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The Weak Commutation Matrices of Matrix with Duplicate Entries in its Secondary Diagonal

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Abstract:-This article discusses the permutation matrix, which is a weak commutation matrix. This weak commutation matrix is determined by compiling the duplicate entry, i.e., two, three, and four which are the same in different positions in the 4×4 secondary diagonal matrix. It uses the property of the transformation vec matrix to vec transpose the matrix for determining the weak commutation matrix of the secondary diagonal matrix. We have 24 weak commutation matrices for the secondary diagonal matrices.

Keywords:-*Weak Commutation Matrix; Duplicate Entries; Secondary Diagonal Entry; Vec Matrix*

I. INTRODUCTION

The vec matrix is a unique operation that can change a matrix into a column vector [1]. It can also be said to change the matrix into a vector by stacking the column vertically [2]. Note that $vec(A)$ and $vec(A^T)$ have the same entries, but the composition of the elements is different. If the matrix A, $m\times n$, and its transpose is matrix A^T, then the vectors $vec(A)$ and $vec(A^T)$ are $mn\times 1$. The *commutation matrix* is the permutation matrix that changes the $vec(A)$ to $vec(A^T)$ [3, 4].

A unique commutation matrix that transforms $vec(A)$ to $vec(A^T)$ for any matrix $m\times n$. But then, by [5], states that for the matrices in the Kronecker quaternion group found in [6] are several matrices that are like a commutation matrix. So, [7] defines the *weak commutation matrix.*

The organization of this paper is as follows: Basic Theory; it is introduced a lot of basic concepts and notations of vec, permutation matrix, Kronecker product, and commutation matrix will be used in the section Result and Discussion. The section Result and Discussion discuss the the transformation of the permutation matrix on the vec matrix and vec transpose matrix, that is, the weak commutation matrix for the secondary diagonal matrix. The first step is to determine various the secondary diagonal matrix with duplicate entries (matrices in (3.1), (3.2), and (3.3) in Section III). Next, it presents the weak commutation matrix of the matrices.

II. BASIC THEORY

- *This Section Presents Some Definitions, Properties and Theorems Related to Commutation Matrix.*

- ***Definition 2.1*** *[8] Let $A=[a_{ij}]$ be an $m\times n$ matrix, and A_j the column of A. The $vec(A)$ is the n column vector, i.e*

$$vec(A)=\begin{bmatrix}A_1\\A_2\\\vdots\\A_n\end{bmatrix}$$

Let S_n denote the set of all permutation of the n element set $[n]:=\{1,2,\dots,n\}$. A permutation is one-to-one function from $[n]$ onto $[n]$. Permutation of finite sets are usually given by listing of each element of the domain and its corresponding functional value. For example, we define a permutation σ of the set $[n]:=\{1,2,3,4,5,6,7\}$ by specifying $\sigma(1)=7$, $\sigma(2)=1$, $\sigma(3)=3$, $\sigma(4)=6$, $\sigma(5)=2$, $\sigma(6)=4$, $\sigma(7)=5$ A more convenient way to express this correspondence is towrite σ in array form as

$$\sigma=\begin{bmatrix}1&2&3&4&5&6&7\\7&1&3&6&2&4&5\end{bmatrix} \qquad (2.1)$$

There is another notation commonly used to specify permutation. It is called cycle notation. Cycle notation has theoretical advantages in that specific essential properties of the permutation can be readily determined when cycle notation is used. For example, permutation in (2.1) can be written as $\sigma=(1\ 7\ 5\ 2)(4\ 6)$. For detail see [9].

- ***Theorem 2.2*** [9] *Let π and σ be a permutation in S_n, then $P(\pi)P(\sigma)=P(\pi\sigma)$.*

If σ is a permutation, we have the identity matrix as follows:

- ***Definition 2.3*** *[10] Let σ be a permutation in S_n. Define the permutation matrix $P(\sigma)=[\delta_{i,\sigma(j)}]$, $\delta_{i,\sigma(j)}=ent_{i,j}(P(\sigma))$ where*

$$\delta_{i,\sigma(j)}=\begin{cases}1 & \text{if } i=\sigma(j)\\0 & \text{if } i\neq\sigma(j)\end{cases}$$

Example 2.1 Let $n:=\{1,2,3,4\}$ and $\sigma=(1\ 4\ 2)$.

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$P(142) = [\delta_{i,\sigma(j)}]$ and $\delta_{i,\sigma(j)} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j) \end{cases}$

(1 to 4; 4 to 2; 3 to 3; 2 to 1 ; $\sigma(1) = 4$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 2$)

$ent_{ij}(P(\sigma)) = \delta_{i,\sigma(j)}$

$ent_{11}(P(\sigma)) = \delta_{1,\sigma(1)} = \delta_{14} = 0\ (\sigma(1) = 4);$

$ent_{12}(P(\sigma)) = \delta_{1,\sigma(2)} = \delta_{11} = 1\ (\sigma(2) = 1);$

$ent_{13}(P(\sigma)) = \delta_{1,\sigma(3)} = \delta_{13} = 0\ (\sigma(3) = 3);$

$ent_{14}(P(\sigma)) = \delta_{1,\sigma(4)} = \delta_{12} = 0\ (\sigma(4) = 2);$

$ent_{21}(P(\sigma)) = \delta_{2,\sigma(1)} = \delta_{24} = 0\ (\sigma(1) = 4);$

$ent_{22}(P(\sigma)) = \delta_{2,\sigma(2)} = \delta_{21} = 0\ (\sigma(2) = 1);$

$ent_{23}(P(\sigma)) = \delta_{2,\sigma(3)} = \delta_{23} = 0\ (\sigma(3) = 3);$

$ent_{24}(P(\sigma)) = \delta_{2,\sigma(4)} = \delta_{22} = 1\ (\sigma(4) = 2);$

$ent_{31}(P(\sigma)) = \delta_{3,\sigma(1)} = \delta_{34} = 0\ (\sigma(1) = 4);$

$ent_{32}(P(\sigma)) = \delta_{3,\sigma(2)} = \delta_{31} = 0\ (\sigma(2) = 1);$

$ent_{33}(P(\sigma)) = \delta_{3,\sigma(3)} = \delta_{33} = 1\ (\sigma(3) = 3);$

$ent_{34}(P(\sigma)) = \delta_{3,\sigma(4)} = \delta_{32} = 0\ (\sigma(4) = 2);$

$ent_{41}(P(\sigma)) = \delta_{4,\sigma(1)} = \delta_{44} = 1\ (\sigma(1) = 4);$

$ent_{42}(P(\sigma)) = \delta_{4,\sigma(2)} = \delta_{41} = 0\ (\sigma(2) = 1);$

$ent_{43}(P(\sigma)) = \delta_{4,\sigma(3)} = \delta_{43} = 0\ (\sigma(3) = 3);$

$ent_{44}(P(\sigma)) = \delta_{4,\sigma(4)} = \delta_{42} = 0\ (\sigma(4) = 2);$

$$P(142) = \begin{bmatrix} \delta_{14} & \delta_{11} & \delta_{13} & \delta_{12} \\ \delta_{24} & \delta_{21} & \delta_{23} & \delta_{22} \\ \delta_{34} & \delta_{31} & \delta_{33} & \delta_{32} \\ \delta_{44} & \delta_{41} & \delta_{43} & \delta_{42} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- ***Definition 2.4*** [1] *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Then Kronecker product of A and B, denoted by $A \otimes B \in \mathbb{R}^{mp\times nq}$ and is defined to be matrix*

$$A\otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

- ***Theorem 2.6*** [1] *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{n\times t}$ and $D \in \mathbb{R}^{q\times s}$. Then $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$.*

- ***Definition 2.8*** [11] *Let I_n be the identity matrix and $\boldsymbol{e}_{im}$ is an m −dimensional column vector that has 1 in the i^{th} position and 0's elsewhere; that is*

$\boldsymbol{e}_{i,m} = [0, 0, \dots, 0, 1, 0, \dots, 0]^T$ *and* $I_n \otimes \boldsymbol{e}_{i,m}{}^T = a_{ij}\boldsymbol{e}_{i,m}{}^T, a_{ij} \in I_n$.

The commutation matrix, denoted by $K_{m,n}$ is given by

$$K_{m,n} = \begin{bmatrix} I_n \otimes \boldsymbol{e}_{1m}{}^T \\ I_n \otimes \boldsymbol{e}_{2m}{}^T \\ \vdots \\ I_n \otimes \boldsymbol{e}_{mm}{}^T \end{bmatrix}$$

Example 2.2. Matrix $K_{3,2}$ is

$$K_{3,2} = \begin{bmatrix} I_2 \otimes \boldsymbol{e}_{13}{}^T \\ I_2 \otimes \boldsymbol{e}_{23}{}^T \\ I_2 \otimes \boldsymbol{e}_{33}{}^T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [1 \ \ 0 \ \ 0] \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [0 \ \ 1 \ \ 0] \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [0 \ \ 0 \ \ 1] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1[1 \ \ 0 \ \ 0] & 0[1 \ \ 0 \ \ 0] \\ 0[1 \ \ 0 \ \ 0] & 1[1 \ \ 0 \ \ 0] \end{bmatrix} \\ \begin{bmatrix} 1[0 \ \ 1 \ \ 0] & 0[0 \ \ 1 \ \ 0] \\ 0[0 \ \ 1 \ \ 0] & 1[0 \ \ 1 \ \ 0] \end{bmatrix} \\ \begin{bmatrix} 1[0 \ \ 0 \ \ 1] & 0[0 \ \ 0 \ \ 1] \\ 0[0 \ \ 0 \ \ 1] & 1[0 \ \ 0 \ \ 1] \end{bmatrix} \end{bmatrix}$$

Specipically

$$K_{3,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The definition of the commutation matrix is given differently by [8], i.e.

$$K_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n}(H_{ij}\otimes H_{ij}{}^T)$$

The matrix H_{ij} can be conveniently expressed in terms of the column from the identity matrices I_m and I_n. If $\boldsymbol{e}_{i,m}$ is the i-th column of I_m and $\boldsymbol{e}_{j,n}$ is the j-th column of I_n, then $H_{ij} = \boldsymbol{e}_{i,m}\boldsymbol{e}_{j,n}{}^T$.

III. RESULT AND DISCUSSION

This paper aims to determine the weak commutation matrix of the following matrices, i.e.,

The arbitrary matrix 4×4 with two duplicate entries in its secondary diagonal:

$$\begin{bmatrix} & & & * \\ & & * & \\ & & & \\ & & & \end{bmatrix}, \begin{bmatrix} & & & * \\ & & & \\ & * & & \\ & & & \end{bmatrix}, \begin{bmatrix} & & & * \\ & & & \\ & & & \\ * & & & \end{bmatrix}, \begin{bmatrix} & & & \\ & & * & \\ & * & & \\ & & & \end{bmatrix}, \begin{bmatrix} & & & \\ & & * & \\ & & & \\ * & & & \end{bmatrix}, \begin{bmatrix} & & & \\ & & & \\ & * & & \\ * & & & \end{bmatrix} \quad (3.1)$$

$$\begin{bmatrix} & & & * \\ & & * & \\ & * & & \\ & & & \end{bmatrix}, \begin{bmatrix} & & & * \\ & & * & \\ & & & \\ * & & & \end{bmatrix}, \begin{bmatrix} & & & * \\ & & & \\ & * & & \\ * & & & \end{bmatrix}, \begin{bmatrix} & & & \\ & & * & \\ & * & & \\ * & & & \end{bmatrix} \quad (3.2)$$

$$\begin{bmatrix} & & & * \\ & & * & \\ & * & & \\ * & & & \end{bmatrix} \quad (3.3)$$

- Note: Symbol * is Duplicate Entries

- ***Definition 3.1*** [7] *Let A be an $m \times n$ matrix. Then the weak commutation matrix of A, denoted by P^*, is a permutation matrix in which $P^* vec(A) = vec(A^T)$, where P^* is an $mn \times mn$ matrix.*

- *Proposition 3.2*

a. For any matrix in (3.1), (3.2), and (3.3), the weak commutation matrix for A is
$$P(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$

b. For $a_{14} = a_{23}$, $a_{14} = a_{23} = a_{32}$, $a_{14} = a_{23} = a_{41}$, and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix of A is
$$P(2\ 5)(3\ 9)(4\ 13\ 7\ 10)(8\ 14)(12\ 15)$$

c. For $a_{14} = a_{32}$, $a_{14} = a_{23} = a_{32}$, $a_{14} = a_{32} = a_{41}$, and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A is
$$P(2\ 5)(3\ 9)(4\ 13\ 10\ 7)(8\ 14)(12\ 15)$$

d. For $a_{14} = a_{41}$, $a_{14} = a_{23} = a_{41}$, $a_{14} = a_{32} = a_{41}$, and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A is
$$P(2\ 5)(3\ 9)(7\ 10)(8\ 14)(12\ 15)$$

e. For $a_{23} = a_{32}$, $a_{14} = a_{23} = a_{32}$, $a_{23} = a_{32} = a_{41}$, and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A is
$$P(2\ 5)(3\ 9)(4\ 13)(8\ 14)(12\ 15)$$

f. For $a_{23} = a_{41}$, $a_{14} = a_{23} = a_{41}$, $a_{23} = a_{32} = a_{41}$, and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A is
$$P(2\ 5)(3\ 9)(4\ 7\ 10\ 13)(8\ 14)(12\ 15)$$

g. For $a_{32} = a_{41}$, $a_{14} = a_{32} = a_{41}$, $a_{23} = a_{32} = a_{41}$, and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A is
$$P(2\ 5)(3\ 9)(4\ 10\ 7\ 13)(8\ 14)(12\ 15)$$

h. For $a_{14} = a_{23} = a_{32}$ and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A are
$$P(2\ 5)(3\ 9)(4\ 13\ 7)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 13\ 10)(8\ 14)(12\ 15)$$

i. For $a_{14} = a_{23} = a_{41}$ and $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A are
$$P(2\ 5)(3\ 9)(7\ 10\ 13)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 7\ 10)(8\ 14)(12\ 15)$$

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j. For $a_{14} = a_{32} = a_{41}$ dan $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A are

$$P(2\ 5)(3\ 9)(7\ 13\ 10)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 10\ 7)(8\ 14)(12\ 15)$$

k. For $a_{23} = a_{32} = a_{41}$ dan $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A are

$$P(2\ 5)(3\ 9)(4\ 7\ 13)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 10\ 13)(8\ 14)(12\ 15)$$

l. For $a_{14} = a_{23} = a_{32} = a_{41}$, the weak commutation matrix for A are

$$P(2\ 5)(3\ 9)(4\ 10)(7\ 13)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 10)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 10\ 13\ 7)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 7\ 13\ 10)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 7)(10\ 13)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(4\ 7)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(7\ 13)(8\ 14)(12\ 15)$$
$$P(2\ 5)(3\ 9)(10\ 13)(8\ 14)(12\ 15)$$

➢ *Proof.*

For a. it is use Definition 2.8, we have

$$K_{4,4} = \begin{pmatrix} I_4 \otimes \boldsymbol{e}_{14}{}^T \\ I_4 \otimes \boldsymbol{e}_{24}{}^T \\ I_4 \otimes \boldsymbol{e}_{34}{}^T \\ I_4 \otimes \boldsymbol{e}_{44}{}^T \end{pmatrix} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} = P(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15) = P^{*}$$

Next, we have $P(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)vec(A) = vec(A^T)$, and proven a.

For b.– l., Since there is a duplicate entry, the position of the permutation in the matrix can move each other for the same entry and not change the results of $P^{*}vec(A) = vec(A^T)$. We directly proven that $P(1\ 6)(2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)\ vec(A) = vec(A^T)$ for A with condition b, and so on.

IV. CONCLUSION

The weak commutation matrix is a matrix that transforms $vec(A)$ to $vec(A^T)$ by paying attention to the same entries in different positions or in duplicate entries. This paper provides a way to determine the weak commutation matrix with certain conditions. In this case, we choose an arbitrary matrix, 4×4, by selecting the duplicate entries in different position in the secondary diagonal. It is found that 24 weak commutation matrices for the matrices.

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