A Density Matrix Description based Calculation of Arithmetic Mean of a Set of Replicate Measurements using an Iterative Method Involving Target Mean Estimates and Associated Convex Quadratic Programming Problem Formulations

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Abstract:- The research article presents a mathematical framework for Density matrix description based determination of Arithmetic Mean of a set of replicate measurements, using iteratively updated value of a Target mean estimate and formulation of associated Convex Quadratic optimization problems. The solution of the iteratively defined Convex Quadratic optimization problems results in a sequence of optimal estimates of the Arithmetic mean which is used to determine the Arithmetic mean associated with the set of replicate measurements, either as a single iteration optimal value that satisfied the imposed tolerance condition or as the average over the entire set of the generated optimal Arithmetic mean estimates.

Keywords:- *Density Matrix description associated with mathematical constructs, Completely Positive Trace Preserving transformations, Kraus operators, Basic form Arithmetic mean associated with a set of replicate measurements, General form Arithmetic mean associated with a set of replicate measurements, Convex Quadratic Programming Problem, Karush-Kuhn-Tucker Optimality conditions.*

I. INTRODUCTION

A Density matrix based approach to determination of Arithmetic mean and associated Standard deviation has been previously presented [12], in this approach, a density matrix description is constructed based on the data points constituting the replicate set of measurements and under the effect of appropriate CPTP transformations, it is transformed into a density matrix description whose diagonal elements provide accurate estimate to weightage associated with the individual data points and hence, provide an accurate estimate to arithmetic mean associated with the set of replicate measurements.

In most real scenarios however, the data generation mechanism associated with an experimental system is extremely complex and hence, its approximation in terms of density matrix descriptions and associated CPTP transformation schematic frameworks is not an easy task. In most situations, one has to resort to intuitively selecting a CPTP schematic or a candidate set of plausible CPTP schematic frameworks, and base their estimate of the Arithmetic mean on these pre-assumed considerations about the structure of the mathematical elements constituting the CPTP transformation schematic framework.

The present research initiative focuses on an iterative approach to estimate the arithmetic mean associated with a set of replicate measurements using an arbitrary chosen standard estimate of central tendency, which is termed as the Target mean estimate. The Target mean estimate acts as a reference value and it is used in formulating an associated minimization problem in terms of a Convex Quadratic Optimization formulation; the solution of the optimization problem is then used to construct an optimal estimate of the arithmetic mean associated with the replicate set of measurements.

The formulated updating rule for iterative re-specification of the target mean estimate allows for the Convex QPP formulations to be constructed and solved iteratively, thereby generating a sequence of optimal estimates for the arithmetic mean. The iterative scheme either terminates by satisfying the tolerance condition imposed or runs through the full set of allowed number of iterations, the optimal arithmetic mean estimates determine the final arithmetic mean estimate either as the value corresponding to a particular iteration where the termination condition is satisfied, or, as an average over the complete set of optimal estimated arithmetic mean values.

The research paper discusses determination of the arithmetic mean of a replicate set of measurements using two distinct mathematical frameworks: The General form based framework and the Basic form based framework, the Basic form based approach utilizes a pre-specified set of orthogonal matrices in constructing the CPTP schematic while the General form based approach has the flexibility in choosing arbitrarily the orthogonal matrices to constitute the associated CPTP schematic setup.

The numerical case studies section of the research paper presents an illustration of the numerical and computational procedures associated with the mathematical frameworks discussed in the present research study. The paper concludes with a section on discussion of some of the key aspects of the presented mathematical frameworks and insights into possible directions for related follow up research studies.

II. NOTATIONS

- N denotes the set of all Natural numbers
- R denotes the Real number field
- $M_{n\times n}(R)$ denotes the Real Matrix Space of order 'n'
- $R^n(R)$ denotes the Real co-ordinate space of dimension 'n'
- $|v\rangle \in R^n(R)$, $|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{bmatrix}_{n\times 1}$, $\langle v| = [v_1 \quad v_2 \quad . \quad . \quad v_n]_{1\times n}$, $v_1, v_2, ..., v_n \in R$
- $|a\rangle \in R^n(R)$, $|b\rangle \in R^n(R)$, $H_{n\times n} \in M_{n\times n}(R)$ where $H_{n\times n} = [h_{ij}]_{n\times n}$, therefore $\langle a|H_{n\times n}|b\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij} a_i b_j$
- $|e_1\rangle = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}_{n\times 1}$, $|e_2\rangle = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{bmatrix}_{n\times 1}$,..., $|e_n\rangle = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$, $|n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$, $|n_1\rangle = \begin{bmatrix} 1 \\ 2 \\ . \\ . \\ n \end{bmatrix}_{n\times 1}$, $|n_2\rangle = \begin{bmatrix} n \\ n-1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$
- $\langle n|n\rangle = n$, $\langle n_1|n_1\rangle = \langle n_2|n_2\rangle = \frac{1}{6}n(n+1)(2n+1) = n_0$, $|\hat{n}\rangle = \frac{1}{\sqrt{n}}|n\rangle$, $|\hat{n}_1\rangle = \frac{1}{\sqrt{n_0}}|n_1\rangle$ and $|\hat{n}_2\rangle = \frac{1}{\sqrt{n_0}}|n_2\rangle$
- $|\psi_1\rangle = |\hat{n}\rangle$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0_{(n-2)\times 1} \end{bmatrix}_{n\times 1}$, $|\psi_3\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0_{(n-3)\times 1} \end{bmatrix}_{n\times 1}$ .. up to $|\psi_n\rangle = \frac{1}{\sqrt{n(n-1)}}\begin{bmatrix} |n-1\rangle \\ -(n-1) \end{bmatrix}_{n\times 1}$
- $I_{n\times n}$ denotes the Identity matrix of order 'n'
- A^T denotes the Transpose of the matrix A
- $(X_{n\times n})^{-1}$ denote the Proper Inverse of the Invertible matrix $X_{n\times n}$, i.e. $(X_{n\times n})^{-1}X_{n\times n} = X_{n\times n}(X_{n\times n})^{-1} = I_{n\times n}$
- $|Y\rangle \in R^f(R)$, then $|Y\rangle \geq 0_{f\times 1}$ implies that the vector $|Y\rangle$ is an element wise non-negative vector
- $|y|$ denotes the modulus of the scalar y
- 'θ' denotes the Tuning parameter, where $\theta \in [0,1]$
- 'I_{tot}' denotes the maximum number of Iterations in the computational procedure associated with the Iterative evolution framework.
- $\rho_R(X, \hat{X}|\theta)_{n\times n}$ denotes the Reference form Density Matrix description associated with the set of replicate measurements
- $\rho_B(X, \hat{X}|\theta)_{n\times n}$ denotes the Basic form Density Matrix description associated with the set of replicate measurements
- $\rho_S(X, \hat{X}|\theta)_{n\times n}$ denotes the Standard form Density Matrix description associated with the set of replicate measurements
- $\rho_G(X, \hat{X}|\theta)_{n\times n}$ denotes the General form Density Matrix description associated with the set of replicate measurements
- $|w(B|\theta)\rangle$ denotes the Basic form Weightage vector associated with the set of replicate measurements
- $|w(G|\theta)\rangle$ denotes the General form Weightage vector associated with the set of replicate measurements
- $\overline{x}(B|\theta)$ denotes the Basic form Arithmetic mean associated with the set of replicate measurements
- $\overline{x}(G|\theta)$ denotes the General form Arithmetic mean associated with the set of replicate measurements
- " CQPP" is the abbreviation of Convex Quadratic Programming Problem

- A " Completely Positive Trace Preserving " transformation is abbreviated as a CPTP transformation
- $\left|w_{opt}(G|\theta|\hat{I})\right\rangle$ denotes the Optimal General form Weightage vector, for the iteration $\hat{I}$, associated with the set of replicate measurements
- $\bar{x}_{opt}(G|\theta|\hat{I})$ denotes the Optimal General form Arithmetic mean, for the iteration $\hat{I}$, associated with the set of replicate measurements
- γ_G denotes the Tolerance fraction under consideration which is associated with determination of termination criteria for the General form based Iterative framework, $\gamma_G \in (0,1]$
- T_G denotes the Tolerance magnitude associated with determination of termination criteria for the General form based Iterative framework
- $\left|w_{opt}(B|\theta|\hat{I})\right\rangle$ denotes the Optimal Basic form Weightage vector, for the iteration $\hat{I}$, associated with the set of replicate measurements
- $\bar{x}_{opt}(B|\theta|\hat{I})$ denotes the Optimal Basic form Arithmetic mean, for the iteration $\hat{I}$, associated with the set of replicate measurements
- γ_B denotes the Tolerance fraction under consideration which is associated with determination of termination criteria for the Basic form based Iterative framework, $\gamma_B \in (0,1]$
- T_B denotes the Tolerance magnitude associated with determination of termination criteria for the Basic form based Iterative framework

III. MATHEMATICAL FRAMEWORK

Mathematical formulation of the different categories of Density Matrix descriptions associated with a set of replicate measurements

We consider the replicate set of measurements $x_1, x_2 ...., x_n$ associated with a measurable quantity constituting an experimental system, we have $x_j \in R$ $\forall j = 1,2,....,n$, $x_1 \le x_2 \le .... \le x_n$ where $n \ge 2$ and $n \in N$

We define the data vectors $\left|X\right\rangle$ and $\left|\hat{X}\right\rangle$ in accordance to the mathematical formalism presented previously [12], as following:

$$\left|X\right\rangle = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} , \quad \left|\hat{X}\right\rangle = \begin{bmatrix} x_n \\ x_{n-1} \\ . \\ . \\ x_1 \end{bmatrix}_{n\times 1} \quad \text{therefore} \quad \left\langle X|X\right\rangle = \left\langle \hat{X}\middle|\hat{X}\right\rangle = \sum_{j=1}^{n} x_j^{\,2} = m_X$$

We define the following sets of orthogonal matrices:

- $U(1)_{n\times n} = I_{n\times n} - 2\left|\hat{n}\right\rangle\left\langle\hat{n}\right|$, $U(2)_{n\times n} = I_{n\times n} - 2\left|\hat{n}_1\right\rangle\left\langle\hat{n}_1\right|$ and $U(3)_{n\times n} = I_{n\times n} - 2\left|\hat{n}_2\right\rangle\left\langle\hat{n}_2\right|$, therefore $U(j)_{n\times n} = (U(j)_{n\times n})^T = (U(j)_{n\times n})^{-1}, \ \forall j = 1,2,3$
- $\Phi(j)_{n\times n} = I_{n\times n} - 2\left|\psi_j\right\rangle\left\langle\psi_j\right|$ where $j = 1,2,....,n$, therefore $\Phi(j)_{n\times n} = (\Phi(j)_{n\times n})^T = (\Phi(j)_{n\times n})^{-1}$ $\forall j = 1,2,....,n$
- $V(j)_{n\times n} \in M_{n\times n}(R)$ where $j = 1,2,....,n$, $(V(j)_{n\times n})^T = (V(j)_{n\times n})^{-1}$ $\forall j = 1,2,....,n$

We define the following symmetric, positive semi-definite or positive definite matrices:

- $\Omega(X,\hat{X})_{n\times n} = (\frac{1}{2})(\left|X\right\rangle\left\langle X\right| + \left|\hat{X}\right\rangle\left\langle\hat{X}\right|)$, therefore $trace(\Omega(X,\hat{X})_{n\times n}) = m_X$
- $q_j \ge 0 \ \ \forall j = 1,2,....,n$ and $\sum_{j=1}^{n} q_j = 1$, $\Omega_B(X,\hat{X})_{n\times n} = \sum_{j=1}^{n} q_j \Phi(j)_{n\times n} \Omega(X,\hat{X})_{n\times n} (\Phi(j)_{n\times n})^T$
- $trace(\Omega_B(X,\hat{X})_{n\times n}) = trace(\Omega(X,\hat{X})_{n\times n}) = m_X$

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- $\rho_R(X,\hat{X}\,|\,\theta)_{n\times n}=(\frac{1}{n+\theta m_X})(I_{n\times n}+\theta\Omega(X,\hat{X})_{n\times n})$, $\rho_R(X,\hat{X}\,|\,\theta)_{n\times n}$ is symmetric, positive definite $\forall\theta\in[0,1]$
- $\rho_B(X,\hat{X}\,|\,\theta)_{n\times n}=(\frac{1}{n+\theta m_X})(I_{n\times n}+\theta\Omega_B(X,\hat{X})_{n\times n})=\sum_{j=1}^{n}q_j\Phi(j)_{n\times n}\rho_R(X,\hat{X}\,|\,\theta)_{n\times n}(\Phi(j)_{n\times n})^T$, therefore $\rho_B(X,\hat{X}\,|\,\theta)_{n\times n}$ is symmetric, positive definite $\forall\theta\in[0,1]$
- $trace(\rho_B(X,\hat{X}\,|\,\theta)_{n\times n})=trace(\rho_R(X,\hat{X}\,|\,\theta)_{n\times n})=1$, $\forall\theta\in[0,1]$

Under the considerations of the mathematical framework presented previously [12], we have the following associated set of results:

- $\Omega_S(X,\hat{X})_{n\times n}=(\frac{1}{4})\Omega(X,\hat{X})_{n\times n}+(\frac{1}{4})\sum_{j=1}^{3}U(j)_{n\times n}\Omega(X,\hat{X})_{n\times n}(U(j)_{n\times n})^T$, therefore $trace(\Omega_S(X,\hat{X})_{n\times n})=m_X$
- $p_j\geq 0\quad\forall j=1,2,....,n$ and $\sum_{j=1}^{n}p_j=1$, $\Omega_G(X,\hat{X})_{n\times n}=\sum_{j=1}^{n}p_jV(j)_{n\times n}\Omega_S(X,\hat{X})_{n\times n}(V(j)_{n\times n})^T$, therefore $trace(\Omega_G(X,\hat{X})_{n\times n})=trace(\Omega_S(X,\hat{X})_{n\times n})=trace(\Omega(X,\hat{X})_{n\times n})=m_X$
- $\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}=(\frac{1}{n+\theta m_X})(I_{n\times n}+\theta\Omega_S(X,\hat{X})_{n\times n})$, $\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}$ is symmetric, positive definite $\forall\theta\in[0,1]$
- $\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}=(\frac{1}{n+\theta m_X})(I_{n\times n}+\theta\Omega_G(X,\hat{X})_{n\times n})=\sum_{j=1}^{n}p_jV(j)_{n\times n}\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}(V(j)_{n\times n})^T$, $\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}$ is symmetric, positive definite $\forall\theta\in[0,1]$
- $trace(\rho_S(X,\hat{X}\,|\,\theta)_{n\times n})=trace(\rho_G(X,\hat{X}\,|\,\theta)_{n\times n})=1$, $\forall\theta\in[0,1]$

➢ *Mathematical formulation of Basic form and General form Weightage vectors and associated set of analytical results*

$\rho_B(X,\hat{X}\,|\,\theta)_{n\times n}=\sum_{j=1}^{n}q_j\hat{\rho}_B(j\,|\,X,\hat{X}\,|\,\theta)_{n\times n}$, $\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}=\sum_{j=1}^{n}p_j\hat{\rho}_G(j\,|\,X,\hat{X}\,|\,\theta)_{n\times n}$ where
$\hat{\rho}_B(j\,|\,X,\hat{X}\,|\,\theta)_{n\times n}=\Phi(j)_{n\times n}\rho_R(X,\hat{X}\,|\,\theta)_{n\times n}(\Phi(j)_{n\times n})^T$, $j=1,2,....,n$ and we have
$\hat{\rho}_G(j\,|\,X,\hat{X}\,|\,\theta)_{n\times n}=V(j)_{n\times n}\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}(V(j)_{n\times n})^T$, $j=1,2,....,n$

$\pi_{rs}(B\,|\,j\,|\,X,\hat{X}\,|\,\theta)=\langle e_r\,|\,\hat{\rho}_B(j\,|\,X,\hat{X}\,|\,\theta)_{n\times n}\,|\,e_s\rangle$, $\pi_{rs}(G\,|\,j\,|\,X,\hat{X}\,|\,\theta)=\langle e_r\,|\,\hat{\rho}_G(j\,|\,X,\hat{X}\,|\,\theta)_{n\times n}\,|\,e_s\rangle$ Where $r=1,2,....,n$, $s=1,2,....,n$ for every $j\in\{1,2,....,n\}$

We have the following set of results:

- $\left|\pi(B\,|\,1\,|\,X,\hat{X}\,|\,\theta)\right\rangle=\begin{bmatrix}\pi_{11}(B\,|\,1\,|\,X,\hat{X}\,|\,\theta)\\ \pi_{22}(B\,|\,1\,|\,X,\hat{X}\,|\,\theta)\\ .\\ .\\ \pi_{nn}(B\,|\,1\,|\,X,\hat{X}\,|\,\theta)\end{bmatrix}_{n\times 1}$, $\left|\pi(B\,|\,2\,|\,X,\hat{X}\,|\,\theta)\right\rangle=\begin{bmatrix}\pi_{11}(B\,|\,2\,|\,X,\hat{X}\,|\,\theta)\\ \pi_{22}(B\,|\,2\,|\,X,\hat{X}\,|\,\theta)\\ .\\ .\\ \pi_{nn}(B\,|\,2\,|\,X,\hat{X}\,|\,\theta)\end{bmatrix}_{n\times 1}$,

$$...., \left|\pi(B|n|X,\hat{X}|\theta)\right\rangle = \begin{bmatrix} \pi_{11}(B|n|X,\hat{X}|\theta) \\ \pi_{22}(B|n|X,\hat{X}|\theta) \\ . \\ . \\ \pi_{nn}(B|n|X,\hat{X}|\theta) \end{bmatrix}_{n\times 1}, \left|\pi(G|1|X,\hat{X}|\theta)\right\rangle = \begin{bmatrix} \pi_{11}(G|1|X,\hat{X}|\theta) \\ \pi_{22}(G|1|X,\hat{X}|\theta) \\ . \\ . \\ \pi_{nn}(G|1|X,\hat{X}|\theta) \end{bmatrix}_{n\times 1},$$

$$\left|\pi(G|2|X,\hat{X}|\theta)\right\rangle = \begin{bmatrix} \pi_{11}(G|2|X,\hat{X}|\theta) \\ \pi_{22}(G|2|X,\hat{X}|\theta) \\ . \\ . \\ \pi_{nn}(G|2|X,\hat{X}|\theta) \end{bmatrix}_{n\times 1}, ..., \left|\pi(G|n|X,\hat{X}|\theta)\right\rangle = \begin{bmatrix} \pi_{11}(G|n|X,\hat{X}|\theta) \\ \pi_{22}(G|n|X,\hat{X}|\theta) \\ . \\ . \\ \pi_{nn}(G|n|X,\hat{X}|\theta) \end{bmatrix}_{n\times 1}$$

Therefore we have $\left\langle n \middle| \pi(B|j|X,\hat{X}|\theta)\right\rangle = \left\langle n \middle| \pi(G|j|X,\hat{X}|\theta)\right\rangle = 1$ for every $j \in \{1,2,....,n\}$

- We define the following matrices:

$$\Delta_B(X,\hat{X}|\theta)_{n\times n} = \left[\left|\pi(B|1|X,\hat{X}|\theta)\right\rangle \quad \left|\pi(B|2|X,\hat{X}|\theta)\right\rangle \quad . \quad \left|\pi(B|n|X,\hat{X}|\theta)\right\rangle\right],$$

$$\Delta_G(X,\hat{X}|\theta)_{n\times n} = \left[\left|\pi(G|1|X,\hat{X}|\theta)\right\rangle \quad \left|\pi(G|2|X,\hat{X}|\theta)\right\rangle \quad . \quad \left|\pi(G|n|X,\hat{X}|\theta)\right\rangle\right]$$

- We define $\left|P\right\rangle = \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ p_n \end{bmatrix}_{n\times 1}$, $\left|Q\right\rangle = \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ q_n \end{bmatrix}_{n\times 1}$ therefore $\left\langle n \middle| P\right\rangle = \left\langle n \middle| Q\right\rangle = 1$, $\left|P\right\rangle \geq 0_{n\times 1}$ and $\left|Q\right\rangle \geq 0_{n\times 1}$

- $$\left|\delta_B(X,\hat{X}|\theta)\right\rangle = \begin{bmatrix} \delta_{1,B}(X,\hat{X}|\theta) \\ \delta_{2,B}(X,\hat{X}|\theta) \\ . \\ . \\ \delta_{n,B}(X,\hat{X}|\theta) \end{bmatrix}_{n\times 1} = (\Delta_B(X,\hat{X}|\theta)_{n\times n})^T \left|X\right\rangle = \begin{bmatrix} \left\langle \pi(B|1|X,\hat{X}|\theta) \middle| X\right\rangle \\ \left\langle \pi(B|2|X,\hat{X}|\theta) \middle| X\right\rangle \\ . \\ . \\ \left\langle \pi(B|n|X,\hat{X}|\theta) \middle| X\right\rangle \end{bmatrix}_{n\times 1},$$

We define $H_B(X,\hat{X}|\theta)_{n\times n} = I_{n\times n} + \left|\delta_B(X,\hat{X}|\theta)\right\rangle\left\langle\delta_B(X,\hat{X}|\theta)\right|$

$$\left|\delta_G(X,\hat{X}|\theta)\right\rangle = \begin{bmatrix} \delta_{1,G}(X,\hat{X}|\theta) \\ \delta_{2,G}(X,\hat{X}|\theta) \\ . \\ . \\ \delta_{n,G}(X,\hat{X}|\theta) \end{bmatrix}_{n\times 1} = (\Delta_G(X,\hat{X}|\theta)_{n\times n})^T \left|X\right\rangle = \begin{bmatrix} \left\langle \pi(G|1|X,\hat{X}|\theta) \middle| X\right\rangle \\ \left\langle \pi(G|2|X,\hat{X}|\theta) \middle| X\right\rangle \\ . \\ . \\ \left\langle \pi(G|n|X,\hat{X}|\theta) \middle| X\right\rangle \end{bmatrix}_{n\times 1}$$

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We define $H_G(X,\hat{X}\,|\,\theta)_{n\times n} = I_{n\times n} + \left|\delta_G(X,\hat{X}\,|\,\theta)\right\rangle\left\langle\delta_G(X,\hat{X}\,|\,\theta)\right|$

- $\left|w(B\,|\,\theta)\right\rangle = \begin{bmatrix} w_1(B\,|\,\theta) \\ w_2(B\,|\,\theta) \\ . \\ . \\ w_n(B\,|\,\theta) \end{bmatrix}_{n\times 1} = \Delta_B(X,\hat{X}\,|\,\theta)_{n\times n}\left|Q\right\rangle$, therefore we have:

$$\bar{x}(B\,|\,\theta) = \left\langle X\,\middle|\,w(B\,|\,\theta)\right\rangle = \left\langle X\,\middle|\,\Delta_B(X,\hat{X}\,|\,\theta)_{n\times n}\,\middle|\,Q\right\rangle = \left\langle\delta_B(X,\hat{X}\,|\,\theta)\,\middle|\,Q\right\rangle$$

$\left|w(G\,|\,\theta)\right\rangle = \begin{bmatrix} w_1(G\,|\,\theta) \\ w_2(G\,|\,\theta) \\ . \\ . \\ w_n(G\,|\,\theta) \end{bmatrix}_{n\times 1} = \Delta_G(X,\hat{X}\,|\,\theta)_{n\times n}\left|P\right\rangle$, therefore we have:

$$\bar{x}(G\,|\,\theta) = \left\langle X\,\middle|\,w(G\,|\,\theta)\right\rangle = \left\langle X\,\middle|\,\Delta_G(X,\hat{X}\,|\,\theta)_{n\times n}\,\middle|\,P\right\rangle = \left\langle\delta_G(X,\hat{X}\,|\,\theta)\,\middle|\,P\right\rangle$$

➢ *The Mathematical formulation of the associated Iterative scheme*

$\hat{I}$ denotes the Iteration Index, we have $\hat{I} \in \{1,2,....,I_{tot}\}$, $I_{tot} \in N$ and $I_{tot} \geq 2$

- *The General form based framework*
- $\hat{\mu}_G(\hat{I})$ denotes the General form Target mean estimate for the iteration $\hat{I}$, we define the following associated variables:

$\left|C_G(X,\hat{X}\,|\,\theta\,|\,\hat{I})\right\rangle = -\hat{\mu}_G(\hat{I})\left|\delta_G(X,\hat{X}\,|\,\theta)\right\rangle$, $\varepsilon_G(\hat{I}) = (\frac{1}{2})(\hat{\mu}_G(\hat{I}))^2$

- The Convex Quadratic Programming Problem associated with the General form based framework is as following:

<u>(CQPP)</u> minimize $(\frac{1}{2})\left\langle P(\hat{I})\,\middle|\,H_G(X,\hat{X}\,|\,\theta)_{n\times n}\,\middle|\,P(\hat{I})\right\rangle + \left\langle C_G(X,\hat{X}\,|\,\theta\,|\,\hat{I})\,\middle|\,P(\hat{I})\right\rangle + \varepsilon_G(\hat{I})$

Subject to $\left\langle n\,\middle|\,P(\hat{I})\right\rangle = 1$, $\left|P(\hat{I})\right\rangle \geq 0_{n\times 1}$ where $\hat{I} \in \{1,2,....,I_{tot}\}$

The Karush-Kuhn-Tucker (abbreviated as KKT) conditions [13,15] associated with the CQPP are as following:

1. Stationarity condition:

$H_G(X,\hat{X}\,|\,\theta)_{n\times n}\left|P(\hat{I})\right\rangle + \left|C_G(X,\hat{X}\,|\,\theta\,|\,\hat{I})\right\rangle - \left|\chi(\hat{I})\right\rangle + \omega(\hat{I})\left|n\right\rangle = 0_{n\times 1}$, we have

- $\left|\chi(\hat{I})\right\rangle = \begin{bmatrix} \chi_1(\hat{I}) \\ \chi_2(\hat{I}) \\ . \\ . \\ \chi_n(\hat{I}) \end{bmatrix}_{n\times 1}$, $\chi_1(\hat{I}),.....,\chi_n(\hat{I})$ denotes the KKT multipliers associated with the Inequality constraints,

$\omega(\hat{I})$ denotes the KKT multiplier associated with the equality constraint.

- Primal feasibility condition: $\left|P(\hat{I})\right\rangle \geq 0_{n\times 1}$, $\left\langle n\,\middle|\,P(\hat{I})\right\rangle = 1$

- Dual feasibility condition: $\left|\chi(\hat{I})\right\rangle \geq 0_{n\times 1}$
- Complementary slackness condition: $\chi_j(\hat{I})p_j(\hat{I}) = 0$ for every $j = 1, 2, ..., n$
- $\left|P_{opt}(\hat{I})\right\rangle$ be an optimal solution of the CQPP, we therefore have: $\left|w_{opt}(G\,|\,\theta\,|\,\hat{I})\right\rangle = \Delta_G(X, \hat{X}\,|\,\theta)_{n\times n}\left|P_{opt}(\hat{I})\right\rangle$, $\bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I}) = \left\langle X\,\middle|\,w_{opt}(G\,|\,\theta\,|\,\hat{I})\right\rangle$
- The updating rule for the Target mean estimate:
- $\hat{\mu}_G(\hat{I}+1) = (\frac{1}{2})(\hat{\mu}_G(\hat{I}) + \bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I}))$ for $\hat{I} = 1, 2, ..., I_{tot} - 1$
- The termination criteria for the Iterative scheme:
- $u_G = \max(\delta_{1,G}(X, \hat{X}\,|\,\theta), ..., \delta_{n,G}(X, \hat{X}\,|\,\theta))$, $d_G = \min(\delta_{1,G}(X, \hat{X}\,|\,\theta), ..., \delta_{n,G}(X, \hat{X}\,|\,\theta))$
- $t_G = u_G - d_G$, therefore we have $t_G \geq 0$, we define: $T_G = \gamma_G t_G$
- If $\left|\bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I}) - \bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I}+1)\right| \leq T_G$ for some $\hat{I} \in \{1, 2, ..., I_{tot} - 1\}$, then the Iterative scheme is terminated at the conclusion of the Iteration $\hat{I}+1$, in this situation, the estimate of General form Arithmetic mean for the replicate set of measurements, denoted as $\bar{x}(G\,|\,\theta)$, is calculated as $\bar{x}(G\,|\,\theta) = \bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I})$
- If $\left|\bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I}) - \bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I}+1)\right| > T_G$ $\forall \hat{I} \in \{1, 2, ..., I_{tot} - 1\}$, then the Iterative scheme is terminated at the conclusion of the Iteration $\hat{I} = I_{tot}$, in this situation, the estimate of General form Arithmetic mean for the replicate set of measurements is calculated as follows: $\bar{x}(G\,|\,\theta) = (\frac{1}{I_{tot}})\sum_{i=1}^{I_{tot}}\bar{x}_{opt}(G\,|\,\theta\,|\,\hat{I} = i)$
- *The Basic form based framework*
- $\hat{\mu}_B(\hat{I})$ denotes the Basic form Target mean estimate for the iteration $\hat{I}$, we define the following associated variables:
- $\left|C_B(X, \hat{X}\,|\,\theta\,|\,\hat{I})\right\rangle = -\hat{\mu}_B(\hat{I})\left|\delta_B(X, \hat{X}\,|\,\theta)\right\rangle$, $\varepsilon_B(\hat{I}) = (\frac{1}{2})(\hat{\mu}_B(\hat{I}))^2$
- The Convex Quadratic Programming Problem formulation and associated set of results:

<u>(CQPP)</u> minimize $(\frac{1}{2})\left\langle Q(\hat{I})\middle|H_B(X, \hat{X}\,|\,\theta)_{n\times n}\middle|Q(\hat{I})\right\rangle + \left\langle C_B(X, \hat{X}\,|\,\theta\,|\,\hat{I})\middle|Q(\hat{I})\right\rangle + \varepsilon_B(\hat{I})$

Subject to $\left\langle n\middle|Q(\hat{I})\right\rangle = 1$, $\left|Q(\hat{I})\right\rangle \geq 0_{n\times 1}$ where $\hat{I} \in \{1, 2, ..., I_{tot}\}$

The Karush-Kuhn-Tucker conditions [13, 15] associated with the CQPP are as following:
$\sigma_1(\hat{I}), \sigma_2(\hat{I}), ..., \sigma_n(\hat{I})$ be the set of KKT multipliers associated with the Inequality constraints and $\nu(\hat{I})$ be the KKT multiplier associated with the equality constraint, we define the vector $\left|\Sigma(\hat{I})\right\rangle$ as follows: $\left|\Sigma(\hat{I})\right\rangle = \begin{bmatrix}\sigma_1(\hat{I})\\ \sigma_2(\hat{I})\\ .\\ .\\ \sigma_n(\hat{I})\end{bmatrix}_{n\times 1}$

- Stationarity condition: $H_B(X,\hat{X}\,|\,\theta)_{n\times n}\left|Q(\hat{I})\right\rangle+\left|C_B(X,\hat{X}\,|\,\theta\,|\,\hat{I})\right\rangle-\left|\Sigma(\hat{I})\right\rangle+v(\hat{I})\left|n\right\rangle=0_{n\times 1}$
- Primal feasibility condition: $\left\langle n\middle|Q(\hat{I})\right\rangle=1$, $\left|Q(\hat{I})\right\rangle\geq 0_{n\times 1}$
- Dual feasibility condition: $\left|\Sigma(\hat{I})\right\rangle\geq 0_{n\times 1}$
- Complementary slackness condition: $\sigma_j(\hat{I})q_j(\hat{I})=0$ for every $j=1,2,....,n$
- $\left|Q_{opt}(\hat{I})\right\rangle$ be an optimal solution of the CQPP, we therefore have: $\left|w_{opt}(B\,|\,\theta\,|\,\hat{I})\right\rangle=\Delta_B(X,\hat{X}\,|\,\theta)_{n\times n}\left|Q_{opt}(\hat{I})\right\rangle$, $\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I})=\left\langle X\middle|w_{opt}(B\,|\,\theta\,|\,\hat{I})\right\rangle$
- The updating rule for the Target mean estimate:
- $\hat{\mu}_B(\hat{I}+1)=(\frac{1}{2})(\hat{\mu}_B(\hat{I})+\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I}))$ Where $\hat{I}=1,2,....,I_{tot}-1$
- The termination criteria for the Iterative scheme:
- $u_B=\max(\delta_{1,B}(X,\hat{X}\,|\,\theta),....,\delta_{n,B}(X,\hat{X}\,|\,\theta))$, $d_B=\min(\delta_{1,B}(X,\hat{X}\,|\,\theta),....,\delta_{n,B}(X,\hat{X}\,|\,\theta))$
- $t_B=u_B-d_B$, therefore we have $t_B\geq 0$, $T_B=\gamma_B t_B$
- If $\left|\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I})-\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I}+1)\right|\leq T_B$ for some $\hat{I}\in\{1,2,....,I_{tot}-1\}$, then the estimate of General form Arithmetic mean for the replicate set of measurements, denoted as $\bar{x}(B\,|\,\theta)$, is given as: $\bar{x}(B\,|\,\theta)=\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I})$
- In this case, the Iterative scheme is terminated at the conclusion of the Iteration $\hat{I}+1$
- If $\left|\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I})-\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I}+1)\right|>T_B$ $\forall\hat{I}\in\{1,2,....,I_{tot}-1\}$, then we have: $\bar{x}(B\,|\,\theta)=(\frac{1}{I_{tot}})\sum_{i=1}^{I_{tot}}\bar{x}_{opt}(B\,|\,\theta\,|\,\hat{I}=i)$
- In this case, the Iterative scheme is terminated at the conclusion of iteration $\hat{I}=I_{tot}$

IV. NUMERICAL CASE STUDIES

- The numerical computations are performed using the Scilab 5.4.1 computational platform
- In the numerical studies discussed in this section, the sample size is n=6, the parameter I_{tot} is set as: $I_{tot}=5$ and the parameter ‘θ’ is set as: $\theta=\frac{1}{2}$
- In the numerical studies, the initial target mean estimate is taken as the median of the set of replicate measurements
- The Convex Quadratic Programming problem formulations are solved using the Linear Quadratic Programming built-in Solver: qp_solve
- $\hat{I}$ denotes the Iteration Index, therefore $\hat{I}\in\{1,2,....,I_{tot}\}$

The CPTP transformation scheme utilized in the discussion of the General form based framework is as follows:

$$\left|q_1\right\rangle=\frac{1}{\sqrt{n}}\left|n\right\rangle\quad,\quad\left|q_2\right\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0_{(n-2)\times 1}\end{bmatrix}_{n\times 1}\quad\text{up to}\quad\left|q_n\right\rangle=\frac{1}{\sqrt{n(n-1)}}\begin{bmatrix}\left|n-1\right\rangle\\-(n-1)\end{bmatrix}_{n\times 1}$$

$$Q_{n\times n}=\begin{bmatrix}\left|q_1\right\rangle & \left|q_2\right\rangle & . & . & \left|q_n\right\rangle\end{bmatrix}\quad,\quad(Q_{n\times n})^T=(Q_{n\times n})^{-1}\ \text{and we have:}$$

$$V(1)_{n\times n}=Q_{n\times n}\quad,\quad V(2)_{n\times n}=Q_{n\times n}Q_{n\times n}=(Q_{n\times n})^2\quad\text{up to}\quad V(n)_{n\times n}=(Q_{n\times n})^n$$

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We have $p_j \geq 0 \quad \forall j = 1, 2, ..., n \quad , \quad \sum_{j=1}^{n} p_j = 1$

Example 1

x_1	x_2	x_3	x_4	x_5	x_6
1.84	1.85	1.86	1.86	1.88	2.55

- *Numerical results associated with the General form based framework*

$$\Delta_G(X, \hat{X} \mid \theta = \tfrac{1}{2})_{6\times 6} = \begin{bmatrix} 0.451648 & 0.213367 & 0.347853 & 0.333997 & 0.259233 & 0.420942 \\ 0.109208 & 0.117015 & 0.190938 & 0.072351 & 0.175802 & 0.083111 \\ 0.062675 & 0.123946 & 0.140518 & 0.134172 & 0.208841 & 0.092207 \\ 0.062852 & 0.120087 & 0.077241 & 0.133190 & 0.179231 & 0.163309 \\ 0.108914 & 0.158871 & 0.064650 & 0.115929 & 0.080980 & 0.123878 \\ 0.204703 & 0.266713 & 0.178800 & 0.210361 & 0.095913 & 0.116553 \end{bmatrix}$$

$$\left| \delta_G(X, \hat{X} \mid \theta = \tfrac{1}{2}) \right\rangle = \begin{bmatrix} 1.993298 \\ 2.041772 \\ 1.975799 \\ 2.000064 \\ 1.920857 \\ 1.933649 \end{bmatrix}_{6\times 1}$$, pre-chosen value of γ_G : $\gamma_G = 0.5$

$$H_G(X, \hat{X} \mid \theta = \tfrac{1}{2})_{6\times 6} = \begin{bmatrix} 4.973238 & 4.069861 & 3.938356 & 3.986724 & 3.828841 & 3.85434 \\ 4.069861 & 5.168833 & 4.034131 & 4.083675 & 3.921952 & 3.948071 \\ 3.938356 & 4.034131 & 4.903781 & 3.951724 & 3.795227 & 3.820502 \\ 3.986724 & 4.083675 & 3.951724 & 5.000256 & 3.841837 & 3.867422 \\ 3.828841 & 3.921952 & 3.795227 & 3.841837 & 4.689692 & 3.714264 \\ 3.85434 & 3.948071 & 3.820502 & 3.867422 & 3.714264 & 4.738999 \end{bmatrix}$$

- $\left| c_G(X, \hat{X} \mid \theta = \tfrac{1}{2} \mid \hat{I} = 1) \right\rangle = (-1) \begin{bmatrix} 3.707535 \\ 3.797696 \\ 3.674986 \\ 3.720119 \\ 3.572794 \\ 3.596588 \end{bmatrix}_{6\times 1}$, $\left| c_G(X, \hat{X} \mid \theta = \tfrac{1}{2} \mid \hat{I} = 2) \right\rangle = (-1) \begin{bmatrix} 3.823551 \\ 3.916534 \\ 3.789984 \\ 3.836529 \\ 3.684594 \\ 3.709132 \end{bmatrix}_{6\times 1}$

- $\left|P_{opt}(\hat{I}=1)\right\rangle = \begin{bmatrix} 0.164836 \\ 0.159194 \\ 0.166873 \\ 0.164049 \\ 0.173269 \\ 0.171780 \end{bmatrix}_{6\times 1}$, $\left|P_{opt}(\hat{I}=2)\right\rangle = \begin{bmatrix} 0.165742 \\ 0.162893 \\ 0.166771 \\ 0.165345 \\ 0.170000 \\ 0.169249 \end{bmatrix}_{6\times 1}$
- $\hat{\mu}_G(\hat{I}=1)=1.86$, $\bar{x}(G\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)=1.976406$, $\hat{\mu}_G(\hat{I}=2)=1.918203$, $\bar{x}(G\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=2)=1.976984$
- $\left|\bar{x}(G\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)-\bar{x}(G\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=2)\right|=0.000578<T_G=0.060457$, therefore the Iterative procedure terminates at conclusion of $\hat{I}=2$
- $\bar{x}(G\,|\,\theta=\frac{1}{2})=\bar{x}(G\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)=1.98$ (up to 2 decimal places)

➢ *Numerical results associated with the Basic form based framework*

$$\Delta_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times 6}=\begin{bmatrix} 0.145239 & 0.153183 & 0.178587 & 0.186178 & 0.189461 & 0.203010 \\ 0.177092 & 0.194162 & 0.142044 & 0.147417 & 0.149889 & 0.163094 \\ 0.177669 & 0.152655 & 0.179370 & 0.146957 & 0.149407 & 0.162662 \\ 0.177669 & 0.152655 & 0.152655 & 0.172103 & 0.149407 & 0.162662 \\ 0.177092 & 0.153183 & 0.153183 & 0.153183 & 0.167673 & 0.163333 \\ 0.145239 & 0.194162 & 0.194162 & 0.194162 & 0.194162 & 0.145239 \end{bmatrix}$$

$$\left|\delta_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})\right\rangle=\begin{bmatrix} 1.959081 \\ 1.99203 \\ 1.992043 \\ 1.991838 \\ 1.992037 \\ 1.95779 \end{bmatrix}_{6\times 1}$$, pre-chosen value of γ_B : $\gamma_B=0.5$

$$H_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times 6}=\begin{bmatrix} 4.837999 & 3.902549 & 3.902574 & 3.902172 & 3.902562 & 3.83547 \\ 3.902549 & 4.968184 & 3.98621 & 3.967801 & 3.968198 & 3.899978 \\ 3.902574 & 3.98621 & 4.968236 & 3.967827 & 3.968224 & 3.900003 \\ 3.902172 & 3.967801 & 3.967827 & 4.967418 & 3.967815 & 3.899601 \\ 3.902562 & 3.968198 & 3.968224 & 3.967815 & 4.968212 & 3.899991 \\ 3.83547 & 3.899978 & 3.900003 & 3.899601 & 3.899991 & 4.832944 \end{bmatrix}$$

$\left|c_B(X,\hat{X}\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)\right\rangle=(-1)\begin{bmatrix} 3.643891 \\ 3.705176 \\ 3.705201 \\ 3.704818 \\ 3.705189 \\ 3.64149 \end{bmatrix}_{6\times 1}$, $\left|c_B(X,\hat{X}\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=2)\right\rangle=(-1)\begin{bmatrix} 3.762045 \\ 3.825318 \\ 3.825343 \\ 3.824948 \\ 3.825331 \\ 3.759567 \end{bmatrix}_{6\times 1}$

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$$\left|Q_{opt}(\hat{I}=1)\right\rangle = \begin{bmatrix} 0.169287 \\ 0.165312 \\ 0.165311 \\ 0.165336 \\ 0.165312 \\ 0.169443 \end{bmatrix}_{6\times1} , \quad \left|Q_{opt}(\hat{I}=2)\right\rangle = \begin{bmatrix} 0.167979 \\ 0.165989 \\ 0.165988 \\ 0.166000 \\ 0.165988 \\ 0.168057 \end{bmatrix}_{6\times1}$$

- $\hat{\mu}_B(\hat{I}=1)=1.86$, $\bar{x}(B\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)=1.980622$, $\hat{\mu}_B(\hat{I}=2)=1.920311$, $\bar{x}(B\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=2)=1.980713$
- $\left|\bar{x}(B\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)-\bar{x}(B\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=2)\right|=0.000090<T_B=0.017126$, therefore the Iterative procedure terminates at conclusion of $\hat{I}=2$
- $\bar{x}(B\,|\,\theta=\frac{1}{2})=\bar{x}(B\,|\,\theta=\frac{1}{2}\,|\,\hat{I}=1)=1.98$ (up to 2 decimal places)

➢ *Example 2*

x_1	x_2	x_3	x_4	x_5	x_6
12.24	12.24	12.26	12.28	12.29	13.98

<u>Numerical results associated with the General form based framework</u>

$$\Delta_G(X,\hat{X}\,|\,\theta=\frac{1}{2})_{6\times6} = \begin{bmatrix} 0.593422 & 0.2446 & 0.441186 & 0.425139 & 0.311164 & 0.549446 \\ 0.084087 & 0.094120 & 0.201759 & 0.025281 & 0.175908 & 0.044875 \\ 0.012480 & 0.103178 & 0.130669 & 0.118865 & 0.226601 & 0.052323 \\ 0.010556 & 0.093359 & 0.035150 & 0.116933 & 0.188911 & 0.161335 \\ 0.077839 & 0.146601 & 0.010215 & 0.082927 & 0.039738 & 0.100002 \\ 0.221616 & 0.318142 & 0.181021 & 0.230855 & 0.057678 & 0.092020 \end{bmatrix}$$

$$\left|\delta_G(X,\hat{X}\,|\,\theta=\frac{1}{2})\right\rangle = \begin{bmatrix} 12.63018 \\ 12.80669 \\ 12.55951 \\ 12.65289 \\ 12.35444 \\ 12.41261 \end{bmatrix}_{6\times1}$$

, pre-chosen value of γ_G : $\gamma_G=0.1$

$$H_G(X,\hat{X}\,|\,\theta=\frac{1}{2})_{6\times6} = \begin{bmatrix} 160.5213 & 161.7508 & 158.6288 & 159.8082 & 156.0387 & 156.7735 \\ 161.7508 & 165.0114 & 160.8458 & 162.0417 & 158.2195 & 158.9646 \\ 158.6288 & 160.8458 & 158.7412 & 158.914 & 155.1656 & 155.8963 \\ 159.8082 & 162.0417 & 158.914 & 161.0956 & 156.3193 & 157.0554 \\ 156.0387 & 158.2195 & 155.1656 & 156.3193 & 153.6321 & 153.3508 \\ 156.7735 & 158.9646 & 155.8963 & 157.0554 & 153.3508 & 155.073 \end{bmatrix}$$

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$$\left|c_G(X,\hat{X}\mid\theta=\tfrac{1}{2}\mid\hat{I}=1)\right\rangle=(-1)\begin{bmatrix}154.9723\\157.1381\\154.1051\\155.2509\\151.5889\\152.3028\end{bmatrix}_{6\times1},\quad \left|c_G(X,\hat{X}\mid\theta=\tfrac{1}{2}\mid\hat{I}=2)\right\rangle=(-1)\begin{bmatrix}156.6338\\158.8229\\155.7574\\156.9155\\153.2142\\153.9357\end{bmatrix}_{6\times1}$$

$$\left|P_{opt}(\hat{I}=1)\right\rangle=\begin{bmatrix}0.150672\\0.104228\\0.169266\\0.144696\\0.223223\\0.207915\end{bmatrix}_{6\times1},\quad \left|P_{opt}(\hat{I}=2)\right\rangle=\begin{bmatrix}0.157700\\0.131665\\0.168124\\0.154350\\0.198371\\0.189790\end{bmatrix}_{6\times1}$$

- $\hat{\mu}_G(\hat{I}=1)=12.27$, $\overline{x}(G\mid\theta=\tfrac{1}{2}\mid\hat{I}=1)=12.53311$, $\hat{\mu}_G(\hat{I}=2)=12.40156$, $\overline{x}(G\mid\theta=\tfrac{1}{2}\mid\hat{I}=2)=12.54905$
- $\left|\overline{x}(G\mid\theta=\tfrac{1}{2}\mid\hat{I}=1)-\overline{x}(G\mid\theta=\tfrac{1}{2}\mid\hat{I}=2)\right|=0.015939<T_G=0.045226$, therefore the Iterative procedure terminates at conclusion of $\hat{I}=2$
- $\overline{x}(G\mid\theta=\tfrac{1}{2})=\overline{x}(G\mid\theta=\tfrac{1}{2}\mid\hat{I}=1)=12.53$ (up to 2 decimal places)

➢ *Numerical results associated with the Basic form based framework*

$$\Delta_B(X,\hat{X}\mid\theta=\tfrac{1}{2})_{6\times6}=\begin{bmatrix}0.152662 & 0.158910 & 0.174084 & 0.178011 & 0.179552 & 0.187741\\0.173736 & 0.182052 & 0.151947 & 0.155389 & 0.156737 & 0.164816\\0.173602 & 0.159038 & 0.173969 & 0.155520 & 0.156866 & 0.164957\\0.173602 & 0.159038 & 0.159038 & 0.170118 & 0.156870 & 0.164971\\0.173736 & 0.158910 & 0.158910 & 0.158910 & 0.167923 & 0.164853\\0.152662 & 0.182052 & 0.182052 & 0.182052 & 0.182052 & 0.152662\end{bmatrix}$$

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$$\left|\delta_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})\right\rangle=\begin{bmatrix}12.52473\\12.57426\\12.57456\\12.57463\\12.57458\\12.52377\end{bmatrix}_{6\times1}$$, pre-chosen value of γ_B : $\gamma_B=0.1$

$$H_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times6}=\begin{bmatrix}157.869 & 157.4892 & 157.493 & 157.4939 & 157.4933 & 156.8569\\157.4892 & 159.112 & 158.1157 & 158.1167 & 158.116 & 157.4771\\157.493 & 158.1157 & 159.1195 & 158.1204 & 158.1198 & 157.4809\\157.4939 & 158.1167 & 158.1204 & 159.1213 & 158.1207 & 157.4818\\157.4933 & 158.116 & 158.1198 & 158.1207 & 159.12 & 157.4812\\156.8569 & 157.4771 & 157.4809 & 157.4818 & 157.4812 & 157.8449\end{bmatrix}$$

$$\left|c_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)\right\rangle=(-1)\begin{bmatrix}153.6785\\154.2861\\154.2898\\154.2907\\154.2901\\153.6667\end{bmatrix}_{6\times1}, \quad \left|c_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)\right\rangle=(-1)\begin{bmatrix}155.4745\\156.0892\\156.0929\\156.0939\\156.0932\\155.4625\end{bmatrix}_{6\times1}$$

$$\left|Q_{opt}(\hat{I}=1)\right\rangle=\begin{bmatrix}0.176137\\0.161934\\0.161848\\0.161827\\0.161842\\0.176413\end{bmatrix}_{6\times1}, \quad \left|Q_{opt}(\hat{I}=2)\right\rangle=\begin{bmatrix}0.171418\\0.164292\\0.164249\\0.164239\\0.164246\\0.171556\end{bmatrix}_{6\times1}$$

- $\hat{\mu}_B(\hat{I}=1)=12.27$, $\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)=12.55679$, $\hat{\mu}_B(\hat{I}=2)=12.41339$, $\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)=12.55727$
- $\left|\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)-\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)\right|=0.000481<T_B=0.005086$, therefore the Iterative procedure terminates at conclusion of $\hat{I}=2$
- $\bar{x}(B\,|\,\theta=\tfrac{1}{2})=\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)=12.56$ (up to 2 decimal places)

➢ *Example 3*

x_1	x_2	x_3	x_4	x_5	x_6
115	115	116	117	117	126

➢ *Numerical results associated with the General form based framework*

$$\Delta_G(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times6}=\begin{bmatrix}0.598602 & 0.248270 & 0.445627 & 0.430541 & 0.315081 & 0.554433\\ 0.083712 & 0.093554 & 0.201216 & 0.023475 & 0.174249 & 0.044345\\ 0.010882 & 0.102176 & 0.130859 & 0.118322 & 0.226014 & 0.050065\\ 0.008282 & 0.091017 & 0.034141 & 0.116216 & 0.190094 & 0.161164\\ 0.075993 & 0.143886 & 0.007371 & 0.079818 & 0.038787 & 0.098746\\ 0.222530 & 0.321097 & 0.180785 & 0.231628 & 0.055774 & 0.091247\end{bmatrix}$$

$$\left|\delta_G(X,\hat{X}\,|\,\theta=\tfrac{1}{2})\right\rangle=\begin{bmatrix}117.6273\\ 119.104\\ 117.2025\\ 118.0583\\ 116.2973\\ 116.5736\end{bmatrix}_{6\times1}$$, pre-chosen value of γ_G : $\gamma_G=0.05$

$$H_G(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times6}=\begin{bmatrix}13837.17 & 14009.88 & 13786.21 & 13886.87 & 13679.73 & 13712.23\\ 14009.88 & 14186.77 & 13959.29 & 14061.22 & 13851.48 & 13884.39\\ 13786.21 & 13959.29 & 13737.43 & 13836.73 & 13630.34 & 13662.72\\ 13886.87 & 14061.22 & 13836.73 & 13938.76 & 13729.86 & 13762.48\\ 13679.73 & 13851.48 & 13630.34 & 13729.86 & 13526.06 & 13557.19\\ 13712.23 & 13884.39 & 13662.72 & 13762.48 & 13557.19 & 13590.4\end{bmatrix}$$

$$\left|c_G(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)\right\rangle=(-1)\begin{bmatrix}13703.58\\ 13875.62\\ 13654.09\\ 13753.79\\ 13548.63\\ 13580.82\end{bmatrix}_{6\times1},\quad \left|c_G(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)\right\rangle=(-1)\begin{bmatrix}13715.88\\ 13888.08\\ 13666.36\\ 13766.14\\ 13560.8\\ 13593.02\end{bmatrix}_{6\times1}$$

$$\left|P_{opt}(\hat{I}=1)\right\rangle=\begin{bmatrix}0.100508\\ 0\\ 0.189385\\ 0.010312\\ 0.378807\\ 0.320988\end{bmatrix}_{6\times1},\quad \left|P_{opt}(\hat{I}=2)\right\rangle=\begin{bmatrix}0.116479\\ 0\\ 0.191089\\ 0.040761\\ 0.350105\\ 0.301567\end{bmatrix}_{6\times1}$$

- $\hat{\mu}_G(\hat{I}=1)=116.5$, $\bar{x}(G\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)=116.7093$, $\hat{\mu}_G(\hat{I}=2)=116.6046$, $\bar{x}(G\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)=116.7803$
- $\left|\bar{x}(G\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)-\bar{x}(G\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)\right|=0.071037<T_G=0.140338$, therefore the Iterative procedure terminates at conclusion of $\hat{I}=2$

- $\overline{x}(G\,|\,\theta=\tfrac{1}{2})=\overline{x}(G\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)=117$ (up to nearest integer)

  - *Numerical results associated with the Basic form based framework*

$$\Delta_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times 6}=\begin{bmatrix}0.158935 & 0.161821 & 0.171403 & 0.173119 & 0.173253 & 0.178035\\ 0.171253 & 0.174969 & 0.158580 & 0.160188 & 0.160279 & 0.165003\\ 0.169812 & 0.16321 & 0.170018 & 0.161587 & 0.161673 & 0.166432\\ 0.169812 & 0.16321 & 0.16321 & 0.168315 & 0.161699 & 0.166485\\ 0.171253 & 0.161821 & 0.161821 & 0.161821 & 0.168127 & 0.165109\\ 0.158935 & 0.174969 & 0.174969 & 0.174969 & 0.174969 & 0.158935\end{bmatrix}$$

$$\left|\delta_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})\right\rangle=\begin{bmatrix}117.6002\\117.7379\\117.7447\\117.7465\\117.746\\117.5779\end{bmatrix}_{6\times 1}$$, pre-chosen value of γ_B : $\gamma_B=0.1$

$$H_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2})_{6\times 6}=\begin{bmatrix}13830.31 & 13846.01 & 13846.81 & 13847.02 & 13846.95 & 13827.19\\ 13846.01 & 13863.22 & 13863.02 & 13863.23 & 13863.17 & 13843.38\\ 13846.81 & 13863.02 & 13864.82 & 13864.03 & 13863.97 & 13844.18\\ 13847.02 & 13863.23 & 13864.03 & 13865.24 & 13864.18 & 13844.39\\ 13846.95 & 13863.17 & 13863.97 & 13864.18 & 13865.12 & 13844.33\\ 13827.19 & 13843.38 & 13844.18 & 13844.39 & 13844.33 & 13825.56\end{bmatrix}$$

$$\left|c_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)\right\rangle=(-1)\begin{bmatrix}13700.43\\13716.47\\13717.26\\13717.47\\13717.41\\13697.83\end{bmatrix}_{6\times 1},\quad \left|c_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)\right\rangle=(-1)\begin{bmatrix}13768.34\\13784.46\\13785.26\\13785.47\\13785.41\\13765.73\end{bmatrix}_{6\times 1},$$

$$\left|c_B(X,\hat{X}\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=3)\right\rangle=(-1)\begin{bmatrix}13803.36\\13819.52\\13820.32\\13820.53\\13820.47\\13800.74\end{bmatrix}_{6\times 1}$$

$$\left|Q_{opt}(\hat{I}=1)\right\rangle=\begin{bmatrix}0.272914\\0.113868\\0.106005\\0.103949\\0.104567\\0.298696\end{bmatrix}_{6\times1},\quad \left|Q_{opt}(\hat{I}=2)\right\rangle=\begin{bmatrix}0.221448\\0.139444\\0.135389\\0.134329\\0.134648\\0.234742\end{bmatrix}_{6\times1},\quad \left|Q_{opt}(\hat{I}=3)\right\rangle=\begin{bmatrix}0.194912\\0.152630\\0.150540\\0.149993\\0.150158\\0.201766\end{bmatrix}_{6\times1}$$

- $\hat{\mu}_B(\hat{I}=1)=116.5$, $\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)=117.655$, $\hat{\mu}_B(\hat{I}=2)=117.0775$, $\hat{\mu}_B(\hat{I}=3)=117.3753$, $\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=3)=117.6823$
- $\left|\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=1)-\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)\right|=0.018024>T_B=0.016861$,
- $\left|\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)-\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=3)\right|=0.009293<T_B=0.016861$, therefore the Iterative procedure terminates at conclusion of $\hat{I}=3$
- $\bar{x}(B\,|\,\theta=\tfrac{1}{2})=\bar{x}(B\,|\,\theta=\tfrac{1}{2}\,|\,\hat{I}=2)=118$ (up to nearest integer)

V. DISCUSSION AND CONCLUSION

The Basic form based and the General form based mathematical framework involves generating the substitute data vectors $\left|\delta_B(X,\hat{X}\,|\,\theta)\right\rangle$ and $\left|\delta_G(X,\hat{X}\,|\,\theta)\right\rangle$ respectively, from the data vector $\left|X\right\rangle$. This is achieved under the effect of formulated CPTP transformation schematic frameworks on Density Matrix descriptions generated from the data vector $\left|X\right\rangle$: The Reference form Density matrix description $\rho_R(X,\hat{X}\,|\,\theta)_{n\times n}$ and the Standard form Density matrix description $\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}$, respectively. In the case of CPTP schematic associated with the Basic form based approach the set of orthogonal matrices are pre-specified and fixed while the CPTP schematic associated with the General form based approach involves an arbitrary set of orthogonal matrices, therefore, using an appropriate set of orthogonal matrices a CPTP schematic framework can be constructed which produces data refinement through the formulated substitute vector $\left|\delta_G(X,\hat{X}\,|\,\theta)\right\rangle$. It can be observed that each component of the substitute Data vectors is a convex combination of the data points $x_1,x_2....,x_n$.

The substitute data vectors $\left|\delta_G(X,\hat{X}\,|\,\theta)\right\rangle$ and $\left|\delta_B(X,\hat{X}\,|\,\theta)\right\rangle$ define the symmetric Positive definite matrices $H_G(X,\hat{X}\,|\,\theta)_{n\times n}$ and $H_B(X,\hat{X}\,|\,\theta)_{n\times n}$ respectively, together with the Target mean estimates $\hat{\mu}_G$ and $\hat{\mu}_B$, Convex Programming Problem formulations are constructed for both the General form and the Basic form based framework whose respective solutions leads to determination of the optimal estimate of the General form Arithmetic mean $\bar{x}(G\,|\,\theta)$ and the Basic form Arithmetic mean $\bar{x}(B\,|\,\theta)$.

The mathematical framework discussed in the paper defines a rule for iterative updating of the target mean estimates; this allows for the CQPP formulations to be constructed and solved for, in an iterative sense and thus leads to generating of optimal solutions corresponding to each such iteration. The pre-specified tolerance levels allows either for the Iterative scheme to be terminated in a finite number of iterations or run through the maximum specified upper limit for the number of iterations , which is denoted as I_{tot}. In the former case, the estimate of the General form or the Basic form Arithmetic mean is the optimal estimate associated with the particular iteration at which the tolerance condition is satisfied; the latter case involves taking the average over the optimal estimates corresponding to the individual iterations, as the estimate of the respective Arithmetic means.

The numerical case studies discussed in the paper attempts to provide a numerical illustration of the mathematical framework presented; In these studies the sample size 'n' is pre-chosen to be equal to 6 and $I_{tot}=5$, the numerical case studies discussed are found to satisfy the termination condition within 2 to 3 iterations, the initial target mean estimate is taken as the median of the replicate set of measurements for both the General and Basic form based frameworks, it is observed that the Arithmetic mean estimate associated with the General and Basic form based framework are in good agreement with each other, for each of the numerical examples comprising the numerical case study setup.

In conclusion, it is emphasized that the mathematical framework discussed in the paper attempts to provide an estimate of General form and the Basic form Arithmetic means of a set of replicate measurements generated from an experimental system, under the scenario where there is lack of information/lack of proper understanding about the data generation mechanism underlying the experimental system: the presented approach is based on empirical selection of the CPTP transformation schematic frameworks, the target mean estimate value being taken as a reference value and a minimization problem, formulated as a CQPP, provides an optimal estimate of the Arithmetic mean based on the selected target mean estimate. A better approximation of the CPTP transformation schematic, which better models the data generation behavior of the experimental system, is expected to provide a more accurate estimate of the results. In follow up studies, focus will be directed at better understanding and at obtaining insights into the intricacies involving data generation by experimental systems and their behavior being modeled through appropriate CPTP transformations and the mathematical framework discussed in the present research endeavor and previously undertaken, related research studies.

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