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New Exact Solution of a Class of Kuramoto Sivashinsky (KS) Equations using Double Reduction Theory

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Abstract:- Conservation laws and symmetries of partial differential equations (PDEs) are very useful in finding new methods for reducing PDEs. In this paper, we study the conservation laws and symmetries of a class of a famous fourth-order Kuramoto Sivashinsky (KS) equation. The invariance properties of the conserved vectors with the Lie point symmetry generators are examined using the Double reduction method. With the Double reduction method, the equation is reduced into solvable PDEs or even ordinary differential equations. Some of these reductions yielded some important differential equations that have been investigated by many reseachers. Furthermore, we obtain important and nontrivial solution in terms of generalized Hypergeometric function which possesses significant features in evolution phenomena. Our results not only contributed extra features to the already existing solutions in literature but are also useful in the analysis of wave propagation in plasma, solid state and fluid physics.

Keywords:- *Lie Symmetries; Conservation Laws; Double Reduction; Exact Solutions.*

I. INTRODUCTION

A large number of global physical systems are defined by nonlinear partial differential equations (NLPDEs). These equations are very necessary because they can show case the real features in many course of applications, for example, gas dynamics, fluid mechanics, relativity, thermodynamics, combustion theory, biology and many others. It is very hard to solve analytically NLPDEs of real life problems. Obtaining exact solutions of the NLPDEs is a very crucial task and plays a vital role in nonlinear sciences.

A lot of effective methods for determining exact solutions of NLPDEs have been found and developed in the past few years. Some of these methods are the inverse scattering transform method [1], Bäcklund transformation [2], Darboux transformation [3], Hirota's bilinear method [4], the homogeneous balance method [5], the extended tanh method [6], the exp-function method [7] and Lie group analysis. [8,9,10,11].Marius Sophus Lie (1842-1899) originally developed Lie group analysis. His study resulted in the current theory of what is now globally as a group called Lie. Subsequently, numerous studies have been produced in literature on the subject of Lie groups applied to differential equations in regard to the Lie point symmetries by the equation understudy. One parameter point transformation which permit the differential equation to be invariant is the Lie point symmetries of a differential equation. The invariance property of symmetries is essential in the sense that mapping a differential equation from one form to another maintaining its fundamental properties without alteration. Symmetries are used for reduction of order of scalar ODEs [8]. For PDEs, symmetries is applied to minimize the existing PDEs system to ODEs that could yield an exact solution to the PDE when solved.

In the mid1980s, Peter G. L. Leach and his student, Fazal M. Mahomed [12] introduced Lie group theory to South Africa. Afterwards, this line of study researcher from various field of study started showcasing interest in which differential equation play a vital role. Lie group theory has vast applications in solid-state mechanics, modern physics, fluid mechanics, biological and physical systems, to name a few. Lately, lots of research have been carried out on the applications of Lie group theory to PDEs in various fields of engineering and natural sciences. These include linearization of ODEs and PDEs, generating new solutions from existing ones, construction of equivalence groups, solving group classification problems, reductions of PDEs (by invariant or similarity solutions), solving initial and boundary value problems, construction of generalized local symmetries and nonlocal symmetries, approximate symmetries, symmetries of difference equations, symmetries of functional differential equations, symmetries of stochastic differential equations, symmetries of integro-differential equations, symmetries of geodesic equations, construction of conservation laws, construction of invariants of algebraic and differential equations etc.

The conservation laws contribute immensely in the analysis of differential equations. Conservation law is used to describe physical conserved quantities such as energy, momentum, mass and angular momentum, as well as charge and other constants of motion [13] . They have been utilized in the study of the uniqueness, stability and existence of solutions of NLPDEs [8]. In addition, they have been used in the use and development of numerical methods [14]. The integrability of a PDE is significant on the presences of a large number of conservation laws of the PDE [8]. Various methods of applying conservation laws on PDEs have been elaborated on. The study of conservation laws is closely connected to that of Lie symmetries due to the work of Emmy Noether [15]. Noether's theorem gives a constructive and stylish way of solving conservation laws for a system of PDEs that has a Lagrangian formulation [16]. In the calculus of variations, the main problem is finding the Lagrangian, such that an Euler-Lagrange equation is generated from a differential equation. This issues is known as the inverse problem in the calculus of variations [15, 17].

Conservation laws can also be obtained through other means without making use of a Lagrangian. These include the direct method introduced by Anco and Bluman[18,19], the multipliers approach which involves writing a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations [20] and the partial Noether approach, introduced by Kara and Mahomed [21]. The Noether approach for differential equations with "a partial Lagrangian" is the same as the partial Noether approach. Noether symmetry is associated with conservation law for variational systems which has already been established.

Recently, the thought of relating Noether symmetries with conservation laws was extended to Lie-Bäcklund symmetries [22] and non-local symmetries [23]. The association of a conserved vector with symmetry results to double reduction theory for PDEs which was done on GB equation and VB equations by Okeke Et. al [24] to obtain exact solution.

In many physical problems like quantum field theory and solid mechanics etc. there emerges a very distinguished and remarkable NLPDEs, which is known as Kuramoto-Sivashinsky (KS) equation [25]

$$u_{tt}+\alpha u_{xxxx}-\Upsilon(u_x^n)_x = 0 \tag{1}$$

The KS equation (1) is a fourth order NLPDE derived by Yoshinki Kuramoto and Gregory Sivashinsky. They used the equation to model the diffusive instability in a flame front in the late 1970's [26].

The extensive use of conservation law and symmetry has helped to reduce the NLPDEs to ODEs which in many cases are not complicated to solve. Okeke Et.al [27] used various other methods including the extended-tanh method and Lie analysis method on equation (1) to obtain the exact solutions.

Thus, we will show that an invariant of a conservation law under Lie point symmetry leads to double reduction of the KS equation (1) which can be easily solvedto obtain new exact solutions.

II. METHODOLOGY

A. Definition And Notations

This section contains definitions, notations and theorems used in this work.

A k^{th}-order ($k \geq 1$) system of s partial differential equations of n-independent variables $x = (x^1, x^2, ..., x^n)$ and m-dependent variables $u = (u^1, u^2, ..., u^m)$ is defined by

$$E^\sigma (x, u, u_{(1)}, ..., u_{(k)}) = 0, \ \sigma = 1, ..., s, \tag{2}$$

where $u_{(1)}, ..., u_{(k)}$ denote the collection of all first, second, ..., k^{th}-order partial derivatives.

- **Definition1.** The Euler operator, for each α, is defined by

$$\frac{\delta}{\partial u^\alpha} = \frac{\partial}{\partial u^\alpha}+\sum_{s\geq1}(-1^s)\, D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_{1\dots is}}} \ ,\alpha = 1,\dots,m \tag{3}$$

The Euler operator is also sometimes referred to as the Euler-Lagrange operator [18] where

$$D_i = \frac{\delta}{\partial x^i}+u_i^\alpha\frac{\delta}{\partial u^\alpha}+u_{ij}^\alpha\frac{\delta}{\partial u_j^\alpha}+\dots, \quad i=1, 2,\dots,n \tag{4}$$

is the total derivative operator with respect to x^i. The Euler-Lagrange equations associated with (2) are

$$\frac{\partial L}{\partial u^\alpha} = 0, \ \alpha = 1,2, ...,m \tag{5}$$

where L is referred to as a Lagrangian of (2)

- **Definition 2**. The Lie-Bäcklund or generalized operator is given by

$$X = \xi\frac{\partial}{\partial x_i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}. \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{6}$$

where $\mathcal{A}$ is taken to be the universal vector space of differential functions.An expanded form of the operator (6) is the infinite formal sum given as

$$X = \xi\frac{\partial}{\partial x_i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\sum_{s\geq1}\zeta^\alpha_{i_1},\dots,_{is}\frac{\partial}{\partial u^\alpha_{i_{1\dots is}}} \tag{7}$$

where the $\varsigma^\alpha_{i_1}, \dots ,_{is}$ can be determined from

$$\varsigma^\alpha_{i_1} = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi_j)\,,$$

$$\varsigma^\alpha_{i_1,} \dots,_{is} = D_{is}\,\varsigma^\alpha_{i_1,} \dots,_{is-1} - u^\alpha_{ji_{1,\dots,I\,s-1}} D_{is}\,(\xi^j),\ s>1. \tag{8}$$

The Lie point symmetry of equation (2) is a generator X of the form (7) that satisfies

$$X^{[k]}\,E_{|E=0} = 0, \tag{9}$$

where $X^{[k]}$ is the k^{th} prolongation of X.

$$X^{[k]} = \xi(x,u)\frac{\partial}{\partial x_i}+\eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}+\zeta^\alpha_{i_1}(x, u, u_{(1)})\frac{\partial}{\partial u_i^\alpha}+\dots +\zeta^\alpha_{i_1}\dots,_{ik}(x, u, u_{(k)})\frac{\partial}{\partial u^\alpha_{i_{1\dots ik}}} \tag{10}$$

A Lie-Bäcklund operator X of the form (7) is called a Noether symmetry generator associated with a Lagrangian$L$ of (5) if there exists a vector $B = (B^1, B^2, \dots, B^n)$ such that

$$XL + LD_i(\xi^i) = D_i(B^i) \tag{11}$$

- **Definition 3:** A conserved vector of (2) is n-tuple

$T = (T^1, T^2, \dots, T^n), T^j = T^j(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) \in \mathcal{A}, j = 1, \dots, n$ such that

$D_i T^i = 0$ (12)

is satisfied for all solutions of (2).

Remark: Once the conditions in Definition 3 are met, (12) is called a conservation law for (2).

B. Symmetry and conservation law relation

- **Definition 4**[22]A Lie-Bäcklund symmetry generator X of the form (7) is associated with a conserved vector T of the system (2) if X and T satisfy the conditions

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, i = 1, \dots, n \quad (13)$$

The conserved vectors T^i are found using equations (12) and (13)

C. Double reduction of PDEs

a) Main Theorems

The first theorem will help us to investigate whether the symmetry X is associated with a conserved vector T.

Theorem 1: [22]Suppose that X is any Lie Bäcklund symmetry of equation (2) and $T^i, i = 1, \dots, n$ are the components of its conserved vectors. Then

$$T^{*i} = (T^i, X) = X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, i = 1, \dots, n \quad (14)$$

form the components of a conserved vector of (2), i.e.,

$$D_i T^i|_{(2)} = 0$$

Theorem 2. [28] Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (2). Then under a contact transformation, there exist functions $\widetilde{T^i}$ such that $J D_i T^i = \widetilde{D_i}\widetilde{T^i}$ where $\widetilde{T^i}$ is given as

$$\begin{pmatrix} \widetilde{T^1} \\ \widetilde{T^2} \\ \vdots \\ \widetilde{T^n} \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \widetilde{T^1} \\ \widetilde{T^2} \\ \vdots \\ \widetilde{T^n} \end{pmatrix} \quad (15)$$

in which

$$A = \begin{pmatrix} \widetilde{D_1}x_1 & \widetilde{D_1}x_2 & \dots & \widetilde{D_1}x_n \\ \widetilde{D_2}x_1 & \widetilde{D_2}x_2 & \cdots & \widetilde{D_2}x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D_n}x_1 & \widetilde{D_n}x_2 & \cdots & \widetilde{D_n}x_n \end{pmatrix} \quad (16)$$

$$A^{-1} = \begin{pmatrix} D_1\widetilde{x_1} & D_1\widetilde{x_2} & \dots & D_1\widetilde{x_n} \\ D_2\widetilde{x_1} & D_2\widetilde{x_2} & \cdots & D_2\widetilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n\widetilde{x_1} & D_n\widetilde{x_2} & \cdots & D_n\widetilde{x_n} \end{pmatrix} \quad (17)$$

and J = det (A).

Theorem 3.[28] (fundamental theorem on double reduction) Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (2). Then under a similarity transformation of a symmetry X of the form (7) for the PDE, there exist functions $\widetilde{T^i}$ such that X is still a symmetry for the PDE satisfying $\widetilde{D_i}\widetilde{T^i} = 0$
and

$$\begin{pmatrix} X\widetilde{T^1} \\ X\widetilde{T^2} \\ \vdots \\ X\widetilde{T^n} \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \quad (18)$$

Where

$$A = \begin{pmatrix} \widetilde{D_1}x_1 & \widetilde{D_1}x_2 & \dots & \widetilde{D_1}x_n \\ \widetilde{D_2}x_1 & \widetilde{D_2}x_2 & \cdots & \widetilde{D_2}x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D_n}x_1 & \widetilde{D_n}x_2 & \cdots & \widetilde{D_n}x_n \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} D_1\widetilde{x_1} & D_1\widetilde{x_2} & \dots & D_1\widetilde{x_n} \\ D_2\widetilde{x_1} & D_2\widetilde{x_2} & \cdots & D_2\widetilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n\widetilde{x_1} & D_n\widetilde{x_2} & \cdots & D_n\widetilde{x_n} \end{pmatrix}$$

and J = det(A).

Theorem 4.[28] A PDE of order n with two independent variables, which admits symmetry X that is associated with a conserved vector T, can be reduced to an ODE of order $n - 1$, namely T^r= k, where T^r is defined in (15).

D. Lie point symmetry of KS equation

The Lie point symmetries of the KS equation (1) takes the form

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \quad (19)$$

The operator X satisfies the Lie symmetry criteria (9)

$$X^{[4]}[u_{tt} + \alpha u_{xxxx} - Y(u_x^n)_x]\big|_{(1)} = 0 \quad (20)$$

$X^{[4]}$ is the fourth prolongation of the operator X and can be calculated from (7). When equation (20) is expanded and separated with respect to the powers of different derivatives of u, an overdetermined system in the unknown coefficients of ξ^1, ξ^2 and η are obtained. Solving the over determined system for $\xi^1(t, x, u), \xi^2(t, x, u)$ and $\eta(t, x, u)$, gives rise to

$$\xi^1(t, x, u) = C_1 + tC_3 \quad (21)$$

$$\xi^2(t, x, u) = C_2 + \frac{1}{2}xC3 \quad (22)$$

$$\eta(t, x, u) = \frac{1}{2}\left(\frac{n-3}{n-1}\right)uC3 + C4 + C5 \quad (23)$$

where C_1, C_2, C_3, C_4 and C_5[29] are arbitrary constants.

Based on the structure of solutions (21) --(23), unique cases of equation (1) namely, (*i*) n = 1 and (*ii*) n = 3 are considered. Hence, the symmetries of equation (1) for these two cases shall be examined. From equations (21)– (23) we obtained a five-dimensional Lie algebra spanned by the following basis

$$X_{11} = \frac{\partial}{\partial t}, \; X_{12} = \frac{\partial}{\partial x}, \; X_{13} = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}\left(\frac{n-3}{n-1}\right)u\frac{\partial}{\partial u}$$

$X_{14}=\frac{\partial}{\partial u}$, $X_{15}=t\frac{\partial}{\partial u}$ (24)

b) *Unique cases.*

a) n =1

Linear wave equation is obtained in this case and is given by

$$u_{tt}+\alpha u_{xxxx}\text{-}\Upsilon u_{xx}=0 \quad (25)$$

Equation (24) produces four Lie point symmetries given by

$$X_{21}=\frac{\partial}{\partial t}\ ,\ X_{22}=\frac{\partial}{\partial x}\ ,\ X_{23}=u\ \frac{\partial}{\partial u} \quad (26)$$

Including an infinite symmetry$X_{24}=F(t,x)\frac{\partial}{\partial u}$, where $F(t,x)$is the solution of equation (25). This symmetry is always presence whenever the equation in question is linear. We discover the misplacement of invariance under dilations in time and space (X_{23}). This misplacement is indeed well taking care of by the infinite-dimensional Lie sub algebra (X_{24}).

b) n=3

In this case, equation (1) reduces to a acclaimed PDE, the modified Boussinesq equation [29].

$$u_{tt}+\alpha u_{xxxx}\text{-}\Upsilon(u_x^3)_x=0, \quad (27)$$

which was shown in the recognized Fermi-Pasta-Ulam problem.

Its symmetries are

$$X_{31}=\frac{\partial}{\partial t}\ ,\ X_{32}=\frac{\partial}{\partial x}\ ,\ X_{33}=t\ \frac{\partial}{\partial t}+\frac{1}{2}x\ \frac{\partial}{\partial x}\ ,\ X_{34}=\frac{\partial}{\partial u}\ ,\ X_{35}=t\frac{\partial}{\partial u} \quad (28)$$

Equation (27) is very useful in the study of the behavior of systems which are mainly linear but a non-linearity is presented as a perturbation. It can also be found in other physical applications.

E. Conservation Laws.

In this section, we give short details on the use of symmetry generators for variational equations to obtain the conserved vectors of equation (1) via Noether's theorem. A vector $T=T^1,T^2,\dots,T^n$ is conserved if it satisfies

$$D_iT^i=0 \quad (29)$$

For all result of the equation in question [30].

A Lagrangian of equation (1) is

$$L=\frac{1}{2}u_t^2-\frac{1}{2}\alpha u_{xx}^2-\frac{\gamma}{n-1}u_x^{n+1}. \quad (30)$$

Substitution of the Lagrangian (30) into the equation (11) and using each of the symmetries of equation (24) result to Noether symmetries of equation (1). The generator X_{15}does not satisfy criteria (11) which imply that it is not variational via this Lagrangian. Therefore, it will not give rise to conservation law. Finally, we obtain four conservation laws for equation (1) via Noether's theorem. presented below [29]

.X_{11}

$T_{11}^t=\frac{1}{2}(\alpha u_{xx}^2+u_t^2)+\frac{1}{n-1}\gamma u_x^{n+1}$

$T_{11}^x=\alpha(u_tu_{xxx}-u_{xx}u_{xt}-\gamma u_x^nu_t)$ (31)

X_{12}

$T_{12}^t=u_xu_t$

$T_{12}^x=\alpha(u_xu_{xxx}-\frac{1}{2}u_{xx}^2)-\frac{1}{2}u_t^2-\frac{1}{n-1}\gamma nu_x^{n+1}$ (32)

X_{13}

$T_{13}^t=u_t$

$T_{13}^x=-\gamma u_x^n+\alpha u_{xxx}$ (33)

X_{14}

$T_{14}^t=t\,u_t\text{-}u$

$T_{14}^x=t(-\gamma u_x^n+\alpha u_{xxx})$. (34)

Conservation laws for (i) n=1and (ii) n=3.

(i) n =1

In this case, the Lagrangian is given by

$L=\frac{1}{2}u_t^2-\frac{1}{2}\alpha u_{xx}^2-\frac{1}{2}\gamma u_x^2$. (35)

As a result of the linearity nature of the wave equation (25), all the symmetries (26) in conjunction with the infinite one form Noether symmetries and are variational. The conserved vectors obtain are

X_{21}

$T_{21}^t=\frac{1}{2}(\alpha u_{xx}^2+u_t^2+\gamma u_x^2)$

$T_{21}^x=\alpha(u_xu_{xxx}\text{-}u_{xx}u_{xt}-\gamma u_xu_t)$ (35)

X_{22}

$T_{22}^t=u_xu_t$

$T_{22}^x=\alpha(u_xu_{xxx}\text{-}\frac{1}{2}u_{xx}^2)\text{-}\frac{1}{2}(u_t^2+\gamma u_x^2)$ (36)

X_{23}

$T_{23}^t=u_t$

$T_{23}^x=\alpha u_{xxx}\text{-}\gamma u_x$ (37)

X_{24}

$T_{24}^t=fu_t-uf_t$

$T_{24}^x=\alpha(u_xf_{xx}\text{-}uf_{xxx}+fu_{xxx}-f_xu_{xx})+\gamma(uf_x\text{-}fu_x)$ (38)

(ii) n= 3

By using n = 3 into the results of the general case in (31-34) we obtain the conserved vectors of the modified Boussinesq equation (26)

III. DOUBLE REDUCTION AND EXACT SOLUTION OF KURAMOTOSIVASHINSKY (KS) EQUATION

In this part, we apply Double reduction technique to equation (1) to find the exact solution. To achieve this, we apply the relationship between the Lie point symmetries and the conservation law of equation (1) to get its doubly reduced equation which can be quickly resolved to obtain the exact solution.

A double reduction of (1) by $<X_{11},X_{12}>$

Firstly, we show that X_{11} and X_{12} are associated with $T=(T_{13}^t,T_{13}^x)$using Theorem 3.1 for $i=1,2$, which is given by

$$T^{*} = X\begin{pmatrix}T^t\\T^x\end{pmatrix} + (D_t\,\xi^t + D_x\,\xi^x)\begin{pmatrix}T^t\\T^x\end{pmatrix} - \begin{pmatrix}D_t\,\xi^t & D_x\,\xi^t\\D_t\,\xi^x & D_x\,\xi^x\end{pmatrix}\begin{pmatrix}T^t\\T^x\end{pmatrix} \quad (39)$$

For X_{11} we obtain

$$\begin{matrix}T_{13}^{*t}\\T_{13}^{*x}\end{matrix} = X_{11}{}^{[3]}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} + (0+0)\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} = \begin{pmatrix}U_1\\U_2\end{pmatrix},$$

where the prolongation of $X_{11}{}^{[3]}$ from (10) is given by $\frac{\partial}{\partial t}$.

$$U_1 = \left(\frac{\partial}{\partial t}\right)u_t = 0$$

and

$$U_2 = \left(\frac{\partial}{\partial t}\right)(-\gamma u_x^n + \alpha u_{xxx}) = 0.$$

Therefore, X_{11} is associated with $T = (T_{13}^t, T_{13}^x)$.
Similarly, for X_{12},

$$\begin{matrix}T_{13}^{*t}\\T_{13}^{*x}\end{matrix} = X_{12}{}^{[3]}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} + (0+0)\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} = \begin{pmatrix}U_1\\U_2\end{pmatrix}, \quad (40)$$

where the prolongation of $X_{12}{}^{[3]}$ from (10) is given by $\frac{\partial}{\partial x}$.

$$U_1 = \left(\frac{\partial}{\partial x}\right)u_t = 0$$

and

$$U_2 = \left(\frac{\partial}{\partial x}\right)(-\gamma u_x^n + \alpha u_{xxx}) = 0.$$

Therefore, X_{12} is also associated with $T = (T_{13}^t, T_{13}^x)$.

We can get a reduced conserved form for equation of (1) since

$X_{11}$and X_{12} are both associated symmetries of $T = (T_{13}^t, T_{13}^x)$.

We now look at a linear combination of X_{11} and X_{12} of the form $X = X_{11} + cX_{12}$ (c is an arbitrary constant).

That is

$$X = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}. \quad (41)$$

Foremost, we investigate the association between X and $T = (T_{13}^t, T_{13}^x)$by putting the important information into the association matrix in Theorem 1

$$\begin{matrix}T_{13}^{*t}\\T_{13}^{*x}\end{matrix} = X^{[3]}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} + (D_t\,(1) + D_x\,(C))\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} - \begin{pmatrix}D_t\,(1) & D_x\,(1)\\D_t\,(C) & D_x\,(C)\end{pmatrix}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} = \begin{pmatrix}U_1\\U_2\end{pmatrix} \quad (42)$$

$$X^{[3]}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} + (0+0)\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} = \begin{pmatrix}U_1\\U_2\end{pmatrix} \quad (43)$$

$$U_1 = \left(\frac{\partial}{\partial t} + C\frac{\partial}{\partial x}\right)u_t = 0$$

$$U_2 = \left(\frac{\partial}{\partial t} + C\frac{\partial}{\partial x}\right)(-\gamma u_x^n + \alpha u_{xxx}) = 0$$

Therefore, X is associated with $T = (T_{13}^t, T_{13}^x)$.

Since there is an association, double reduction method can be used to obtain a solution. Next, we transform X to a new canonical form in (r, s w).

$$Y = \frac{\partial}{\partial s}. \quad (44)$$

The generator is of the form

$$Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}. \quad (45)$$

We select X(r) =0, X(s) = 1, X(w) = 0,without any loss of generality. For which we obtain the invariance condition

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{0} = \frac{ds}{1} \quad (46)$$

Solving the above characteristic equation (46) we find new canonical coordinates

$$s = t$$
$$r = x - ct \quad (47)$$
$$w(r) = u.$$

The inverse canonical coordinates from (47) are given by

$$t = s$$
$$x = r + cs \quad (48)$$
$$u = w(r).$$

The computation of A and $(A^{-1})^T$from (16), (17) and (48) is given by

$$A = \begin{pmatrix}D_r t & D_r x\\D_s t & D_s x\end{pmatrix} = \begin{pmatrix}\frac{dt}{dr} & \frac{dx}{dr}\\\frac{dt}{ds} & \frac{dx}{ds}\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & c\end{pmatrix} \quad (49)$$

$$A^{-1} = \begin{pmatrix}D_t r & D_x r\\D_t s & D_x s\end{pmatrix} = \begin{pmatrix}\frac{dr}{dt} & \frac{dr}{dx}\\\frac{ds}{dt} & \frac{ds}{dx}\end{pmatrix} = \begin{pmatrix}-c & 1\\1 & 0\end{pmatrix} = (A^{-1})^T \quad (50)$$

and

$$J = \det(A) = -1. \quad (51)$$

The terms of the new dependent variable w(r) in equation (47) which are first and second derivatives of u are

$$U_t = \frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial t} = w_r\,(-c)$$

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$$=-cw_r$$

$$U_{tt}=\frac{\partial U_t}{\partial t}=\frac{\partial(-cw_r)}{\partial t}=c^2w_{rr}(52)$$

$$U_x=\frac{\partial u}{\partial x}=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial r}\frac{\partial r}{\partial x}=w_r$$

$$U_{xx}=\frac{\partial U_x}{\partial x}=\frac{\partial(w_r)}{\partial t}=w_{rr} \qquad (53)$$

Applying Theorem 2 for $i=1,2$, which is given by

$$\begin{pmatrix}T_{13}^r\\T_{13}^s\end{pmatrix}=J(A^{-1})^T \quad \begin{pmatrix}T_{13}^t\\T_{13}^x\end{pmatrix} \qquad (54)$$

and substituting equation (50 -53) into (54) we have

$$\begin{pmatrix}T_{13}^r\\T_{13}^s\end{pmatrix}=(-1)\begin{pmatrix}-c & 1\\1 & 0\end{pmatrix}\begin{pmatrix}u_t\\-\gamma u_x^n+\alpha u_{xxx}\end{pmatrix}$$

$$\begin{pmatrix}T_{13}^r\\T_{13}^s\end{pmatrix}=\begin{pmatrix}cu_t-(-\gamma u_x^n+\alpha u_{xxx})\\u_t\end{pmatrix}$$

$$\begin{pmatrix}T_{13}^r\\T_{13}^s\end{pmatrix}=\begin{pmatrix}c(-cw_r)+\gamma w_r^n-\alpha w_{rrr}\\-cw_r\end{pmatrix} \qquad (55)$$

The new reduced conserved form is given by

$$D_rT_{13}^r=0. \qquad (56)$$

This implies that

$$T_{13}^r=K \qquad (57)$$

which is given as

$$-c^2w_r+\gamma w_r^n-\alpha w_{rr}=K \qquad (58)$$

where $K\in\mathbb{R}$ is a constant.

Setting the constant K=0 in equation (58) and solving we obtain the solution in terms of a special power function called Hyper geometric function given as

$$\boldsymbol{w}=\boldsymbol{K_2}-\frac{\boldsymbol{\alpha}}{\boldsymbol{c^2}}\left[\boldsymbol{\gamma}+\boldsymbol{e}^{-c^2(n-1)\left(K_1-\frac{r}{\alpha}\right)}\right]^{\frac{1}{1-n}}$$

$$\boldsymbol{2F_1}\left[\boldsymbol{1},\frac{\boldsymbol{1}}{\boldsymbol{n-1}},\frac{\boldsymbol{n}}{\boldsymbol{n-1}},\frac{\gamma}{c^2}\left(\boldsymbol{\gamma}+\boldsymbol{e}^{-c^2(n-1)\left(K_1-\frac{r}{\alpha}\right)^{n-1}}\right)^{\frac{1}{1-n}}\right] \qquad (59)$$

K_1,K_2 are constants of integration.

In terms of the variables u,x,t, we derive the solution of (1) as

$$\boldsymbol{u(x,t)}=\boldsymbol{K_2}-\frac{\boldsymbol{\alpha}}{\boldsymbol{c^2}}\left[\boldsymbol{\gamma}+\boldsymbol{e}^{-c^2(n-1)\left(K_1-\frac{x-ct}{\alpha}\right)}\right]^{\frac{1}{1-n}}$$

$$\boldsymbol{2F_1}\left[\boldsymbol{1},\frac{\boldsymbol{1}}{\boldsymbol{n-1}},\frac{\boldsymbol{n}}{\boldsymbol{n-1}},\frac{\gamma}{c^2}\left(\boldsymbol{\gamma}+\boldsymbol{e}^{-c^2(n-1)\left(K_1-\frac{x-ct}{\alpha}\right)^{n-1}}\right)^{\frac{1}{1-n}}\right]$$

(60)

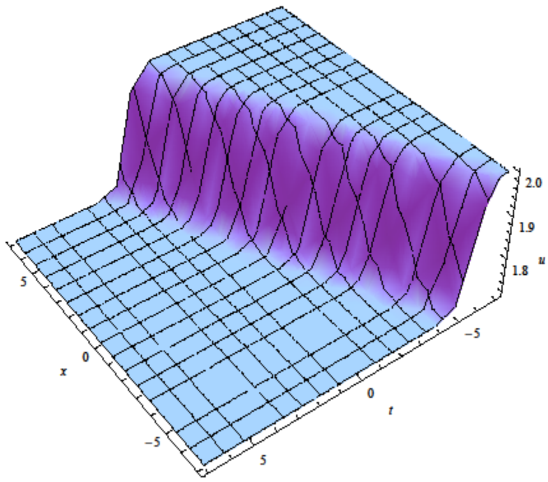

Fig. 1: 3D graph for Dynamical behaviour of Kink type wave solutions given by Eq. (60), for $n=\boldsymbol{\alpha}=\boldsymbol{c}=\boldsymbol{\gamma}=\boldsymbol{K_2}=\boldsymbol{K_1}=\mathbf{2},-\mathbf{7}\le\boldsymbol{x}\le\mathbf{7},-\mathbf{7}\le\boldsymbol{t}\le\mathbf{7}$.

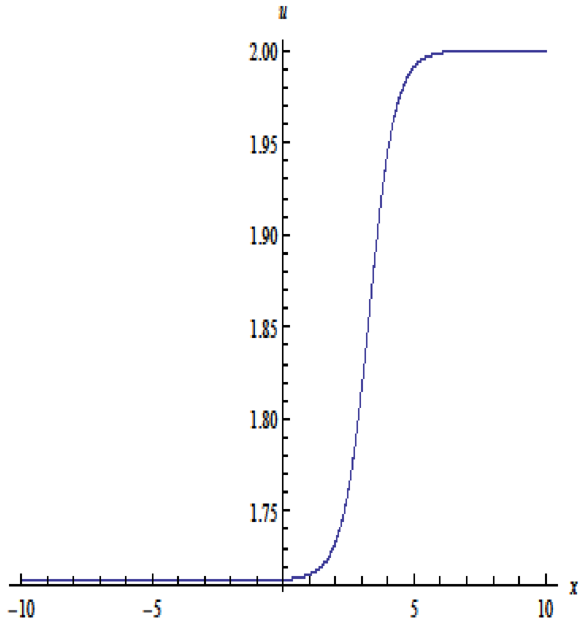

Fig. 2: 2D graph for Dynamical behaviour of Kink type wave solutions given by Eq. (60), with $n=\boldsymbol{\alpha}=\boldsymbol{c}=\boldsymbol{\gamma}=\boldsymbol{K_2}=\boldsymbol{K_1}=\mathbf{2},\ -\mathbf{10}\le\boldsymbol{x}\le\mathbf{10}$.

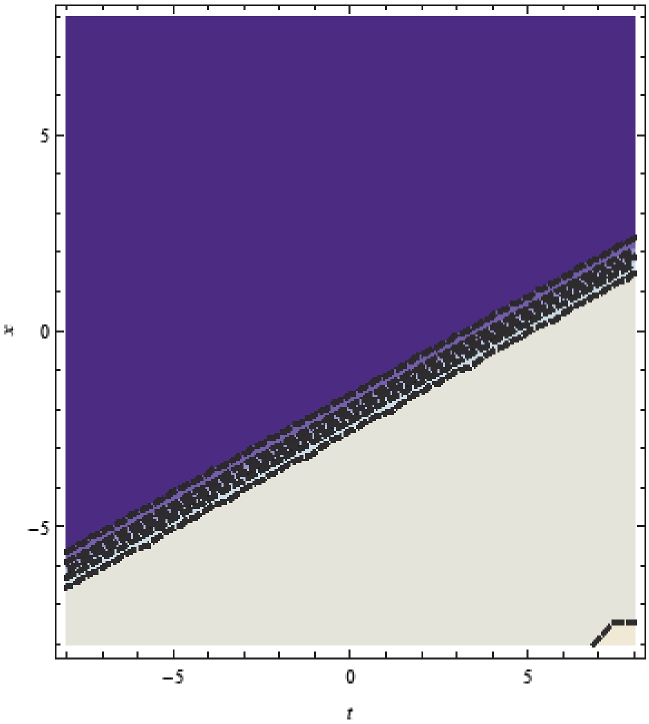

Fig. 3: Density plot for Dynamical behavior of Kink type wave solutions given by Eq. (60), with$n = \boldsymbol{\alpha} = \boldsymbol{c} = \boldsymbol{\gamma} = \boldsymbol{K_2} = \boldsymbol{K_1} = \mathbf{2},\ -\mathbf{7} \le \boldsymbol{x} \le \mathbf{7}, -\mathbf{7} \le \boldsymbol{t} \le \mathbf{7}$.

IV. CONCLUSION

In this paper, we investigated a reputed nonlinear partial differential equation (NLPDE) known as Kuramoto Sivanshinky (KS) equation. We implemented a renowned unified Double reduction method for PDEs to extract solutions of the KS equation. Interestingly, important Soliton like wave solutions were obtained. To comprehend the dynamical character of these solutions, we constructed the graphs of the solution surfaces for some special parameter values. The graphs of the wave solutions regarding (KS) equation, by us are quite novel and latest findings (Figures 1–3).

V. RECOMMENDATION

The results of this work will be of great importance in mathematical physics, engineering sciences and other scientific real-time application fields. It can be seen by the results of this work that the Double reduction method is a very powerful technique and is worthy of being studied further. In future, there are other methods like extended tanh function methods that will be implemented to obtain several other forms of solutions. The KS equation with perturbation terms will also be studied. The results of those researches will give an edge over the current and former results.

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