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Multiple Split Equilibrium Fixed Point Problem Involving Finite Families of $\eta-$demimetric and Quasi-nonexpansive Mappings

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Abstract:- Let H_i : (i = 1, ..., k) be Hilbert spaces, S_i : $H_i \to H_i$ (i = 1, ..., k) be η_i- demimetric; We obtained a common solution to a multiple split equilibrium problem, a finite family of simultaneous equilibrium problem, and a common fixed Point of a finite family of nonlinear mappings. We developed an algorithm and established sufficient condition for it's strong convergence to such a common solution. MSC(2010):- 47H10, 47J25.

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I. INTRODUCTION

Let E be a normed linear space, $K \subset E$. A self-mapping T on K is said to be Lipschitzian if $\exists$ L ≥ 0 such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in K. \quad (1)$$

If $L = 1$ then T is called non-expansive, and if $L < 1$ then the mapping T is called a contraction.

A point $x_0 \in K$ is called a fixed point of a mapping $T : K \to K$ if $Tx_0 = x_0$. We denote the set of fixed points of T by $Fix(T)$, that is, $Fix(T) = \{x \in K | Tx = x\}$. If in (1) $y \in Fix(T)$, and $L = 1$ then T is called *quasi-nonexpansive*.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *total asymptotically non-expansive* if and only if there exist two sequences $\{\mu_n\}_{n\geq 1}$, $\{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and non decreasing continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ such that for all $x, y \in D(T)$,

$$||T^n x - T^n y|| \leq ||x - y|| + \mu_n \varphi(||x - y||) + \eta_n \;\; n \geq 1. \quad (2)$$

Clearly, total asymptotically nonexpansive mapping is a generalization of nonexpansive maps.

Let K be a nonempty, closed and convex subset of a smooth Banach space E and let η and p be real numbers such that $\eta \in (-\infty, 0)$ and $1 < p < +\infty$.

A map $T : K \to E$ with $Fix(T)$ not equal 0 is called $\eta - demietric$ if, for any $x \in K$ and $x^* \in Fix(T)$, wehave,

$$\langle x - x^*, J_E^p(x - Tx)\rangle \geq \frac{1-\eta}{2} ||x - Tx||^p \quad (3)$$

And thus in a Hilbert space we have,

$$\langle x - x^*, x - Tx\rangle \geq \frac{1-\eta}{2} ||x - Tx||^p \quad (4)$$

Let $T : K \to K$ be a mapping and I be the identity mapping of K, we say that $(I - T)$ is demiclosed at zero if for any sequence $\{x_n\}_{n\geq 1}$ in K such that $\{x_n\}_{n\geq 1}$

Converges weakly to x and $x_n - Tx_n \to 0$, as $n \to \infty$, we have that $x = Tx$.

Let D_1 and D_2 be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point x satisfying

$$x \in D_1 \text{ such that } Ax \in D_2, \quad (5)$$

Where A is bounded linear operator from H_1 to H_2. A split feasibility problem in finite dimensional Hilbert spaces was first studied by Censor and Elfving [7] for modeling inverse problems which arise in medical image reconstruction, image restoration and radiation therapy treatment planning (see e.g [5], [6], [7]), It is clear that $x \in D_1$ is a solution of the split feasibility problem (5) if and only if $Ax - P_{D_2}Ax = 0$, where P_{D_2} is the metric projection from H_2 onto D_2.

Let K be a closed convex nonempty subset of a real Hilbert space H. Let $f : K \times K \to$ R be a bifunction. The classical equilibrium problem (abbreviated EP) for f is to find $u^* \in K$ such that

$$f(u^*, y) \geq 0 \quad \forall y \in K \tag{6}$$

The set of solutions of the classical equilibrium problem is denoted by $EP(f)$, where $EP(f) = \{u \in K : f(u,y) \geq 0 \ \forall y \in K\}$. The classical equilibrium problem (EP) includes, as special cases, the monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. Furthermore, there are several other problems, for example, the complementarity problems and fixed point problems, which can also be written in the form of the classical equilibrium problem. In other words, the classical equilibrium problem is a unifying model for several problems arising from engineering, physics, statistics, computer science, optimization theory, operations research, economics and countless other fields. For the past 20 years or so, many existence results have been established for various equilibrium problems (see e.g.Blum and Oettli(1994), Flam and Antipin (1997), Mouda (2003), Chang et al (2010), Zegeye et al (2010), Ofoedu and Malonza (2011) andthe references therein).

A bifunction $f : K \times K \to R$ is said to satisfy Condition C, if it satisfies the following conditions:

(C1) $f(x,x) = 0 \ \forall x \in K$;

(C2) f is monotone, in the sense that $f(x,y) + f(y,x) \leq 0 \ \forall x, y \in K$;

(C3) $\lim_{t \to 0^+} \sup f(tz + (1-t)x, y) \leq f(x,y) \forall x, y, z \in K$

(C4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous for all $x \in K$

Let $\Phi : K \to$ R be a proper extended real valued function, where R denotes the real numbers and let $\Theta : K \to H$ be a nonlinear monotone mapping. The gen- eralised mixed equilibrium problem (abbreviated GMEP) for f, Φ and Θ is to find $u^* \in K$ such that

$$f(u^*, y) + \Phi(y) - \Phi(u^*) + (\Theta u^*, y - u^*) \geq 0 \ \forall y \in K \tag{7}$$

Observe that if we define $\Gamma : K \times K \to$ R by

$$\Gamma(x,y) = f(x,y) + \Phi(y) - \Phi(x) + (\Theta x, y - x) \tag{8}$$

Then it could be easily checked that Γ is a bi-function and satisfies properties (C1) to (C4). Thus, the so called generalized mixed equilibrium problem reduces to the classical equilibrium problem for the bifunction Γ.

The Split Equilibrium Fixed Point Problem (SEFPP) for T and S is to find

$x^* \in Fix(T), y^* \in Fix(S)$ such that $Ax^* = By^*$,

Where T and S are nonlinear self maps defined on linear spaces, E_1 and E_2 respec-tively. A and B are bounded linear operators defined respectively from E_1 and E_2 to another linear space E_3. A lot of research works have focused on the Split Equilib- rium Fixed Point Problem (SEFPP) in recent time, Mouda (2014), proposed an al- gorithm in Hilbert spaces, involving quasi-nonexpansive Mappings and provedweak convergence of his Scheme to a solution of (SEFPP). Zhao (2015) solved a SEFPP of quasi-nonexpansive mappings without prior knowledge of operator norms. Moti- vated by the result of Zhao (2015), Shehu et al (2017), proposed a scheme which does not require prior knowledge of the operator norm for quasi-nonexpansive mappings in real Hilbert spaces. Wang and Kim (2017) gave a modified Mann iteration and proved a strong convergence result in Hilbert for demicontractive mappings. Ofoedu and Araka (2019), proposed an iterative Scheme for simultaneous approximation of common solution of equilibrium, fixed point and split equal ity problem involving some $\eta-$demimetric and finite family of quasi-noexpansive mappings.

Let D_i be a nonempty closed convex subset of a linear space X_i (i = 1,...,k) and let D be a nonempty closed convex subset of another linear space X. Let $A_i: X_i \to X$ be a bounded linear operator $\forall\, i = 1,\ldots,k$. The Multiple Split Feasibility Problem (MSFP) is to find $x_i \in D_i$ such that $A_i x_i \in D\ \forall\, i$, that is, find $(x_1,\ldots,x_k) \in \prod_{i=1}^{k} D_i \subset \prod_{i=1}^{k} X_i$ such that $A_i X_i \in D\ \forall\, i = 1,\ldots,k;\ k > 1$. The Multiple Split Equality Problem (MSEP) consists in finding $(x_1,\ldots,x_k) \in \prod_{i=1}^{k} D_i \subset \prod_{i=1}^{k} X_i$ such that $A_i x_i = A_j x_j;\ i,j = 1,\ldots,k, ij$. It is easy to observe that if $m = 1$, then the Multiple Split Feasibility Problem (MSFP) reduces to Split Feasibility Problem (SFP) and if $m = 2$, Multiple Split Equality Problem (MSEP) reduces to Split Equilibrium Problem (SEP).

In this paper, we focus on obtaining a common solution to a Multiple Split Equality problem, a finite family of simultaneous equilibrium problems, and a common fixed Point of a finite collection of a finite families of nonlinear mappings. We develop an algorithm and establish sufficient condition for its strong convergence to such a common solution.

We shall make use of the following lemmas in the sequel.

Lemma 1.1 [26] Let E_1, E_2 be uniformly smooth, uniformly convex real Banach spaces. Let $E = E_1 \times E_2$ and $1 < p < \infty$. For arbitrary $x = (x_1, x_2) \in E$, define the mapping $J_E^p: E \to E^*$ by $J_E^p x := (J_{E_1}^p x_1, J_{E_2}^p x_2)$ so that for arbitrary $w_1 = (u_1,u_2), w_2 = (v_1,v_2) \in E$. The generalized duality pairing $(.,.)$ is given by

$$(w_1, J_E^p(w_2)) = (u_1, J_E^p(u_2)) + (v_1, J_E^p(v_2))$$

Then J^p is single valued generalized duality mapping on E.

Lemma 1.2 [24] Let K be a nonempty closed convex subset of a real smooth, strictly convex and reflexive Banach space E. Let $f : K \times K \to R$ be a bifunction satisfying condition C. Let $\rho : K \to E^*$ be a monotone mapping and let $\Phi : K \to R$ be a lower semi-continuous convex function. For $r > 0$ and any $x \in E$, define a map $G_r^F : E \to 2^K$ as follows,

$$G_r^F(x) = \{z \in K: f(z,y) + \Phi(y) - \Phi(x) + \langle y - z, \rho x\rangle + \frac{1}{r}\langle y - z, J_E^p z - J_E^p x\rangle \geq 0\ \forall\, y \in K\} \quad (9)$$

Then the following hold:

- G_r^F
- G_r^F

Is single-valued

Is firmly nonexpansive-type mapping, that is, for any $x, y \in E$,

$$\langle G_r^F x - G_r^F y, J_E^p G_r^F x - J_E^p G_r^F y\rangle \leq \langle G_r^F x - G_r^F y, J_E^p x - J_E^p y\rangle$$

- $\mathrm{Fix}(G_r^F) = GMEP(f,\Phi,\rho)$
- $GMEP(f,\Phi,\rho)$ is closed and convex

Where $F : E \times E \to \mathbb{R}$ is defined by $F(x,y) = f(x,y) + \Phi(y) - \Phi(x) + \langle y - z, \rho x\rangle$

Remark 1.1 Let K be a nonempty closed convex subset of a Hilbert space H. Let $f : K \times K \to R$ be a bifunction satisfying condition C. Let $\rho : K \to H$ be a Monotone mapping and let $\Phi : K \to R$ be a lower semi-continuous convex function.

For $r > 0$ and any $x \in H$, the map $G_r^F: H \to 2^K$ is nonexpansive. Recall that for an equilibrium problem(EP), $\Phi = 0, \rho = 0$.

Lemma 1.3 (compare with Lemma 2.4 of Chang et al(2010)) Let K be a nonempty closed convex subset of a real Hilbert space H. Let $f_i : K \times K \to R$ be finite family of bifunction satisfying conditions (C1)-(C4) for each $i \in I = \{1,2,\ldots,m\}$ then for all $r > 0$ and $x \in H$, there exists $u \in K$ such that

$$f_i(u,y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0\ \forall\, y \in K, i \in I \quad (10)$$

Moreover, if for all $x \in H$ we define $G_{ir}: H \to 2^K$ by

$$G_{ir}(x) = \left\{u \in K : f_i(x,y) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0 \; \forall \; y \in K\right\} \tag{11}$$

Then the following hold:

- *G_{ir} is single-valued for all $r \geq 0$ $i \in I$*
- *$Fix(G_{ir}) = EP(f_i)$ for all $r > 0$*
- *$EP(f_i)$ is closed and convex*

Lemma 1.4 (see eg [19]) Let K be a nonempty closed convex subset of a real smooth, strictly convex and reflexive Banach space E. Let $f : K \times K \to R$ be a bifunction satisfying condition C. Let $\rho : K \to E^*$ be a monotone mapping and let $\Phi : K \to R$ be alower semi-continuous convex function. For $r > 0$, define a map $G_r^F : E \to 2^K$ as in Lemma 1.2, then for all $s,t > 0$ and for all $x \in K$;

$$||G_s^F x - G_t^F x|| \leq \frac{|s-t|}{s}(||J_E^p G_s^F x|| + ||J_E^p x||) \tag{12}$$

Lemma 1.5 Let E be a real normed linear space with single valued generalized duality mapping and let $1 < p < \infty$. Then for all $x, y \in E$ the following inequality holds.

$$||x+y||^p \leq ||x||^p + p\langle y, J_E^p(x+y)\rangle$$

Let $E = H$ for $x, y, z \in H$, the following also holds

- $||x-y+z||^2 - 2\langle z, x-y\rangle \geq ||x-y||^2$
- $||x+y||^2 = ||x||^2 + 2\langle y, x\rangle + ||y||^2$

Lemma 1.6 For any x, y, z in a real Hilbert space H and a real number $\lambda \in [0,1]$,

$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda||x-z||^2 + (1-\lambda)||y-z||^2 - \lambda(1-\lambda)||x-y||^2$.

Lemma 1.7 [24] Let K be a closed convex nonempty subset of a real Hilbert space

H. Let $x \in H$, then $x_0 = P_K x$ if and only if

$$\langle z - x_0, x - x_0\rangle \leq 0 \; \forall \; z \in K$$

Lemma 1.8 (see [8]) Let E be a reflexive Banach space with weakly continuous normalised duality mapping. Let K be a closed convex subset of E and let T be a uniformly continuous total asymptotically nonexpansive mapping from K into itself with bounded orbit, then $(I - T)$ is demiclosed at zero.

Lemma 1.9 [16] Let $\{\Gamma_n\}$ be sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1} \; \forall \; j \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows

$$\tau(n) = \max\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\},$$

Where $n_0 \in N$ and that the set $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\}$ is not empty, then the following hold (i) $\tau(n_0) \leq \tau(n_0 + 1)$ and $\tau(n) \to \infty$ as $n \to \infty$ (ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n+1)}$ and $\Gamma_n \leq \Gamma_{\tau(n+1)} \; \forall \; n \in N$.

Lemma 1.10 (see eg [9]) Let a_n be sequence of nonegative real numbers satisfying the following relation:

$$a_{n+1} \leq a_n - \alpha_n a_n + \delta_n, n \geq n_0,$$

Where $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ and $\{\delta_n\}_{n\geq 1} \subset \mathbb{R}$ satisfying the following conditions:

$\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$ and $\limsup_{n\to\infty} \delta_n \leq 0$ then $\lim_{n\to\infty} a_n = 0$

II. MAIN RESULT

Theorem 2.1 Let $H_i : (i = 1,\dots,k)$ be Hilbert spaces, $S_i : H_i \to H_i \ (i = 1,\dots,k)$ be η_i-demimetric; $\eta_i \in (-\infty, 1)$ such that $I - S_i$ is demiclosed at $0 \,\forall\, i$. Let $T_{ij} : H_i \to H_i \ (j = 0,1,\dots,m_i;\ i = 1,\dots,k)$ be a finite collection of finite families of uniformly continuous quasi-nonexpansive maps such that $I - T_{ij}$ is demiclosed at 0 for each i and each j. Let $f_{it} : H_i \to H_i \ (t = 0,1,\dots,n_i;\ i = 1,\dots,k)$ be bifunctions satisfying condition C. Let E be a smooth, strictly convex and reflexive real Banach space, $A_i : H_i \to E\ (i = 1,\dots,k)$ be bounded linear operators with adjoint operators A_i^*,

$$\Omega_1 = \{(x_1, x_2, \dots, x_k) \in \prod_{i=1}^{k} Fix(S_i) : A_i x_i = A_j x_j; (i,j = 1,\dots,k)\},$$

$$\Omega_2 = \{(x_1, x_2, \dots, x_k) \in \prod_{i=1}^{k} H_i : x_i \in \bigcap_{j=1}^{m_i} Fix(T_{ij})\}$$

$$\Omega_3 = \{(x_1, x_2, \dots, x_k) \in \prod_{i=1}^{k} H_i : x_i \in \bigcap_{t=1}^{n_i} EP(f_{it})\} \text{ and}$$

$$\Omega = \bigcap_{l=1}^{3} \Omega_l$$

Starting with an arbitrary $x_{i,0} \in H_i; i = 1,\dots,k$, define the iterative sequence $\{x_{i,n}\}$ by

$$x_{i,n+1} = \alpha_n x_{i,0} + (1-\alpha_n) y_{i,n}$$

$$y_{i,n} = z_{i,n} - \beta_n (z_{i,n} - S_i z_{i,n})$$

$$z_{i,n} = \omega_{i,n} - r_n A_i^* J_E (A_i \omega_{i,n} - A_j \omega_{j,n}), j \neq i$$

$$\omega_{i,n} = \alpha u_{i,n} + (1-\alpha) G_{r_n}^{f_{i,n}} u_{i,n}$$

$$u_{i,n} = \alpha x_{i,n} + (1-\alpha) T_{i,n} x_{i,n} \quad (13)$$

Suppose $\Omega \neq \emptyset$, and that

- $\alpha \in (0,1)$,
- $\{r_n\}_{n\geq 0}$ $\{r_n\}_{n\geq 0}$ is a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n = r_0 > 0$,
- $\{\alpha_n\}_{n\geq 0} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$

Then $\{x_{i,n}\}$ is bounded $\forall\, i \in \{1,2,\dots,k\}$

Proof Let $(x_1^*, x_2^*, \dots, x_k^*) \in \Omega$, from (13), Lemma 1.6 and our conditions on T_{ij}, we have

$$\begin{aligned} \|u_{i,n} - x_i^*\|^2 &= ||\alpha(x_{i,n} - x_i^*) + (1-\alpha)(T_{i,n}x_{i,n} - x_i^*)||^2 \\ &= \alpha||x_{i,n} - x*_i||^2 + (1-\alpha)||T_{i,n}x_{i,n} - x_i^*||^2 \\ &\quad -\alpha(1-\alpha)||x_{i,n} - T_{i,n}x_{i,n}||^2 \\ &\leq \alpha||x_{i,n} - x_i^*||^2 + (1-\alpha)||x_{i,n} - x_i^*||^2 \\ &\quad -\alpha(1-\alpha)||x_{i,n} - T_{i,n}x_{i,n}||^2 \\ &= ||x_{i,n} - x_i^*||^2 - \alpha(1-\alpha)||x_{i,n} - T_{i,n}x_{i,n}||^2 \end{aligned} \quad (14)$$

Also, by Remark 1.1 and hypothesis,

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$$\|\omega_{i,n} - x_i^*\|^2 = \left\|\alpha(u_{i,n} - x_i^*) + (1-\alpha)\left(G_{r_n}^{f_{i,n}} u_{i,n} - x_i^*\right)\right\|^2$$

$$\leq \|x_{i,n} - x_i^*\|^2 - \alpha(1-\alpha)(\left\|u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n}\right\|^2 + \|x_{i,n} - T_{i,n}x_{i,n}\|^2) \quad (15)$$

$$\|z_{i,n} - x_i^*\|^2 \leq \|\omega_{i,n} - r_n A_i^* J_E(A_i\omega_{i,n} - A_j\omega_{j,n}) - x_i^*\|^2 \;\; j \neq i$$

$$= \|\omega_{i,n} - r_n A_i^* J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\|^2 - 2\langle x_i^*, \omega_{i,n}\rangle$$

$$+2r_n\langle A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle + \|x_i^*\|^2$$

$$= \|\omega_{i,n}\|^2 - 2\langle x_i^*, \omega_{i,n}\rangle + \|x_i^*\|^2 + r_n^2\|A_i\|^2\|J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\|^2$$

$$-2r_n\langle A_i\omega_{i,n} - A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle$$

$$= \|\omega_{i,n} - x_i^*\|^2 + r_n^2\|A_i\|^2\|J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\|^2$$

$$-2r_n\langle A_i\omega_{i,n} - A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle$$

$$\leq \|x_{i,n} - x_i^*\|^2 + r_n^2\|A_i\|^2\|J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\|^2$$

$$-2r_n\langle A_i\omega_{i,n} - A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle$$

$$-\alpha(1-\alpha)(\left\|u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n}\right\|^2 + \|x_{i,n} - T_{i,n}x_{i,n}\|^2) \quad (16)$$

Also by 1.5, (16) and hypothesis

$$\|y_{i,n} - x_i^*\|^2 = \|z_{i,n} - \beta_n(z_{i,n} - S_i z_{i,n}) - x_i^*\|^2$$

$$= \|z_{i,n} - x_i^*\|^2 + \beta_n^2\|z_{i,n} - S_i z_{i,n}\|^2$$

$$-2\beta_n\langle (z_{i,n} - S_i z_{i,n}, z_{i,n} - x_i^*)\rangle$$

Since S_i is η_i −demimetric,

$$\|y_{i,n} - x_i^*\|^2 \leq \|z_{i,n} - x_i^*\|^2 - \beta_n(1 - \eta_i - \beta_n)\|z_{i,n} - S_i z_{i,n}\|^2$$

$$\leq \|x_{i,n} - x_i^*\|^2 + r_n^2\|A_i\|^2\|J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\|^2$$

$$-2r_n\langle A_i\omega_{i,n} - A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle$$

$$-\alpha(1-\alpha)(\left\|u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n}\right\|^2 + \|x_{i,n} - T_{i,n}x_{i,n}\|^2)$$

$$-\beta_n(1 - \eta_i - \beta_n)\|z_{i,n} - S_i z_{i,n}\|^2 \quad (17)$$

More so, using Lemma 1.5, (17) and hypothesis

$$\|x_{i,n+1} - x_i^*\|^2 = \|\alpha_n x_{i,0} + (1-\alpha_n)y_{i,n} - x_i^*\|^2$$

$$= \|\alpha_n(x_{i,0} - x_i^*) + (1-\alpha_n)(y_{i,n} - x_i^*)\|^2$$

$$= \alpha_n\|x_{i,0} - x_i^*\|^2 + (1-\alpha_n)\|y_{i,n} - x_i^*\|^2 - \alpha_n(1-\alpha_n)\|y_{i,n} - x_{i,0}\|^2$$

$$\leq \alpha_n \left\| x_{i,0} - x_i^* \right\|^2 + (1-\alpha_n)\left\| y_{i,n} - x_i^* \right\|^2$$

$$\leq \alpha_n \left\| x_{i,0} - x_i^* \right\|^2 + (1-\alpha_n)[\left\| x_{i,n} - x_i^* \right\|^2 + r_n^2 \|A_i\|^2 \left\| J_E(A_i\omega_{i,n} - A_j\omega_{j,n}) \right\|^2$$

$$-2r_n \langle A_i\omega_{i,n} - A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle$$

$$-\alpha(1-\alpha)(\left\| u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n} \right\|^2 + \left\| x_{i,n} - T_{i,n}x_{i,n} \right\|^2)$$

$$-\beta_n(1-\eta_i-\beta_n)\left\| z_{i,n} - S_i z_{i,n} \right\|^2]$$

$$\leq \alpha_n \left\| x_{i,0} - x_i^* \right\|^2 + (1-\alpha_n)\left\| x_{i,n} - x_i^* \right\|^2 + (1-\alpha_n) r_n^2 \|A_i\|^2 \left\| J_E(A_i\omega_{i,n} - A_j\omega_{j,n}) \right\|^2$$

$$-2r_n(1-\alpha_n)\langle A_i\omega_{i,n} - A_i^* x_i^*, J_E(A_i\omega_{i,n} - A_j\omega_{j,n})\rangle$$

$$-(1-\alpha)\alpha(1-\alpha_n)(\left\| u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n} \right\|^2 + \left\| x_{i,n} - T_{i,n}x_{i,n} \right\|^2)$$

$$-(1-\alpha_n)\beta_n(1-\eta_i-\beta_n)\left\| z_{i,n} - S_i z_{i,n} \right\|^2 \qquad (18)$$

Define

$$D_n(x_1^*, x_2^*, \ldots, x_k^*) = \sum_{i=1}^{k} \left\| x_{i,n} - x_i^* \right\|^2$$

So that

$$D_{n+1}(x_1^*, x_2^*, \ldots, x_k^*) \leq \alpha_n \sum_{i=1}^{k} \left\| x_{i,0} - x_i^* \right\|^2 + (1-\alpha_n) \sum_{i=1}^{k} \left\| x_{i,n} - x_i^* \right\|^2$$

$$-(1-\alpha_n) r_n (2k - r_n \sum_{i=1}^{k} \|A_i\|^2) \sum_{i=1}^{k} \left\| A_i\omega_{i,n} - A_j\omega_{j,n} \right\|^2$$

$$-(1-\alpha)\alpha(1-\alpha_n) \sum_{i=1}^{k} (\left\| u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n} \right\|^2 + \left\| x_{i,n} - T_{i,n}x_{i,n} \right\|^2)$$

$$-(1-\alpha_n)\beta_n \textstyle\sum_{i=1}^{k} (1-\eta_i-\beta_n)\left\| z_{i,n} - S_i z_{i,n} \right\|^2 \qquad (19)$$

Since $2k - r_n \sum_{i=1}^{k} \|A_i\|^2 \geq 0$ and $1-\eta_i-\beta_n \geq 0 \ \ \forall\, i = 1, \ldots, k$, then

$$D_{n+1}(x_1^*, x_2^*, \ldots, x_k^*) \leq (1-\alpha_n) D_n(x_1^*, x_2^*, \ldots, x_k^*) + \alpha_n \textstyle\sum_{i=1}^{k} \left\| x_{i,0} - x_i^* \right\|^2 \qquad (20)$$

Using mathematical induction, we show that the sequence $\{D_{n+1}(x_1^*, x_2^*, \ldots, x_k^*)\}_{n\geq 1}$ is bounded. Let $M = \sum_{i=1}^{k} \left\| x_{i,0} - x_i^* \right\|^2$ for $n = 0$, $D_0(x_1^*, \ldots, x_k^*) = M$ so that

$$D_1(x_1^*, \ldots, x_k^*) \leq (1-\alpha_n) D_0(x_1^*, \ldots, x_k^*) + \alpha_n \sum_{i=1}^{k} \left\| x_{i,0} - x_i^* \right\|^2$$

$$= (1-\alpha_n)M + \alpha_n M = M$$

Suppose $D_s(x_1^*, \ldots, x_k^*) \leq M$ for $n = s \geq 1$ then,

$$D_{s+1}(x_1^*, \ldots, x_k^*) \leq (1-\alpha_n) D_n(x_1^*, \ldots, x_k^*) + \alpha_n \sum_{i=1}^{k} \left\| x_{i,0} - x_i^* \right\|^2$$

$$\leq (1-\alpha_n)M + \alpha_n M = M$$

Which says, $\{D_n(x_1^*, \dots, x_k^*)\}_{n\geq 1}$ is bounded and hence $\{x_{i,n}\}_{n\geq 1}$ is bounded $\forall\, i, \dots, k$.

Theorem 2.2 Let $H_i : (i = 1, \dots, k)$ be Hilbert spaces, $S_i : H_i \to H_i$ $(i = 1, \dots, k)$ be η_i-demimetric; $\eta_i \in (-\infty, 1)$ such that $I - S_i$ is demiclosed at $0\ \forall\, i$.

Let $T_{ij} : H_i \to H_i$ $(j = 0, 1, \dots, m_i;\ i = 1, \dots, k)$ be a finite collection of finite families of uniformly continuous quasi-nonexpansive maps such that $I - T_{ij}$ is demiclosed at 0 for each i and each j. Let $f_{it} : H_i \to H_i$ $(t = 0, 1, \dots, n_i;\ i = 1, \dots, k)$ be bifunctions satisfying condition C. Let E be a smooth, strictly convex and reflexive real Banach space.

Let $A_i : H_i \to E$ $(i = 1, \dots, k)$ be bounded linear operators with adjoint operators A_i^*. Let Ω be as in theorem 2.1. Let $\{x_{i,n}\}$ be defined by (13), then $\{x_{i,n}\}_{n\geq 0}$ $i \in \{1,2,\dots,k\}$ converges strongly to an element $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k) \in P_\Omega(x_{1,0}, x_{2,0}, \dots, x_{k,0})$.

where $1 - \eta_i - \beta_n > 0$, $2k - r_n \sum_{i=1}^{k} \|A_i\|^2 > 0$, $T_{i,n} = T_{i,n \bmod m_i}$, and

$f_{i,n} = f_{i,n \bmod m_i}$, $\forall\, i = 1, \dots, k$.

Proof From Theorem 2.1, we have that $\{x_{i,n}\}_{n\geq 1}$ is bounded $\forall\, i = 1, \dots, k$.
Let $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k) \in P_\Omega(x_{1,0}, x_{2,0}, \dots, x_{k,0})$,

From Lemma 1.7

$$\langle (y_1, y_2, \dots, y_k) - (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k), (x_{1,0}, x_{2,0}, \dots, x_{k,0}) - (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k) \rangle$$

$$= \langle (y_1 - \hat{x}_1, y_2 - \hat{x}_2, \dots, y_k - \hat{x}_k), (x_{1,0} - \hat{x}_1, x_{2,0} - \hat{x}_2, \dots, x_{k,0} - \hat{x}_k) \rangle \leq 0 \forall (y_1, y_2, \dots, y_k) \in \Omega$$

From Lemma 1.5, Theorem 2.1, Definition 4 and hypothesis, we have that

$$\|x_{i,n+1} - \hat{x}_i\|^2 = \|\alpha_n x_{i,0} + (1-\alpha_n) y_{i,n} - \hat{x}_i\|^2$$

$$\leq \|\alpha_n x_{i,0} + (1-\alpha_n) y_{i,n} - \hat{x}_i - \alpha_n (x_{i,0} - \hat{x}_i)\|^2 + 2\alpha_n \langle x_{i,0} - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle$$

$$= \|\alpha_n \hat{x}_i + (1-\alpha_n) y_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle x_{i,0} - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle$$

$$= (1-\alpha_n)^2 \|y_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle x_{i,0} - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle$$

$$\leq (1-\alpha_n) \|y_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle x_{i,0} - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle$$

$$\leq (1-\alpha_n) \|x_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle x_{i,0} - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle$$

$$+ (1-\alpha_n) r_n^2 \|A_i\|^2 \| J_E (A_i \omega_{i,n} - A_j \omega_{j,n}) \|^2$$

$$-(1-\alpha_n) 2 r_n \langle A_i \omega_{i,n} - A_i^* \hat{x}_i, J_E (A_i \omega_{i,n} - A_j \omega_{j,n}) \rangle$$

$$-(1-\alpha_n)\alpha(1-\alpha)\left(\left\|u_{i,n} - G_{r_n}^{f_{i,n}} u_{i,n}\right\|^2 + \|x_{i,n} - T_{i,n} x_{i,n}\|^2\right)$$

$$-(1-\alpha_n)\beta_n (1 - \eta_i - \beta_n) \|z_{i,n} - S_i z_{i,n}\|^2$$

Now,

$$D_{n+1}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k) \leq (1-\alpha_n) D_n(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k) + 2\alpha_n \sum_{i=1}^{k} \langle x_{i,0} - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle$$

$$-(1-\alpha_n)\alpha(1-\alpha)\sum_{i=1}^{k}\left\|x_{i,n}-\ T_{i,n}x_{i,n}\right\|^2$$

$$-(1-\alpha_n)\alpha(1-\alpha)\sum_{i=1}^{k}\left\|u_{i,n}-G_{r_n}^{f_{i,n}}u_{i,n}\right\|^2$$

$$-(1-\alpha_n)\beta_n\sum_{i=1}^{k}(1-\eta_i-\beta_n)\left\|z_{i,n}-S_i z_{i,n}\right\|^2$$

$$-(1-\alpha_n)r_n(2k-r_n\sum_{i=1}^{k}\|A_i\|^2)\sum_{i=1,i\neq j}^{k}\left\|A_i\omega_{i,n}-A_j\omega_{j,n}\right\|^2$$

$$\leq(1-\alpha_n)D_n(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)+2\alpha_n\sum_{i=1}^{k}\langle x_{i,0}-\hat{x}_i,x_{i,n}-\hat{x}_i\rangle$$

$$+2\alpha_n\sum_{i=1}^{k}\left\|x_{i,0}-\hat{x}_i\right\|\left\|x_{i,n+1}-x_{i,n}\right\|$$

$$-(1-\alpha_n)\alpha(1-\alpha)\sum_{i=1}^{k}\left\|x_{i,n}-\ T_{i,n}x_{i,n}\right\|^2$$

$$-(1-\alpha_n)\alpha(1-\alpha)\sum_{i=1}^{k}\left\|u_{i,n}-G_{r_n}^{f_{i,n}}u_{i,n}\right\|^2$$

$$-(1-\alpha_n)\beta_n\sum_{i=1}^{k}(1-\eta_i-\beta_n)\left\|z_{i,n}-S_i z_{i,n}\right\|^2$$

$$-(1-\alpha_n)r_n(2k-r_n\textstyle\sum_{i=1}^{k}\|A_i\|^2)\sum_{i=1,i\neq j}^{k}\left\|A_i\omega_{i,n}-A_j\omega_{j,n}\right\|^2 \quad (21)$$

Hence,

$$D_{n+1}(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)\leq(1-\alpha_n)D_n(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)+2\alpha_n\sum_{i=1}^{k}\left\|x_{i,0}-\hat{x}_i\right\|\left\|x_{i,n+1}-x_{i,n}\right\|$$

$$+2\alpha_n\textstyle\sum_{i=1}^{k}\langle x_{i,0}-\hat{x}_i,x_{i,n}-\hat{x}_i\rangle \quad (22)$$

Furthermore, we show that $\{D_n(\hat{x}_1,\dots,\hat{x}_k)\}_{n\geq1}$ converges strongly to zero.

We discern two possible cases

Case 1: Suppose the real sequence $\{D_n(\hat{x}_1,\dots,\hat{x}_k)\}_{n\geq1}$ is nonincreasing for $n\geq n_0$, for some $n_0\in\mathbb{N}$, this implies that $\{D_n(\hat{x}_1,\dots,\hat{x}_k)\}_{n\geq1}$ is monotone and bounded and hence converges. Moreover, using (21) and the fact that $\alpha_n\to0$ as $n\to\infty$, we have that $\forall i\in\{1,2,\dots,k\}$

$$\lim_{n\to\infty}\left\|x_{i,n}-\ T_{i,n}x_{i,n}\right\|=0=\lim_{n\to\infty}\left\|u_{i,n}-G_{r_n}^{f_{i,n}}u_{i,n}\right\|=\lim_{n\to\infty}\left\|z_{i,n}-S_i z_{i,n}\right\|\ \forall\ i$$

$$\lim_{n\to\infty}\left\|A_i\omega_{i,n}-A_j\omega_{j,n}\right\|=0\ \forall i;j\neq i$$

$$\lim_{n\to\infty}\left\|y_{i,n}-z_{i,n}\right\|\ =\lim_{n\to\infty}\beta_n\left\|z_{i,n}-S_i z_{i,n}\right\|=0\,\forall\ i$$

$$\lim_{n\to\infty}\|z_{i,n}-\omega_{i,n}\| = \lim_{n\to\infty} r_n\|A_i\|\|A_i\omega_{i,n}-A_j\omega_{j,n}\| = 0\,\forall\, i\,; j\neq i$$

$$\lim_{n\to\infty}\|\omega_{i,n}-u_{i,n}\| = (1-\alpha)\lim_{n\to\infty}\left\|u_{i,n}-G_{r_n}^{f_{i,n}}u_{i,n}\right\| = 0\,\forall\, i$$
$$\lim_{n\to\infty}\|u_{i,n}-x_{i,n}\| = (1-\alpha)\lim_{n\to\infty}\|x_{i,n}- T_{i,n}x_{i,n}\| = 0\,\forall\, i$$

$$\lim_{n\to\infty}\|x_{i,n+1}-y_{i,n}\| = \lim_{n\to\infty}\alpha_n\|x_{i,0}- y_{i,n}\| = 0\,\forall\, i$$

Now, since

$$\|x_{i,n+1}-x_{i,n}\|\leq\|x_{i,n+1}-y_{i,n}\|+\|y_{i,n}-z_{i,n}\|+\|z_{i,n}-\omega_{i,n}\|+\|\omega_{i,n}-u_{i,n}\|+\|u_{i,n}-x_{i,n}\|$$

We have that

$$\lim_{n\to\infty}\|x_{i,n+1}-x_{i,n}\| = 0\,\forall\, i$$

Claim 1

$$\lim_{n\to\infty} sup\sum_{i=1}^{k}\langle x_{i,0}-\hat{x}_i, x_{i,n}-\hat{x}_i\rangle\leq 0$$

Proof of Claim: Let $\{x_{1,n_l},x_{2,n_l},\dots,x_{k,n_l}\}_{l\geq1}$ be a subsequence of $\{x_{1,n},x_{2,n},\dots,x_{k,n}\}_{n\geq1}$ such that

$$\lim_{n\to\infty} sup\sum_{i=1}^{k}\langle x_{i,0}-\hat{x}_i, x_{i,n}-\hat{x}_i\rangle = \lim_{l\to\infty} sup\sum_{l=1}^{k}\langle x_{i,0}-\hat{x}_i, x_{i,n_l}-\hat{x}_i\rangle$$

So, $\forall\, i$

$$\lim_{n\to\infty} sup\langle x_{i,0}-\hat{x}_i, x_{i,n}-\hat{x}_i\rangle = \lim_{l\to\infty} sup\langle x_{i,0}-\hat{x}_i, x_{i,n_l}-\hat{x}_i\rangle \tag{23}$$

More so, since $H=\prod_{i=1}^{k}H_i$ is a Hilbert space and so reflexive, and $\{x_{1,n_l},x_{2,n_l},\dots,x_{k,n_l}\}_{l\geq1}$ is a bounded sequence in H, $\exists$ a subsequence $\{x_{1,n_{l_t}},x_{2,n_{l_t}},\dots,x_{k,n_{l_t}}\}_{t\geq1}$ of $\{x_{1,n_l},x_{2,n_l},\dots,x_{k,n_l}\}_{l\geq1}$ which converges weakly to $(\hat{x}_1,\dots,\hat{x}_k)\in H$ i.e, $x_{1,n_{l_t}}\rightarrow^{w}\ \hat{x}_i$ as $t\to\infty$. Hence, the subsequences $\{u_{i,n_{l_t}}\}\subset\{u_{i,n_l}\}$; $\{\omega_{i,n_{l_t}}\}\subset\{\omega_{i,n_l}\}$; $\{z_{i,n_{l_t}}\}\subset\{z_{i,n_l}\}$; and $\{y_{i,n_{l_t}}\}\subset\{y_{i,n_l}\}$ converge weakly to $\hat{x}_i\ \forall\, i$. Since $I-S_i$ is demiclosed at $0\,\forall\, i$, we have that $\hat{x}_i\in fix(S_i)\forall\, i$. That is $(\hat{x}_1,\dots,\hat{x}_k)\in\prod_{i=1}^{k}fix(S_i)$.

Now, $\forall\, i$ and $j\neq i$

$$\|A_ix_i^*-A_jx_j^*\|^2=\left\|A_ix_i^*-A_i\omega_{i,n_{l_t}}+A_i\omega_{i,n_{l_t}}+A_j\omega_{j,n_{l_t}}-A_j\omega_{j,n_{l_t}}-A_jx_j^*\right\|^2$$

$$=\left\|A_i\omega_{i,n_{l_t}}-A_j\omega_{j,n_{l_t}}+A_ix_i^*+A_i\omega_{i,n_{l_t}}+A_j\omega_{j,n_{l_t}}-A_jx_j^*\right\|^2$$

$$\leq\left\|A_i\omega_{i,n_{l_t}}-A_j\omega_{j,n_{l_t}}\right\|^2+2\langle A_ix_i^*+A_i\omega_{i,n_{l_t}}+A_j\omega_{j,n_{l_t}}-A_jx_j^*, A_ix_i^*-A_jx_j^*\rangle$$

But $\omega_{i,n_{l_t}}\rightarrow^{w}\ x_i^*$ as $t\to\infty$. So, $A_i\omega_{i,n_{l_t}}\rightarrow^{w}\ A_ix_i^*$ as $t\to\infty$ so that taking limits

on both sides and using the fact that $\lim_{n\to\infty}\|A_i\omega_{i,n}-A_j\omega_{j,n}\|=0\ \forall i; j\neq i$

we have that $A_ix_i^*=A_jx_j^*\ \ (j\neq i)$

Thus, $(x_1^*,x_2^*,\dots,x_k^*)\in\Omega_1$.

Observe that $\lim_{n\to\infty} \left\| x_{i,n_{l_{t+1}}} - x_{i,n_{l_t}} \right\| = 0 \; \forall \; i$ so that

$$\lim_{t\to\infty} \left\| x_{i,n_{l_{t+\tau}}} - x_{i,n_{l_t}} \right\| = 0 \; \forall \; i \,; \tau = 0,1,2, \dots, m_i$$

Now

$$\left\| x_{i,n_{l_t}} - T_{i,n_{l_{t+\tau}}} x_{i,n_{l_t}} \right\| \le \left\| x_{i,n_{l_t}} - x_{i,n_{l_{t+j}}} \right\| + \left\| x_{i,n_{l_{t+j}}} - T_{i,n_{l_{t+j}}} x_{i,n_{l_{t+j}}} \right\|$$

$$+ \left\| T_{i,\delta_j} x_{i,n_{l_{t+j}}} - T_{i,\delta_j} x_{i,n_{l_t}} \right\|$$

So that by uniform continuity of $T_{i,j}$, we have

$$\lim_{t\to\infty} \left\| x_{i,n_{l_t}} - T_{i,n_{l_{t+j}}} x_{i,n_{l_t}} \right\| = 0; j = 0,1 \dots, m_i \; \forall \; i \in \{0,1,2, \dots, k\}.$$

Hence, $x_i^* \in \bigcap_{j=0}^{m_i} fix(T_{i,j})$. Thus $(x_1^*, x_2^*, \dots, x_k^*) \in \Omega_2$.

Next is to show that $(x_1^*, x_2^*, \dots, x_k^*) \in \Omega_3$.

Observe that

$$\lim_{t\to\infty} \left\| u_{i,n_{l_t}} - G_{r_{n_{l_t}}}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} \right\| = 0 \; \forall \; i$$

And by Lemma 1.4

$$\left\| G_{r_0}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} - G_{r_{n_{l_t}}}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} \right\| \le \frac{r_0 - r_{n_{l_t}}}{r_{n_{l_t}}} \left(\left\| G_{r_{n_{l_t}}}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} \right\| + \left\| u_{i,n_{l_t}} \right\| \right).$$

Since $\left\{ u_{i,n_{l_t}} \right\}_{t\ge 0}$ is bounded for each i and $\lim_{t\to\infty} r_{n_{l_t}} = r_0$, we have that $\forall \; i \in \{1,2, \dots, k\}$

$$\lim_{t\to\infty} \left\| G_{r_0}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} - G_{r_{n_{l_t}}}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} \right\| = 0$$

Observe that $\forall \; i \in \{1,2, \dots, k\}$

$$\left\| G_{r_0}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} - u_{i,n_{l_t}} \right\| \le \left\| G_{r_0}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} - G_{r_{n_{l_t}}}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} \right\| + \left\| G_{r_{n_{l_t}}}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} - u_{i,n_{l_t}} \right\|$$

And hence

$$\lim_{t\to\infty} \left\| G_{r_0}^{f_{i,n_{l_t}}} u_{i,n_{l_t}} - u_{i,n_{l_t}} \right\| = 0.$$

Note also that

$$\lim_{t\to\infty} \left\| u_{i,n_{l_t}} - x_{i,n_{l_t}} \right\| = 0 = \lim_{t\to\infty} \left\| x_{i,n_{l_{t+1}}} - x_{i,n_{l_t}} \right\| \quad \forall \; i$$

Now, $\forall \; i \in \{1,2, \dots, k\}$

$$\left\| u_{i,n_{l_t}+1} - u_{i,n_{l_t}} \right\| \le \left\| u_{i,n_{l_t}+1} - x_{i,n_{l_t}+1} \right\| + \left\| x_{i,n_{l_t}+1} - x_{i,n_{l_t}} \right\| + \left\| x_{i,n_{l_t}} - u_{i,n_{l_t}} \right\|,$$

So that

$$\lim_{t\to\infty} \left\| u_{i,n_{l_t}+1} - u_{i,n_{l_t}} \right\| = 0,$$

And so $\forall \; i \in \{1,2, \dots, k\}$

$$\lim_{t\to\infty}\left\|u_{i,n_{l_t}+\tau}-u_{i,n_{l_t}}\right\|=0, \tau=0,1,2,\dots,m_i$$

Now, for $\tau=0,1,2,\dots,m_i$

$$\left\|u_{i,n_{l_t}}-G_{r_0}^{f_{i,n_{l_t+\tau}}}u_{i,n_{l_t}}\right\|\le\left\|u_{i,n_{l_t}}-u_{i,n_{l_t+\tau}}\right\|+\left\|u_{i,n_{l_t+\tau}}-G_{r_0}^{f_{i,n_{l_t+\tau}}}u_{i,n_{l_t+\tau}}\right\|$$

$$+\left\|G_{r_0}^{f_{i,n_{l_t+\tau}}}u_{i,n_{l_t+\tau}}-G_{r_0}^{f_{i,n_{l_t+\tau}}}u_{i,n_{l_t}}\right\|$$

$$\le 2\left\|u_{i,n_{l_t+\tau}}-u_{i,n_{l_t}}\right\|+\left\|u_{i,n_{l_t+\tau}}-G_{r_0}^{f_{i,n_{l_t+\tau}}}u_{i,n_{l_t+\tau}}\right\| \ \forall\, i,$$

So that

$$\lim_{t\to\infty}\left\|u_{i,n_{l_t+\tau}}-G_{r_0}^{f_{i,n_{l_t+\tau}}}u_{i,n_{l_t}}\right\|=0, \tau=0,1,2,\dots,m_i$$

Now $\{f_{i,j}\}_{j=0}^{m_i}$ is a finite set of mappings and $f_{i,n} = f_{i,n \bmod m_i}$

Then $\forall\, j \in \{0,1,\dots,m_i\}\ \exists\, \delta_j \in \{0,1,\dots,m_i\}$ such that $n_{l_t}+\delta_j = j \bmod m_i$,

So that $\forall\, i$

$$\lim_{t\to\infty}\left\|x_{i,n_{l_t}}-G_{r_0}^{f_{i,j}}x_{i,n_{l_t}}\right\|=\lim_{t\to\infty}\left\|x_{i,n_{l_t}}-G_{r_0}^{f_{i,n_{l_t}+\delta_j}}x_{i,n_{l_t}}\right\|=0; (j=0,1,\dots,m_i).$$

So,

$$\lim_{t\to\infty}\left\|x_{i,n_{l_t}}-G_{r_0}^{f_{i,j}}x_{i,n_{l_t}}\right\|=0; (j=0,1,\dots,m_i).$$

Thus, $\left\{(x_{1,n_{l_t}},x_{2,n_{l_t}},\dots,x_{k,n_{l_t}})\right\}_{t\ge1}$ converges weakly to $(x_1^*,x_2^*,\dots,x_k^*)$, so that $(x_1^*,x_2^*,\dots,x_k^*)\in\Omega_3$.

Therefore, by Lemmas 1.1, 1.7 and (23)

$$\limsup_{t\to\infty}\sum_{i=1}^{k}\langle x_{i,0}-x_i^*, x_{i,n}-x_i^*\rangle$$

$$=\lim_{t\to\infty}\langle(x_{1,n_l},x_{2,n_l},\dots,x_{k,n_l})-(x_1^*,x_2^*,\dots,x_k^*),(x_{1,0},x_{2,0},\dots,x_{k,0})-(x_1^*,x_2^*,\dots,x_k^*)\rangle$$

$$=\langle(x_1^*,x_2^*,\dots,x_k^*)-(x_1^*,x_2^*,\dots,x_k^*),(x_{1,0},x_{2,0},\dots,x_{k,0})-(x_1^*,x_2^*,\dots,x_k^*)\rangle\le 0$$

Now, from (21), (22), we have

$$D_{n+1}(x_1^*,x_2^*,\dots,x_k^*)\le(1-\alpha_n)D_n(x_1^*,x_2^*,\dots,x_k^*)+2\alpha_n\sum_{i=1}^{k}\langle x_{i,0}-x_i^*,x_{i,n}-x_i^*\rangle$$

$$+2\alpha_n\sum_{i=1}^{k}\|x_{i,0}-x_i^*\|\|x_{i,n+1}-x_{i,n}\|$$

So that by Lemma 1.10

$\{D_n(x_1^*,x_2^*,\dots,x_k^*)\}_{n\ge0}$ converges strongly to zero as $n\to\infty$. Hence, $(x_{1,n},x_{2,n},\dots,x_{k,n})$ converges strongly to $(x_1^*,x_2^*,\dots,x_k^*)\in\Omega$. This completes the proof of Case 1.

➢ *CASE 2*

Suppose there exists a subsequence $\{D_{n_s}(\hat{x}_1,\dots,\hat{x}_k)\}$ of $\{D_n(\hat{x}_1,\dots,\hat{x}_k)\}$ such that $D_{n_s}(\hat{x}_1,\dots,\hat{x}_k) \le D_{n_s+1}(\hat{x}_1,\dots,\hat{x}_k)\ \forall\ s \in \mathbb{N}$, then by lemma 1.9 there exists a non-decreasing sequence $\{q_r\}_{r\ge1} \subset \mathbb{N}$ such that $(i)\ \lim_{n\to\infty} q_r = \infty$ $(ii)\ D_{q_r}(\hat{x}_1,\dots,\hat{x}_k) \le D_{q_r+1}(\hat{x}_1,\dots,\hat{x}_k)\ \forall\ r \in \mathbb{N}$. Since the sequence $\{x_{i,q_r}\}_{r\ge1}$ $i = 1,\dots,k$ are bounded, we obtain from (21) and using the arguments earlier as $r \to \infty$ $\|x_{i,q_r} - y_{i,q_r}\|, \|y_{i,q_r} - z_{i,q_r}\|, \|z_{i,q_r} - \omega_{i,q_r}\|, \|\omega_{i,q_r} - u_{i,q_r}\| \to 0$ as $r \to \infty\ \forall\ i$ and

$$\|x_{i,q_r+1} - x_{i,q_r}\| \to 0\ as\ r \to \infty.$$

Moreover, $\limsup_{t\to\infty} \sum_{i=1}^{k}\langle x_{i,0} - x_i^*, x_{i,n} - x_i^*\rangle = \lim_{r\to\infty} \sum_{i=1}^{k}\langle x_{i,0} - x_i^*, x_{i,q_r} - x_i^*\rangle \le 0$

Since $D_{q_r}(\hat{x}_1,\dots,\hat{x}_k) \le D_{q_r+1}(\hat{x}_1,\dots,\hat{x}_k)$ $r \in \mathbb{N}$, such that

$$\alpha_{q_r} D_{q_r}(\hat{x}_1,\dots,\hat{x}_k) \le D_{q_r}(\hat{x}_1,\dots,\hat{x}_k) - D_{q_r+1}(\hat{x}_1,\dots,\hat{x}_k) + 2\alpha_{q_r}\sum_{i=1}^{k}\langle x_{i,0} - \hat{x}_i, x_{i,q_r} - \hat{x}_i\rangle$$

$$+2\alpha_{q_r}\sum_{i=1}^{k}\|x_{i,0} - \hat{x}_i\|\|x_{i,q_r+1} - x_{i,q_r}\|$$

Dividing through by α_{q_r} and taking limits as $r \to \infty$ then we would have $D_{q_r}(\hat{x}_1,\dots,\hat{x}_k) \to 0$ as $r \to \infty$ and so $D_{q_r+1}(\hat{x}_1,\dots,\hat{x}_k) \to 0$ as $r \to \infty$

Since $D_r(\hat{x}_1,\dots,\hat{x}_k) \le D_{q_r+1}(\hat{x}_1,\dots,\hat{x}_k)$, then $D_r(\hat{x}_1,\dots,\hat{x}_k) \to 0$ as $r \to \infty$. This implies that $\lim_{t\to\infty}\|x_{i,r} - \hat{x}_i\| = 0\ \forall\ i$

So that $x_{i,r} \to \hat{x}_i$ as $r \to \infty$. Thus, $(x_{1,n}, x_{2,n},\dots,x_{k,n})$ converges to $(\hat{x}_1,\dots,\hat{x}_k)$ as $n \to \infty$. This completes the proof.

III. CONCLUSION

We have obtained a common solution to a Multiple Split Equality problem, a finite family of simultaneous equilibrium problems, and a common fixed Point of a finite collection of a finite families of nonlinear mappings. We have developed an algorithm and established sufficient condition for its strong convergence to such a common solution. This result extends and generalizes a lot of results in this area. To see this, in our theorem, let $k = 2$, we immediately get the result of Ofoedu and Araka (2019) which in turn extends Zegeye (2019) when considered in a real Hibert spaces.

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