = \sum_{j=1}^{\sigma} T(j)_{n\times n} = \sum_{t=1}^{\nu} \hat{T}(t)_{n\times n}$

ISSN No:-2456-2165

- $(G_{s\times m})^H W_{s\times n} = (G_{s\times m})^H I_{s\times s} W_{s\times n} = \sum_{j=1}^{\sigma} E(j)_{m\times n} = \sum_{t=1}^{\nu} \hat{E}(t)_{m\times n}$

- *Numerical Case Study*

- $m = 2, n = 4$ therefore $s = 6$

- In the following numerical computations, we set the value of the parameter ' μ ' as $\mu = \frac{1}{2}$

$$P_{4\times 6} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad Q_{6\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$Z_{6\times 6} = (\frac{1}{8}) \begin{bmatrix} 8 & 0 & 2 & 2 & 3 & 3 \\ 0 & 8 & 2 & 2 & 3 & 3 \\ 2 & 2 & 6 & 2 & 3 & 3 \\ 2 & 2 & 2 & 6 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}, \quad G_{6\times 2} = (\frac{1}{2\sqrt{6}}) \begin{bmatrix} \sqrt{2}+\sqrt{6} & \sqrt{2}-\sqrt{6} \\ \sqrt{2}-\sqrt{6} & \sqrt{2}+\sqrt{6} \\ \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$W_{6\times 4} = (\frac{1}{4\sqrt{6}}) \begin{bmatrix} 2+3\sqrt{6} & 2-\sqrt{6} & 2-\sqrt{6} & 2-\sqrt{6} \\ 2-\sqrt{6} & 2+3\sqrt{6} & 2-\sqrt{6} & 2-\sqrt{6} \\ 2-\sqrt{6} & 2-\sqrt{6} & 2+3\sqrt{6} & 2-\sqrt{6} \\ 2-\sqrt{6} & 2-\sqrt{6} & 2-\sqrt{6} & 2+3\sqrt{6} \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix},$$

$$\Omega_{6\times6}(\mu=\frac{1}{2})=(\frac{1}{16})\begin{bmatrix}16 & 0 & 2 & 2 & 3 & 3\\ 0 & 16 & 2 & 2 & 3 & 3\\ 2 & 2 & 14 & 2 & 3 & 3\\ 2 & 2 & 2 & 14 & 3 & 3\\ 3 & 3 & 3 & 3 & 11 & 3\\ 3 & 3 & 3 & 3 & 3 & 11\end{bmatrix},$$

$$F(1)_{6\times6}=(\frac{1}{6})\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix},\quad F(2)_{6\times6}=(\frac{1}{2})\begin{bmatrix}1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$F(3)_{6\times6}=(\frac{1}{4})\begin{bmatrix}1 & 1 & -1 & -1 & 0 & 0\\ 1 & 1 & -1 & -1 & 0 & 0\\ -1 & -1 & 3 & -1 & 0 & 0\\ -1 & -1 & -1 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$F(4)_{6\times6}=(\frac{1}{12})\begin{bmatrix}1 & 1 & 1 & 1 & -2 & -2\\ 1 & 1 & 1 & 1 & -2 & -2\\ 1 & 1 & 1 & 1 & -2 & -2\\ 1 & 1 & 1 & 1 & -2 & -2\\ -2 & -2 & -2 & -2 & 10 & -2\\ -2 & -2 & -2 & -2 & -2 & 10\end{bmatrix},$$

$$\lambda_1=\frac{9}{4},\ d_1=1;\ \lambda_2=1,\ d_2=1\ ;\ \lambda_3=\frac{1}{2},\ d_3=2\ ;\ \lambda_4=0\ ,\ d_4=2$$

$$\hat{\lambda}_1(\mu=\frac{1}{2})=\frac{13}{8}\ ,\ \hat{\lambda}_2(\mu=\frac{1}{2})=1\ ,\ \hat{\lambda}_3(\mu=\frac{1}{2})=\frac{3}{4}\ ,\ \hat{\lambda}_4(\mu=\frac{1}{2})=\frac{1}{2}$$

$$(G_{6\times2})^H W_{6\times4}=(\frac{1}{2\sqrt{2}})\begin{bmatrix}1+\sqrt{2} & 1-\sqrt{2} & 1 & 1\\ 1-\sqrt{2} & 1+\sqrt{2} & 1 & 1\end{bmatrix},$$

$$E(1)_{2\times4}=(\frac{1}{2\sqrt{2}})\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix},\ E(2)_{2\times4}=(\frac{1}{2})\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\end{bmatrix},\ E(3)_{2\times4}=E(4)_{2\times4}=0_{2\times4}$$

ISSN No:-2456-2165

$$H(1)_{2\times2} = (\frac{1}{2})\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \ , \ H(2)_{2\times2} = (\frac{1}{2})\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} \ , \ H(3)_{2\times2} = H(4)_{2\times2} = 0_{2\times2}$$

$$T(1)_{4\times4} = (\frac{1}{4})\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \ , \ T(2)_{4\times4} = (\frac{1}{2})\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix} \ ,$$

$$T(3)_{4\times4} = (\frac{1}{4})\begin{bmatrix}1 & 1 & -1 & -1\\1 & 1 & -1 & -1\\-1 & -1 & 3 & -1\\-1 & -1 & -1 & 3\end{bmatrix} \ , \ T(4)_{4\times4} = 0_{4\times4}$$

$$R(m=2, n=4 \mid \mu = \frac{1}{2}) = \begin{bmatrix}1.074524 & 0.074524 & 0.574524 & 0.574524\\0.074524 & 1.074524 & 0.574524 & 0.574524\end{bmatrix}$$

$$R(m=2, m=2 \mid \mu = \frac{1}{2}) = \begin{bmatrix}1.3125 & 0.3125\\0.3125 & 1.3125\end{bmatrix}$$

$$R(n=4, n=4 \mid \mu = \frac{1}{2}) = \begin{bmatrix}1.09375 & 0.09375 & 0.21875 & 0.21875\\0.09375 & 1.09375 & 0.21875 & 0.21875\\0.21875 & 0.21875 & 0.96875 & 0.21875\\0.21875 & 0.21875 & 0.21875 & 0.96875\end{bmatrix}$$

$$\Gamma_{6\times6} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1\\0 & 1 & 0 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 0 & 1\\0 & 0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \ , \quad \hat{U}_{6\times6} = \Gamma_{6\times6}[(\Gamma^H\Gamma)^{-1/2}]_{6\times6} \ ,$$

$$\hat{U}_{6\times6} = \begin{bmatrix}0.933333 & -0.066667 & -0.066667 & -0.066667 & -0.066667 & 0.333333\\-0.066667 & 0.933333 & -0.066667 & -0.066667 & -0.066667 & 0.333333\\-0.066667 & -0.066667 & 0.933333 & -0.066667 & -0.066667 & 0.333333\\-0.066667 & -0.066667 & -0.066667 & 0.933333 & -0.066667 & 0.333333\\-0.066667 & -0.066667 & -0.066667 & -0.066667 & 0.933333 & 0.333333\\-0.333333 & -0.333333 & -0.333333 & -0.333333 & -0.333333 & 0.666667\end{bmatrix}$$

$$p_1 = \frac{32}{63} \ , \ p_2 = \frac{16}{63} \ , \ p_3 = \frac{8}{63} \ , \ p_4 = \frac{4}{63} \ , \ p_5 = \frac{2}{63} \ , \ p_6 = \frac{1}{63}$$

$$U(1)_{6\times6} = \hat{U}_{6\times6} \ , \ U(2)_{6\times6} = [\hat{U}\hat{U}]_{6\times6} \ , \ U(3)_{6\times6} = [\hat{U}\hat{U}\hat{U}]_{6\times6} \ , \ U(4)_{6\times6} = [\hat{U}\hat{U}\hat{U}\hat{U}]_{6\times6} \ ,$$

ISSN No:-2456-2165

$U(5)_{6\times6}=[\hat{U}\hat{U}\hat{U}\hat{U}\hat{U}]_{6\times6}$, $U(6)_{6\times6}=[\hat{U}\hat{U}\hat{U}\hat{U}\hat{U}\hat{U}]_{6\times6}$

$$\hat{Z}_{6\times6}=\begin{bmatrix} 0.752245 & -0.247755 & 0.002245 & 0.002245 & 0.127245 & 0.037263 \\ -0.247755 & 0.752245 & 0.002245 & 0.002245 & 0.127245 & 0.037263 \\ 0.002245 & 0.002245 & 0.502245 & 0.002245 & 0.127245 & 0.037263 \\ 0.002245 & 0.002245 & 0.002245 & 0.502245 & 0.127245 & 0.037263 \\ 0.127245 & 0.127245 & 0.127245 & 0.127245 & 0.127245 & 0.037263 \\ 0.037263 & 0.037263 & 0.037263 & 0.037263 & 0.037263 & 1.613775 \end{bmatrix}$$

$$\hat{\Omega}_{6\times6}(\mu=\frac{1}{2})=\begin{bmatrix} 0.876122 & -0.123878 & 0.001122 & 0.001122 & 0.063622 & 0.018632 \\ -0.123878 & 0.876122 & 0.001122 & 0.001122 & 0.063622 & 0.018632 \\ 0.001122 & 0.001122 & 0.751122 & 0.001122 & 0.063622 & 0.018632 \\ 0.001122 & 0.001122 & 0.001122 & 0.751122 & 0.063622 & 0.018632 \\ 0.063622 & 0.063622 & 0.063622 & 0.063622 & 0.563622 & 0.018632 \\ 0.018632 & 0.018632 & 0.018632 & 0.018632 & 0.018632 & 1.306888 \end{bmatrix}$$

$\varepsilon_1=1.620826$, $\delta_1=1$; $\varepsilon_2=1$, $\delta_2=1$; $\varepsilon_3=0.629174$, $\delta_3=1$; $\varepsilon_4=0.5$, $\delta_4=2$;
$\varepsilon_5=0$, $\delta_5=1$

$\hat{\varepsilon}_1(\mu=\frac{1}{2})=1.310413$, $\hat{\varepsilon}_2(\mu=\frac{1}{2})=1$, $\hat{\varepsilon}_3(\mu=\frac{1}{2})=0.814587$, $\hat{\varepsilon}_4(\mu=\frac{1}{2})=0.75$, $\hat{\varepsilon}_5(\mu=\frac{1}{2})=0.5$

$$\hat{F}(1)_{6\times6}=\begin{bmatrix} 0.037711 \\ 0.037711 \\ 0.037711 \\ 0.037711 \\ 0.037711 \\ 0.996438 \end{bmatrix}\begin{bmatrix} 0.037711 & 0.037711 & 0.037711 & 0.037711 & 0.037711 & 0.996438 \end{bmatrix} ,$$

$$\hat{F}(2)_{6\times6}=(\frac{1}{2})\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$\hat{F}(3)_{6\times6}=\begin{bmatrix} 0.445621 \\ 0.445621 \\ 0.445621 \\ 0.445621 \\ 0.445621 \\ -0.084324 \end{bmatrix}\begin{bmatrix} 0.445621 & 0.445621 & 0.445621 & 0.445621 & 0.445621 & -0.084324 \end{bmatrix}$$

ISSN No:-2456-2165

$$\hat{F}(4)_{6\times6}=(\frac{1}{4})\begin{bmatrix}1&1&-1&-1&0&0\\1&1&-1&-1&0&0\\-1&-1&3&-1&0&0\\-1&-1&-1&3&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},$$

$$\hat{F}(5)_{6\times6}=(\frac{1}{20})\begin{bmatrix}1\\1\\1\\1\\-4\\0\end{bmatrix}\begin{bmatrix}1&1&1&1&-4&0\end{bmatrix}$$

$$\hat{E}(1)_{2\times4}=(0.082744)\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}\ ,\ \hat{E}(2)_{2\times4}=(\frac{1}{2})\begin{bmatrix}1&-1&0&0\\-1&1&0&0\end{bmatrix},$$

$$\hat{E}(3)_{2\times4}=(0.270810)\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}\ ,\ \hat{E}(4)_{2\times4}=\hat{E}(5)_{2\times4}=0_{2\times4}$$

$$\hat{H}(1)_{2\times2}=(0.117017)\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}\ ,\ \hat{H}(2)_{2\times2}=(\frac{1}{2})\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1&-1\end{bmatrix}\ ,\ \hat{H}(3)_{2\times2}=(0.382983)\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix},$$

$$\hat{H}(4)_{2\times2}=\hat{H}(5)_{2\times2}=0_{2\times2}$$

$$\hat{T}(1)_{4\times4}=(0.058509)\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}\ ,\ \hat{T}(2)_{4\times4}=(\frac{1}{2})\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\begin{bmatrix}1&-1&0&0\end{bmatrix},$$

$$\hat{T}(3)_{4\times4}=(0.191491)\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}\ ,\ \hat{T}(4)_{4\times4}=(\frac{1}{4})\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{bmatrix}\ ,\ \hat{T}(5)_{4\times4}=0_{4\times4}$$

$$\hat{R}(m=2,n=4\,|\,\mu=\frac{1}{2})=\begin{bmatrix}0.829026&-0.170974&0.329026&0.329026\\-0.170974&0.829026&0.329026&0.329026\end{bmatrix}$$

$$\hat{R}(m=2,m=2\,|\,\mu=\frac{1}{2})=\begin{bmatrix}0.965314&-0.034686\\-0.034686&0.965314\end{bmatrix}$$

ISSN No:-2456-2165

$$\hat{R}(n=4,n=4\,|\,\mu=\frac{1}{2})=\begin{bmatrix} 0.920157 & -0.079843 & 0.045157 & 0.045157 \\ -0.079843 & 0.920157 & 0.045157 & 0.045157 \\ 0.045157 & 0.045157 & 0.795157 & 0.045157 \\ 0.045157 & 0.045157 & 0.045157 & 0.795157 \end{bmatrix}$$

III. DISCUSSION AND CONCLUSION

The CPTP transformation, as formulated in context of the paper, relates the matrix $\hat{Z}_{s\times s}$ to the matrix $Z_{s\times s}$ preserving the hermitian aspect and the matrix trace, but with possible restructuring of the eigenspaces-eigenvalue framework, this results in general, into reformulation of the analytical expressions of the Projection matrices associated with the corresponding mutually orthogonal eigenspaces which further leads to the reformulation of Building-Block matrix components and that of the Correlation component matrices.

The numerical study of the $(m=2,n=4)$ Complex matrix space demonstrates the contrasting features of two formalisms; it can be observed that in case of the standard formalism the orthogonal decomposition of the co-ordinate space C^6 involves four mutually orthogonal eigenspaces of the matrix $Z_{s\times s}$, two of which have dimension equal to one and the other two such subspaces have dimension equal to two, in the case of the generalized formalism which involves orthogonal decomposition of the space C^6 based on the eigenspaces of the generalized core component matrix $\hat{Z}_{s\times s}$, the number of such eigenspaces increases to five, with four of them having dimension equal to one and one eigenspace of dimension equal to two. The set of the associated eigenvalues are also observed to have changed under the effect of the defined CPTP transformation in context of the numerical study.

The generalized formalism increases the scope of the mathematical framework by allowing the intrinsic structure of the core component matrices to be modified by the use of CPTP transformations which changes their relationship with the embedding component matrices which are determined only by the spacer component matrices, thus leading to possibility of a diversified mathematical structure being associated with the framework; changing only the parameter 'μ' over the interval $(0,1]$ allows the numerical forms of the Correlation component matrices to be modified but preserving the intrinsic eigenspace decomposition structure, incorporating a CPTP transformation and changing of the parameter 'μ' allows the numerical forms of the Correlation component matrices to be sampled from a more diverse set of numerical situations.

The paper establishes the importance of the application of appropriate CPTP transformations to enhance the numerical diversity associated with the framework based on the spacer component matrices and the related mathematical elements. Follow up studies focused on understanding the nature of relationships between the analytical forms of the CPTP transformations and the analytical expressions of the mathematical elements pertaining to the framework, is expected to shed more light into the intricacies of the mathematical framework under consideration.

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