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# Power Set of Natural Numbers is Countable

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**Abstract:- This paper explains the Cardinality of the Power Set of Natural numbers. Set of Natural numbers is countable, in the same way the power set of Natural numbers is also countable as every subset of the Power set of Natural numbers is Countable. Prime numbers and Well ordering Principle play a very important role in proving this result. Since every Subset of Natural numbers is Countable, there exists a bijection between the Power Set of Natural numbers and a proper subset of Natural numbers.**

***Keywords:-*** *Subset, Power set , Order of a Power Set of a Set, Well ordering Principle, Prime Numbers, countable set, bijection, Uncountable Set.*

## I. INTRODUCTION

Set N of natural numbers is infinite but countable. The Power Set of Every Finite Set is Countable if A contains 5 elementsi.e O( A) = 5 . then its power set contains O(P( A)) = $2^5$ = 32 elements O(P( A) > O(A) but still countable. In the same way P( N ) is also countable.

- **Lemmas**
  - **Axiom** : Well Ordering Principle : Every non empty Subset of Natural numbers contains its least element.
  - **Axiom** : An infinite set A is said to be Countable if there exists one- one correspondence between A and Set of
  - Natural Numbers or between A and a subset of Natural Numbers.
  - **Theorem** : Every subset of a Countable Set is Countable.
  - **Theorem** : Every countable union of Countable Sets is countable.

## II. MAIN RESULT

$N$ = { 1, 2, 3, 4, ………} is a countable set of Natural Numbers. Every subset of a countable set is also countable.

Consider S = { 1, $p^{k}$, $(p.q)^k$ ; k, p, q ∈ $N$ and p & q are prime numbers with ( p, q ) = 1 } ⊂ $N$ .

As S is a proper Subset of Natural numbers , therefore S is a Countable Set

Hence | S| = |$N$| = $\aleph_0$

P( $N$ ) = { $\phi$, { 1 }, { 2 }, { 3 }, ………. } ( collection of all subsets of natural numbers )

Define f : P ( $N$ ) ⟶ S , such that
$\phi$ ⟶1
{ n } ⟶$2^n$
{ $n_1$ , $n_2$ } ⟶ $3^n$, ( with the help of Well ordering Principle)
{ $n_1$, $n_2$ , $n_3$ } ⟶ $5^n$.
.
$2^n \neq 3^n \neq 5^n \neq$ ….. . .

Since Set of Natural numbers is countable therefore every member of P ( $N$ ) is countable set ,so order of each finite member of P ( $N$ ) is a Whole number and every non empty Subset of Natural numbers whether finite or infinite must contain the smallest element ( well ordering principle),thus collection of all finite subsets of Natural numbers is countable.

Now infinite subsets of Natural numbers can be classified into three categories: 1. Even subsets 2. Odd subsets 3. Mixed Subsets. Every infinite subset of Natural numbers must have a least positive integer

### A. *Infinite Even Subsets*

$E_1$ = { 2,4,6,8,………, $e_{1_k}$ , ………. } is the first infinite Subset of Natural numbers, $E_2$ = { 2,4,6,8, ……….$e_{2_k}$,……… } is the second even subset of Natural numbers where $e_{1_k} < e_{2_k}$

Define the map f: P ($N$) → S

$E_1 \to (2.3)^1$
$E_2 \to (2.3)^2$
$E_3 \to (2.3)^3$
. .
. .
$E_n \to (2.3)^n$
. .
. .
. .

### B. *Infinite Odd Subsets:*

$O_1$ = { 1,3,5,7,9, …… , $\boldsymbol{o_{1_k}}$,……} , $O_2$ = { 1,3,5,7,9,………,$\boldsymbol{o_{2_k}}$ ,………. },…………

$o_{1_k} < o_{2_k}$
Define the map f: P ($N$) → S

$O_1 \to (3.5)^1$
$O_2 \to (3.5)^2$
$O_3 \to (3.5)^3$
. .
. .
$O_n \to (3.5)^n$
. .
. .
. .

### *C. Infinite Mixed Subsets :*

Each Mixed Infinite subset of natural numbers contain Even and odd natural numbers.

$M_1 = \{ 1, 2, 3, 4, 5, \ldots\ldots\ldots\ldots, m_{1_k} \ldots\ldots \} = N$ ,
$M_2 = \{ 1, 2, 3, 4, 5, \ldots\ldots\ldots\ldots, m_{2_k} \ldots\ldots \}$

$m_{1_k} < m_{2_k}$

Define the map f: P $(N) \rightarrow$ S

$M_1 \quad (2.5)^1$

$M_2 \rightarrow (2.5)^2$

$M_3 \rightarrow (2.5)^3$

. .

. .

$M_n \rightarrow (2.5)^n$

. .

. .

. .

Here The map defined f: P $(N) \rightarrow$ S is a one - one map ,But S is Countable as S is a subset of $N$ , Which is Countable Set and every subset of a countable set is Countable.

$S_1 = \{ 1, p^n , 6^n , 10^n , 15^n , \forall n \in N, p \text{ is a prime number} \}$

$S_1$ is a proper subset of S and hence proper subset of $N$

The function $f_1$ : P $(N) \rightarrow S_1$ defined as

$f_1$ ( A) = f (A) $\forall$ A$\in$ P $(N)$ is a Bijection from P $(N)$ to a countable subset of $N$ hence P $(N)$ is countable.

$2^{\aleph_0} = \aleph_0$

## III. CONCLUSION

Another way of writing elements of P $(N)$ = { $\phi$, { 1 }, {2}, {1,2}, {3}, { 1, 3 }, { 2, 3 }, { 1, 2, 3 }, {4}, { 1, 4}, {2, 4 }, {1,2,4 }, {3,4}, { 1,3,4 }, { 2,3,4} , { 1,2,3,4}, {5} ………………}

$\phi$ is the first element of P$(N)$ , second element is Singleton set of first natural number {1}, $\phi \cup \{1\} = \{1\}$ has already written so second natural number is 2 so the third element is {2} , $\phi \cup \{2\} = \{2\}$ already written, so 4th element of P$(N)$ is $\{2\} \cup \{1\} = \{1,2\}$ ,$\{2\} \cup \{2\} = \{2\}$ already written, 5th element is a singleton set containing next natural number i.e. {3} . Whenever a singleton set is written next elements would be the union of that set with previous elements, to ignore those elements which have already been written.

Every Subset of P $(N)$ is Countable as every element of The subset of P $(N)$ can be arranged in a systematic way using the Well ordering Principle. Whether the Power Set of a Set contains more elements but it does not mean that it is uncountable.

P $(N) = K_0 \cup (\cup_{K_{n \in P(N)}} K_n) \cup_{I_{n \in P(N)}} \{I_n\}$ , *where* $I_n$ Is an infinite Subset of $N$ .

$K_n$ = { A ; A$\in$ P$(N)$ & O( A ) = n } , $K_0 = \phi$.

$K_1 \cong N, \qquad I_1 = N$

$K_2 \subset N \text{ X } N I_2 = N_1$

$K_3 \subset N \text{ X } N \text{ X } N I_3 = N_2$

$K_4 \subset N \text{ X } N \text{ X } N \text{ X } N I_4 = N_3$

. .

.

$K_n \subset N^n I_n = N_n$

. .

. .

each $I_n$ is an infinite subset of the set of Natural numbers must having a least positive integer according to the well ordering Principle.

The countable union of Countable Sets is countable. Hence

P $(N)$ is Countable.

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