Homotopy Analysis Laplace Transform Method for Analytical Solution of Porous Medium Equation

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Abstract:- In the present paper, we have derived an analytical solution of a porous medium equation utilizing Homotopy Analysis Laplace Transform Method (HALTM). The suggested approach is an amalgamation of Homotopy Analysis Method and Laplace Transform Method. The analytical solutions have been obtained in terms of convergent series by proper choice of convergence control parameter. The results conclude that this method is extremely simple and efficient. This method is suitable for other nonlinear differential equations.

Keywords:- *Homotopy Analysis Method, Laplace Transformation, Porous medium equation.*

I. INTRODUCTION

Nonlinear differential equations arise in a broad area of wildly different contexts in natural world, such as fluid dynamics, classical mechanics, nanotechnology, economic and physical structures. In this paper, we consider the nonlinear partial differential equation (heat equation) called the porous medium equation[1].

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) \tag{1}$$

where m is a rational number.

Porous medium equation mostly arises in mass and heat transfer problems, flow through porous media, population dispersal and combustion process. Porous medium equation has been studied by many researchers such as J.D.Murray[3], A. D. Polyanin and V. F. Zaitsev [5], J.L.Vazquez [6] , S. Pamuk [4], D. Mishra et al [2].

We will solve the initial value problem given by(1) and the initial condition $w(x,0) = g(x)$ by the Homotopy Analysis Laplace Transform Method (HALTM).

II. METHOD

The general type of a time-dependent differential equation is analyzed to represent the fundamental concept of the Homotopy Analysis Laplace Transform Method as described below:

$$B\big(w(x,t)\big) - h(x,t) = 0 \tag{2}$$

Where$B$ is a differential operator, $w(x,t)$ is an unknown function and$h(x,t)$is a known analytic function. Moreover, we divide B into two parts

$$B\big(w(x,t)\big) = \frac{\partial w(x,t)}{\partial t} + R\big(w(x,t)\big) \tag{3}$$

where $\frac{\partial w(x,t)}{\partial t}$is a simple part that is easy to manage, and Rincludes the rest parts of B. So, equation on the given domain $\gamma : \{0 \le x \le \infty\,; 0 \le t \le T\}$ can be described as follows:

$$\frac{\partial w(x,t)}{\partial t} + R\big(w(x,t)\big) = h(x,t)\,, (x,t) \in \gamma \tag{4}$$

with the initial condition$w(x,0) = g(x)$ for any$x\epsilon[0,\infty)$

Now, we apply Laplace transform on both sides of equation(4), then we obtain

$$\mathscr{L}\left\{\frac{\partial w(x,t)}{\partial t}\right\} + \mathscr{L}\big\{R\big(w(x,t)\big)\big\} = \mathscr{L}\{h(x,t)\} \tag{5}$$

Utilizing the property of the Laplace transform and using initial condition in equation (5), we get

$$\mathscr{L}\{w(x,t)\} = s^{-1}g(x) - s^{-1}\mathscr{L}\big\{R\big(w(x,t)\big)\big\} + s^{-1}\mathscr{L}\{h(x,t)\} \tag{6}$$

The nonlinear operator is considered as

$$N[U(x,t;p)] = \mathscr{L}\{U(x,t;p)\} - s^{-1}g(x) + s^{-1}\mathscr{L}\big\{R\big(U(x,t;p)\big)\big\} - s^{-1}\mathscr{L}\{h(x,t)\} \tag{7}$$

We can create a homotopy that satisfies [1]

$$\mathscr{H}\{U(x,t;p); w_0(x,t), H(x,t), h, p\} = (1-p)(\mathscr{L}\{U(x,t;p) - w_0(x,t)\}) - phH(x,t)N\{U(x,t;p)\} \tag{8}$$

where $w_0(x,t)$ is an first guess of solution of equation (24) that fulfills the initial/boundary constraints (45). p is an embedding parameter.$\hbar$ is a convergence control parameter and $H(x,t)$is an auxiliary function.

Implementing a zero value for the homotopy equation(8), we have

$$\mathscr{H}\{U(x,t;p);w_0(x,t),H(x,t),\hbar,p\}=0 \tag{9}$$

We develop a homotopy of zero order deformation equation as

$$(1-p)(\mathscr{L}\{U(x,t;p)-w_0(x,t)\}) = p\hbar H(x,t)N\{U(x,t;p)\} \tag{10}$$

Putting $p=0$ into the equation (10), we get

$$U(x,t;0)=w_0(x,t) \tag{11}$$

Putting $p=1$ into the equation (10), we get

$$U(x,t;1)=w(x,t) \tag{12}$$

As the embedding parameter grows from zero to unity, the solution progressively shifts from the initial prediction to the desired solution. We have, according to Taylor's theorem,

$$U(x,t;p)=U(x,t;0)+\sum_{k=1}^{\infty}w_k(x,t)p^k \tag{13}$$

where

$$w_k(x,t)=\frac{1}{k!}\frac{\partial^k U(x,t;p)}{\partial p^k}\bigg|_{p=0} \tag{14}$$

If the first prediction, convergence control parameter, and auxiliary function are all chosen correctly, the series converges at $p=1$, and we obtain

$$w(x,t)=w_0(x,t)+\sum_{k=1}^{\infty}w_k(x,t) \tag{15}$$

which unquestionably illustrates one of the solutions to the nonlinear partial differential equation.

By differentiating equation (10)k times with respect to p, diving by $k!$and setting p to be zero, we can obtain the following k th order deformation equation

$$\mathscr{L}\{w_k(x,t)-\chi_k w_{k-1}(x,t)\}=\hbar H(x,t)\mathscr{R}_k(\vec{w}_{k-1}) \tag{16}$$

where

$$\mathscr{R}_k(\vec{w}_{k-1})=\frac{1}{(k-1)!}\frac{\partial^{k-1}N\{U(x,t;p)\}}{\partial p^{k-1}}\bigg|_{p=0} \tag{17}$$

and

$$\chi_k=\begin{cases}0, & k\le 1\\ 1, & k>1\end{cases} \tag{18}$$

Equation (16) yields the following results when the inverse Laplace transform is applied to both sides

$$w_k(x,t)=\chi_k w_{k-1}(x,t)+\hbar\mathscr{L}^{-1}\{H(x,t)\mathscr{R}_k(\vec{w}_{k-1})\} \tag{19}$$

which is an approximate solution of equation (2). The series (15) leads to an exact solution when infinite series converges.

III. THE SOLUTION OF THE TWO SPECIAL CASE OF THE POROUS MEDIUM EQUATION WITH (A)$m=-2$ AND(B)$m=1$

Example-1. Let us take

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left(w^{-2}\frac{\partial w}{\partial x}\right) \tag{20}$$

with the initial condition $w(x,0)=(2c_1x)^{\frac{-1}{2}}$ (21)

Solution:

We can rewrite equation (20) as

$$\frac{\partial w}{\partial t}=w^{-2}\frac{\partial^2 w}{\partial x^2}-2w^{-3}\left(\frac{\partial w}{\partial x}\right)^2 \tag{22}$$

By using Laplace transformation on both side of equation (22), we have

$$\mathscr{L}\left\{\frac{\partial w}{\partial t}\right\}=\mathscr{L}\left\{w^{-2}\frac{\partial^2 w}{\partial x^2}-2w^{-3}\left(\frac{\partial w}{\partial x}\right)^2\right\} \tag{23}$$

Employing the property of Laplace transform, we have

$$s\mathscr{L}\{w(x,t)\}-w(x,0) = \mathscr{L}\left\{w^{-2}\frac{\partial^2 w}{\partial x^2}-2w^{-3}\left(\frac{\partial w}{\partial x}\right)^2\right\} \tag{24}$$

$$\mathscr{L}\{w(x,t)\}=\frac{w(x,0)}{s}+\frac{1}{s}\mathscr{L}\left\{w^{-2}\frac{\partial^2 w}{\partial x^2}-2w^{-3}\left(\frac{\partial w}{\partial x}\right)^2\right\} \tag{25}$$

The nonlinear operator is considered as

$$N[U(x,t;p)]=\mathscr{L}\{U(x,t;p)\}-\frac{1}{s}\left[(2c_1x)^{\frac{-1}{2}}\right]-\frac{1}{s}\mathscr{L}\left\{\left(U(x,t;p)\right)^{-2}\frac{\partial^2 U(x,t;p)}{\partial x^2}-2\left(U(x,t;p)\right)^{-3}\left(\frac{\partial U(x,t;p)}{\partial x}\right)^2\right\} \tag{26}$$

We can create a homotopy as follows:

$$\mathscr{H}\{U(x,t;p);w_0(x,t),H(x,t),\hbar,p\} = (1-p)\mathscr{L}\{U(x,t;p)-w_0(x,t)\} - p\hbar H(x,t)N\{U(x,t;p)\} \tag{27}$$

where $w_0(x,t)=(2c_1x)^{\frac{-1}{2}}$ is an first guess of solution of equation (20) that fulfills the initial constraint (21)

A homotopy of the zero-order deformation equation is generated as

$$(1-p)\mathscr{L}\{U(x,t;p)-w_0(x,t)\} = p\hbar H(x,t)(N\{U(x,t;p)\}) \tag{28}$$

where p is an embedding parameter. $\hbar$is a convergence control parameter and $H(x,t)$is an auxiliary function.

It is obvious that when p is zero and p is one, equation (28) becomes

$$U(x,t;0) = w_0(x,t) = (2c_1 x)^{\frac{-1}{2}} \tag{29}$$

and $U(x,t;1) = w(x,t)$ (30)

We can get k th-order deformation equation as

$$\mathscr{L}\{w_k(x,t) - \chi_k w_{k-1}(x,t)\} = \hbar H(x,t)\mathscr{R}_k(\vec{w}_{k-1}) \tag{31}$$

Where

$$\mathscr{R}_k(\vec{w}_{k-1}) = \frac{1}{(k-1)!}\frac{\partial^{k-1}\{U(x,t;p)\}}{\partial p^{k-1}}\bigg|_{p=0} \tag{32}$$

and

$$\chi_k = \begin{cases} 0, & k \le 1 \\ 1, & j > 1 \end{cases} \tag{33}$$

When both sides of equation (31) are subjected to the inverse Laplace transform, and $H(x,t)$ is set to 1, we obtain

$$w_k(x,t) = \chi_k w_{k-1}(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_k(\vec{w}_{k-1})] \tag{34}$$

Taking $k = 1$ in the above equation(34), we obtain

$$w_1(x,t) = \chi_1 w_0(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_1(\vec{w}_0)]$$

$$w_1(x,t) = \hbar\mathscr{L}^{-1}[\mathscr{R}_1(\vec{w}_0)] \qquad \text{since } \chi_1 = 0$$

$$= \hbar\mathscr{L}^{-1}\left\{\mathscr{L}\{w_0\} - \frac{(2c_1x)^{\frac{-1}{2}}}{s} - \frac{1}{s}\mathscr{L}\left\{{w_0}^{-2}\frac{\partial^2 w_0}{\partial x^2} - 2{w_0}^{-3}\left(\frac{\partial w_0}{\partial x}\right)^2\right\}\right\} \tag{35}$$

$$= -\hbar {c_1}^2 (2c_1x)^{\frac{-3}{2}} t$$

Taking $k = 2$ in the equation(34), we obtain

$$w_2(x,t) = \chi_2 w_1(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_2(\vec{w}_1)]$$

$$w_2(x,t) = w_1(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_2(\vec{w}_1)] \quad \text{since } \chi_2 = 1$$

$$w_2(x,t) = \hbar\left(-{c_1}^2(2c_1x)^{\frac{-3}{2}}\right)t + \hbar^2\left(-{c_1}^2(2c_1x)^{\frac{-3}{2}}t + 3{c_1}^4(2c_1x)^{\frac{-5}{2}}\frac{t^2}{2!}\right) \tag{36}$$

Substituting the values from equations (29), (35)and (36) in equation (15), we have

$$w(x,t) = \sum_{k=0}^{\infty} w_k(x,t) = w_0 + w_1 + w_2 + \cdots$$
$$= (2c_1x)^{\frac{-1}{2}} + 2\hbar\left(-{c_1}^2(2c_1x)^{\frac{-3}{2}}t\right) + \hbar^2\left(-{c_1}^2(2c_1x)^{\frac{-3}{2}}t + 3{c_1}^4(2c_1x)^{\frac{-5}{2}}\frac{t^2}{2!}\right) \tag{37}$$

We draw the $\hbar$ -curves of the 15th order approximation utilizing math software Maple. The $\hbar$-curves of $w_t(1,1), w_{tt}(1,1), w_{ttt}(1,1)$ are given in Fig. 1. To achieve a convergent series solution, we select the appropriate value of the convergence control parameter $\hbar = -1$.

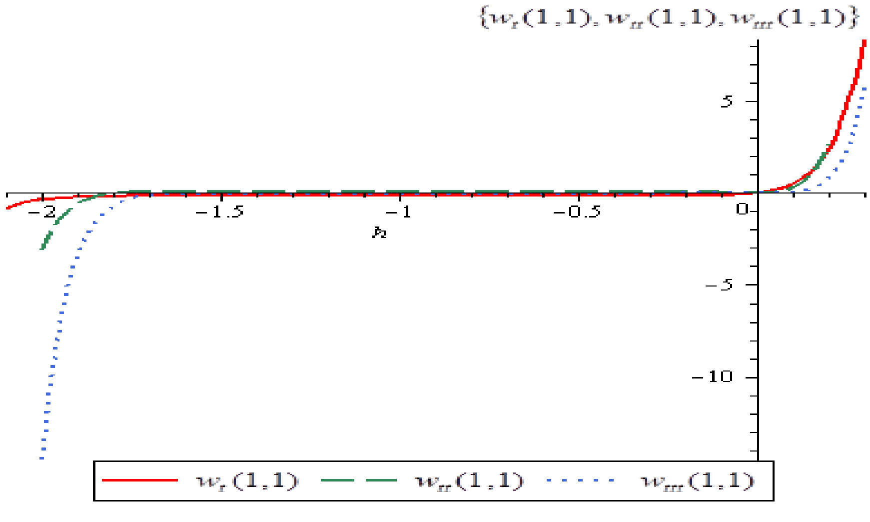

Fig. 1: The $\hbar$ -curves of $\boldsymbol{w_t(1,1), w_{tt}(1,1), w_{ttt}(1,1)}$

Plugging $\hbar = -1$ in the equation (37), we have

$$w(x,t) = (2c_1x)^{\frac{-1}{2}} + \left(-c_1^2(2c_1x)^{\frac{-3}{2}}\right)t + \left(3c_1^4(2c_1x)^{\frac{-5}{2}}\right)\frac{t^2}{2!} + \cdots = (2c_1x - 2c_1^2t)^{\frac{-1}{2}} \quad (38)$$

Example: 2

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) \quad (39)$$

with the initial condition $w(x,0) = \frac{-(x+c_1)^2}{6c_2} + \frac{c_3}{c_2^{\frac{1}{3}}}$ (40)

Solution:

$$\frac{\partial w}{\partial t} = w\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 \quad (41)$$

By using Laplace transformation on both side of equation (41), we have

$$\mathscr{L}\left\{\frac{\partial w}{\partial t}\right\} = \mathscr{L}\left\{w\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2\right\} \quad (42)$$

Employing the property of Laplace transform, we have

$$s\mathscr{L}\{w(x,t)\} - w(x,0) = \mathscr{L}\left\{w\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2\right\} \quad (43)$$

$$\mathscr{L}\{w(x,t)\} = \frac{w(x,0)}{s} + \frac{1}{s}\mathscr{L}\left\{w\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2\right\} \quad (44)$$

The nonlinear operator is considered as

$$N[U(x,t;p)] = \mathscr{L}\{U(x,t;p)\} - \frac{1}{s}\left[\frac{-(x+c_1)^2}{6c_2} + \frac{c_3}{c_2^{\frac{1}{3}}}\right] - \frac{1}{s}\mathscr{L}\left\{U(x,t;p)\frac{\partial^2 U(x,t;p)}{\partial x^2} + U(x,t;p)\left(\frac{\partial U(x,t;p)}{\partial x}\right)^2\right\} \quad (45)$$

We can create a homotopy as follows:

$$\mathscr{H}\{U(x,t;p); w_0(x,t), H(x,t), \hbar, p\} = (1-p)\mathscr{L}\{U(x,t;p) - w_0(x,t)\} - p\hbar H(x,t)N\{U(x,t;p)\} \quad (46)$$

where $w_0(x,t) = (2c_1x)^{\frac{-1}{2}}$ is an first guess of solution of equation(39) that fulfills the initial constraint(40).

A homotopy of the zero-order deformation equation is generated as

$$(1-p)\mathscr{L}\{U(x,t;p) - w_0(x,t)\} = p\hbar H(x,t)(N\{U(x,t;p)\}) \quad (47)$$

where p is an embedding parameter. $\hbar$is a convergence control parameter and $H(x,t)$is an auxiliary function.

It is obvious that when p is zero and p is one, equation (47) becomes

$$U(x,t;0) = w_0(x,t) \quad (48)$$

and

$$U(x,t;1) = w(x,t) \quad (49)$$

We can get k th-order deformation equation as

$$\mathscr{L}\{w_k(x,t) - \chi_k w_{k-1}(x,t)\} = \hbar H(x,t)\mathscr{R}_k(\vec{w}_{k-1}) \quad (50)$$

Where

$$\mathscr{R}_k(\vec{w}_{k-1}) = \frac{1}{(k-1)!}\frac{\partial^{k-1}\{U(x,t;p)\}}{\partial p^{k-1}}\bigg|_{p=0} = \mathscr{L}[w_{k-1}(x,t)] - (1-\chi_k)\frac{1}{s}\left[\frac{-(x+c_1)^2}{6c_2} + \frac{c_3}{c_2^{\frac{1}{3}}}\right] - \frac{1}{s}\mathscr{L}\left\{\sum_{j=0}^{k-1} w_j\frac{\partial^2 w_{k-1-j}}{\partial x^2} + \sum_{j=0}^{k-1}\left(\frac{\partial w_j}{\partial x}\right)\left(\frac{\partial w_{k-1-j}}{\partial x}\right)\right\} \quad (51)$$

And

$$\chi_k = \begin{cases} 0, & k \le 1 \\ 1, & j > 1 \end{cases} \quad (52)$$

When both sides of equation (50) are subjected to the inverse Laplace transform, and $H(x,t)$ is set to 1, we obtain

$$w_k(x,t) = \chi_k w_{k-1}(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_k(\vec{w}_{k-1})] \quad (53)$$

Taking $k = 1$ in the above equation, we obtain

$$w_1(x,t) = \chi_1 w_0(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_1(\vec{w}_0)]$$

$$w_1(x,t) = \hbar\mathscr{L}^{-1}[\mathscr{R}_1(\vec{w}_0)] \qquad \text{since } \chi_1 = 0$$

$$= \hbar\mathscr{L}^{-1}\left\{\frac{\partial w_0}{\partial t} - w_0\frac{\partial^2 w_0}{\partial x^2} - \left(\frac{\partial w_0}{\partial x}\right)^2\right\}$$

Substituting the value of $w_0(x,t) = \frac{-(x+c_1)^2}{6c_2} + \frac{c_3}{c_2^{\frac{1}{3}}}$ in the previous equation, we have

$$w_1(x,t) = \hbar\left(\frac{-(x+c_1)^2c_2^{-2}}{6} + \frac{1}{3}c_3\left(c_2^{\frac{-4}{3}}\right)\right)t \quad (54)$$

Taking $k = 2$ in the equation (53), we have

$$w_2(x,t) = \chi_2 w_1(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_2(\vec{w}_1)]$$

$$w_2(x,t) = w_1(x,t) + \hbar\mathscr{L}^{-1}[\mathscr{R}_2(\vec{w}_1)] \quad \text{since } \chi_2 = 1$$

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$$w_2(x,t) = \hbar\left(\frac{-(x+c_1)^2 c_2^{-2}}{6} + \frac{1}{3}c_3\left(c_2^{\frac{-4}{3}}\right)\right)t + \hbar^2\left(\frac{(x+c_1)^2 c_2^{-2} t}{6} - \frac{1}{3}c_3\left(c_2^{\frac{-4}{3}}\right)t - \frac{1}{3}\frac{(x+c_1)^2 c_2^{-3} t^2}{2!} + \frac{4}{9}c_3\left(c_2^{\frac{-7}{3}}\right)\frac{t^2}{2!}\right) \quad (55)$$

Substituting the values from equations (48),(54) and (55) in equation (1), we have

$$w(x,t) = \sum_{k=0}^{\infty} w_k(x,t) = w_0 + w_1 + w_2 + \cdots$$

$$= \frac{-(x+c_1)^2 c_2^{-1}}{6} + c_3\left(c_2^{\frac{-1}{3}}\right) + 2\hbar\left(\frac{-(x+c_1)^2 c_2^{-2}}{6} + \frac{1}{3}c_3\left(c_2^{\frac{-4}{3}}\right)\right)t + \hbar^2\left(\frac{(x+c_1)^2 c_2^{-2} t}{6} - \frac{1}{3}c_3\left(c_2^{\frac{-4}{3}}\right)t - \frac{1}{3}\frac{(x+c_1)^2 c_2^{-3} t^2}{2!} + \frac{4}{9}c_3\left(c_2^{\frac{-7}{3}}\right)\frac{t^2}{2!}\right) + \cdots \quad (56)$$

We draw the $\hbar$ -curves of the 15th order approximation utilizing math software Maple. The $\hbar$-curves of $w_t(1,0), w_{tt}(1,0), w_{ttt}(1,0)$ are given in figure Fig: 2. To achieve a convergent series solution, we select the appropriate value of the convergence control parameter $\hbar = -1$.

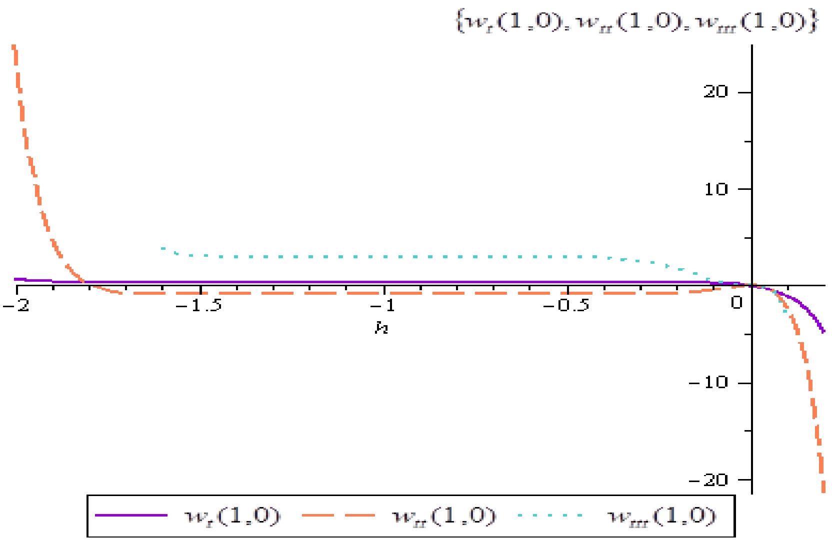

Fig. 2: The $\hbar$ -curves of$w_t(1,0), w_{tt}(1,0), w_{ttt}(1,0)$

Plugging$\hbar = -1$ in the equation (56),we have

$$w(x,t) = \frac{-(x+c_1)^2 c_2^{-1}}{6} + c_3\left(c_2^{\frac{-1}{3}}\right) + \left(\frac{(x+c_1)^2 c_2^{-2}}{6} - \frac{1}{3}c_3\left(c_2^{\frac{-4}{3}}\right)\right)t + \left(-\frac{(x+c_1)^2 c_2^{-3}}{3} + \frac{4}{9}c_3\left(c_2^{\frac{-7}{3}}\right)\right)\frac{t^2}{2!} + \cdots = \frac{-(x+c_1)^2}{6(t+c_2)} + \frac{c_3}{(t+c_2)^{\frac{1}{3}}} \quad (57)$$

IV. CONCLUSION

The primary concern of such paper is to build an analytical solution with various powers of$m$for porous medium equation. Our aim is fulfilled by applying the Homotopy Analysis Laplace Transform Method (HALTM) on the porous medium equation. The major benefit of such a method is that it gives analytical approximation, in most situations an exact solution. This method is easy and short in calculation compare to other available numerical methods. Its convergence rate is depend on convergence control parameter $\hbar$which indicate that the technique is accurate and significantly improves the solution of differential equations over current technique. At last, we suggest that HALTM can be seen as a good refining of the current numerical technique.

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