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In the Heat Diffusion/Conduction Equation, How to Extend the Validity of the Dirichlet Boundary Conditions to More Than One Dimensional Geometric Space

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Abstract:- In fact, the heat equation with Dirichlet boundary conditions has analytical solutions for a number of geometries that involve sufficient symmetry. You might say it's "cheating" since you're using symmetry to reduce the 2D or 3D equation to a simpler 1D problem.

IIt can be thought that more broadly, any arbitrary heat equation can be solved with any desired accuracy using finite difference methods by imposing an arbitrarily small geometric grid on the system and calculating the heat transfer for these grid elements using arbitrarily small time steps.This is not true because the FDM technique fails with Dirichlet BC at the boundaries of the 2D and 3D geometric spatial grid

There is an irreversible error inherent in applying Dirichlet boundary conditions to the heat diffusion equation. Mathematical models like the 3D heat equation are just models that we use to predict reality. Reality doesn't have to bend to fit our mathematical models, however elegant they may be. In this article, we introduce and explain the theory of the so-called Cairo technique where the space-time PDE such as the heat equation can be discarded and the 2D/3D physical situation is translated directly into a stable algorithm at rapid convergence.

The proposed method has many approaches depending on the physical phenomena and therefore has a wide field of applications. We limit our analysis here to the B-matrix approach which has been repeatedly discussed in several previous papers and has proven effective in calculating temperature, electric potential, and sound intensity in boundary value problems.

The proposed procedure operates on a 4D space-time fabric as a unit and the classically defined scalar thermal diffusion coefficient as K/Roh S is reformulated accordingly. Theoretical numerical results in 2D and 3D geometric space are presented where a special algorithm intended to validate the theory is described. Since the proposed technique using transition matrix B is a proper hypothetical thought experiment, the 4D solution of transition matrix chains (B) should exist in a stable, unique, and rapidly convergent form.

I. INTRODUCTION

Below is the general form of the heat diffusion partial differential equation we are investigating, except we use D for the thermal diffusion coefficient instead of Alpha,

d / dt (partial) U (x,y,z,t) = D Nabla^2 U (x,y,z,t) + S (x,y,z,t) .. . .(1)

Subject to the boundary conditions of Dirichlet B C on the limits of the domain of U and the initial conditions IC of U at t=0 that is U (x,y,z,0).

Where,

U(x,y,z,t) represents temperature energy density.

U(x,y,z,t)=K .T(x,y,z,t) , T in degrees Kelvin and KB is Boltzmann constant.

S(x,y,z,t) is the heat energy density source/sink term at the corresponding free nodes in the considered 2D or 3D domain.

It can be shown that the same equation can also describe the spatio-temporal evolution U(x,y,z,t) of the electric potential energy density and the sound kinetic energy density (J/ m ^ 3) as well as the potential energy of the electric field in the bounded media when the expression of the diffusion coefficient D is assumed to be correct [1,2,3].

In classical numerical methods, the source term is not taken into account and Nabla^2 in 3D configuration is expressed numerically by the well-known FDM technique Eq 2.

In effect most classical numerical methods work with finite difference FDM, where the source term is not considered, Nabla2 in 3D configuration is expressed numerically as follows, [4,5]

Nabla^2 U (i, j, k) = {U (i + 1, j, k) + U (i-1, j, k) + U (i, j + 1, k) + U (i, j- 1, k) + U (i, j, k + 1) + U (i, j, k-1) -6 * (U (i, j, k)} /6h^2 . . . .(2)

h is the distance between two adjacent free nodes assumed to be equidistant and the incremental spatial variation dU is expressed in the form,

dU(x,y,z,t) = D Nabla ^ 2 U(x,y,z,t) dt

dt is the time step or time jump is arbitrary chosen in the interval greater than zero and less than the numerical stability condition viz, dt less than or equal (1/2) dx2/D for one dimensional geometrical space [4,5].

It is quite surprising that Equation 1 in 2D and 3D with Dirichlet BC has neither a continuous-time analytical solution nor a numerical solution at discrete time intervals in the classical mathematical sense. The solution exists but is not accessible.

In other words, the exact analytical integral solution and the approximate numerical solution satisfying an arbitrary Dirichlet BC cannot be found mathematically in 4D space (x, y, z, t).

Obviously, the heat equation has analytical solutions for a number of geometries that involve sufficient symmetry to reduce the 2D or 3D equation to a simpler 1D problem (squares and cubes). But this is a kind of cheating since we are imposing a symmetry which is not present in the real physical problem.

The mathematical solution of the Dirichlet boundary condition problem assumes that the heat energy solution propagates from interior nodes to the boundary, which is not correct, contrary to nature, so it can never work.

The reason why FDM technique succeeds for 1D-Dirichlet problems and fails at both 2D and 3D geometric spatial grid limits is that in 1D numerical solution via FDM, an inherent trick is involved and the two BC conditions are added to the matrix[4,5] as sources (here the solution propagates from BC inwards and not the other way around), which is impossible to achieve in 2D and 3D grids.

Therefore, there is a need to provide a technique in which the Dirichlet boundary conditions are physically treated as a source/sink term, as shown in Figures 2, 3, and 4.

This is the reason why the boundary conditions in the proposed numerical technique express the interaction of the system with its environment in a thermodynamic way.

We speculate that the treatment of Dirichlet boundary conditions as lifeless fixed nodes and the solution propagates from interior nodes to BC contrary to nature is the main reason beyond the failure of the classical mathematical solution at l heat diffusion equation in 2D and 3D situations.

We assume that,
i- The classic mathematical spatio-temporal PDE is a method of predicting reality.
ii-Spatiotemporal Shrodinger Quantum Mechanics PDE is a reality prediction method.
iii- The proposed new technique that bypasses the PDE is a way to predict reality.
etc.

This same view was predicted by A. Einstein among other scientists a century ago where their statements on mathematics and partial differential equations predict the judgment mentioned above [6,7].
i-God/nature does not care about our mathematical difficulties. He integrates experimentally.
ii-The partial differential equations has entered theoretical physics as a handmaid, but has gradually become a mistress.

A striking example of the statements of Einstein and his colleagues (statements I, ii) is the equation of diffusion/conduction of heat as a function of time. In reality the solution of equation 1 exists but it is a mathematical impossibility in 2D and 3D geometric space simply because nature does not work purely mathematically [8].

On the other hand, it is possible to run the appropriate and successful numerical modeling algorithm of a hypothetical 3D diffusion experiment, cubic or not, Fig 1.

This is the subject of this article.

In the proposed method based on the above analysis, which we can call the Cairo technique for discrimination purposes, we completely ignore the PDE, Eq1 heat and assume that it was never present [3].

Equation 1 and its FDM numerical solution [4] are used to find the steady-state temperature distribution which serves as a comparison or empirical test for the new numerical technology.

Since the proposed technique using transition matrix B is an appropriate hypothetical thought experiment, the 4D solution of transition matrix chains

(B) must exist in a stable, unique and rapidly convergent form.

In short, a physical experiment or phenomenon described by a spacetime partial differential equation should be better solved by running the correct hypothetical experimental algorithm with life BC than by mathematically solving the PDE with Dirichlet boundary conditions. This is the heart of current article.

II. THEORY

To clarify the subject, we present Figures 1a and b showing the difference of the two methods, the classical numerical mathematical solution and the proposed numerical modeling algorithm of the thought physics experiment in a schematic flowchart.

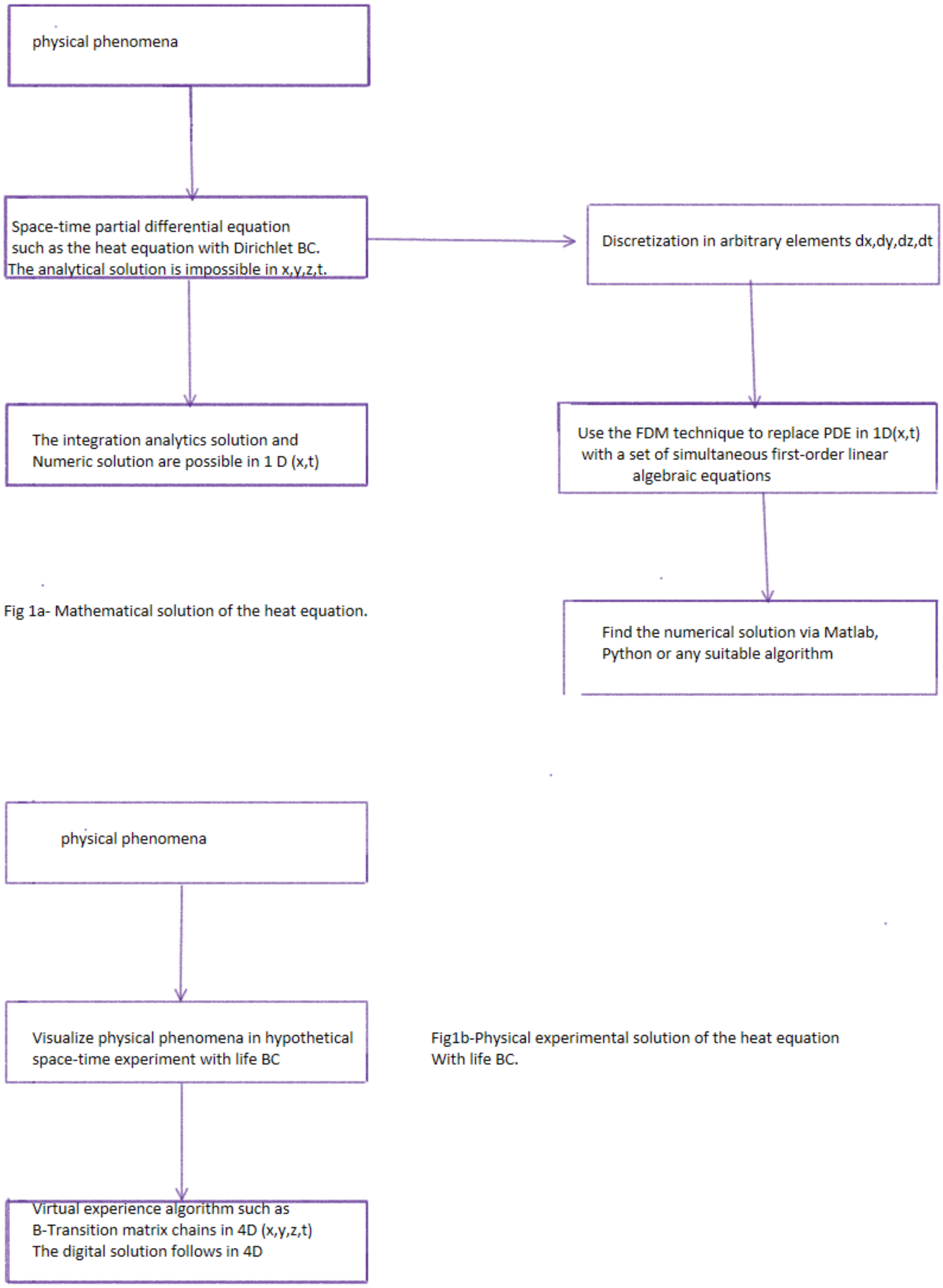

Fig 1a- Mathematical solution of the heat equation.

Fig1b-Physical experimental solution of the heat equation With life BC.

Fig.1a&b. A comparison between the current mathematical method and the proposed technique.

It is obvious that Figure 1b circumvents the PDE and its difficulties in mathematical resolution. The physical phenomenon is translated directly into its corresponding hypothetical experience algorithm.

In the method proposed by what may be called the Cairo technique for distinction, space and time are an inseparable unit or fabric. The value of the time increment or the time jump dt cannot be chosen arbitrarily but is done implicitly according to the dimensions of the mesh expressed in B-matrix and the coefficient of thermal diffusivity D.

The thermal diffusion coefficient D itself which expresses the thermal property of the material is intrinsically included in RO as indicated later in section III Relations 1,2.

Again, in the new technique, the thermal diffusion coefficient D, which is a physical property of the material, is inherent in the procedure and implicitly contained in the main diagonal element RO of the transition matrix B where the space and time are an inseparable unit or fabric. The value of the temporal increment or the temporal jump dt cannot be chosen arbitrarily but is done implicitly according to the dimensions of the grid expressed in B-matrix and of the coefficient of thermal diffusivity D.

The thermal diffusion coefficient D itself which expresses the thermal property of the material is inherently included in RO as shown later in section III Relationships 1,2.

Section III deals with numerical solutions for cubic and non-cubic 3D configurations. We use the 3D thermal diffusion coefficient D to derive Alpha which is the exponent of the exponential temporal rise/fall of U(x,y,z,t) rather than the classical inadequate scalar definition of D as K/Rho.S (in normal conventions).

Again, in the new technique, the thermal diffusion coefficient D, which is a physical property of the material, is inherent in the procedure and implicitly contained in the main diagonal element RO of the transition matrix B.

One way to realize the implications of Figure 1b is to formulate a stochastic transition matrix that takes into account the modified Dirichlet BC as the source/sink term and adequately describes the hypothetical experimental solution in consistent time steps dt. Here comes the role of the stochastic transition B- matrix .

It has been shown previously in papers [3,8] that the numerical solution of the thermal energy density U(x,y,z,t) at time t =N dt (where N is the number of steps or of iterations) is given by ,

UN = D(N) (b+S) + B^N (IC) . . . (3)

Where B is the transition matrix nxn , b is the boundary condition n vector arranged in a proper order and IC is the initial conditions n vector at the considered n points.

For any number of iterations of time step N.

Where B is the nxn transition matrix, b is the n boundary condition vector arranged in an appropriate order, and IC is the n initial condition vector at the n points under consideration.

Note that in equation 3, B^N tends to zero for large N, which means that the initial IC conditions dampen out and we end up with the forced BC solution.

Again, the stochastic transition matrix B nxn with all its entries (bi,j) is well defined as well as its consequent transfer matrices D(N) and E(N) [3,8,9]

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Again, the stochastic transition matrix B nxn with all its entries (bi,j) is well defined as well as its consequent transfer matrices D(N) and E(N) [3,8,9].

The transfer matrix D(N) is given by,

D(N)=B+B2+. . .. . . +BN . . . . (4)

while the transfer matrix E(N) is simply defined as the D-matrix plus the unit matrix I.

E(N)=D(N)+I

In the matrix B power series notations,

E(N)=B0+B+B2+. . . +BN . . . . (5)

.Obviously, B0=I

For sufficiently large number of iteration N steps it can be shown that,

E(N)=(I-B) ^-1 . . . . (6)

Equation 7 below describes the stationary or steady state solution for a sufficiently large number of iterations N,

U(x,y,z)=D.(b+S) . . . . (7)

For any arbitrary boundary conditions vector b and any arbitrary source term S.

Equation 3 is efficient and of great importance because it gives the time-dependent solution of U(x,y,z,t) as a function of BC boundary conditions and IC initial conditions simultaneously in one formula.

It also shows that the boundary condition vector b is treated as the source/sink term vector S.

For nature's solution, the two BC&S are indistinguishable and can be added together.

In order not to worry too much about the details of the theory, we go directly to the numerical examples and their verification in section III.

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III. TRANSIENT SOLUTIONS

Consider the simplest case of 3D geometric configuration. A 3D cube of 8 equidistant free nodes and 8 boundary conditions as shown in figure2.

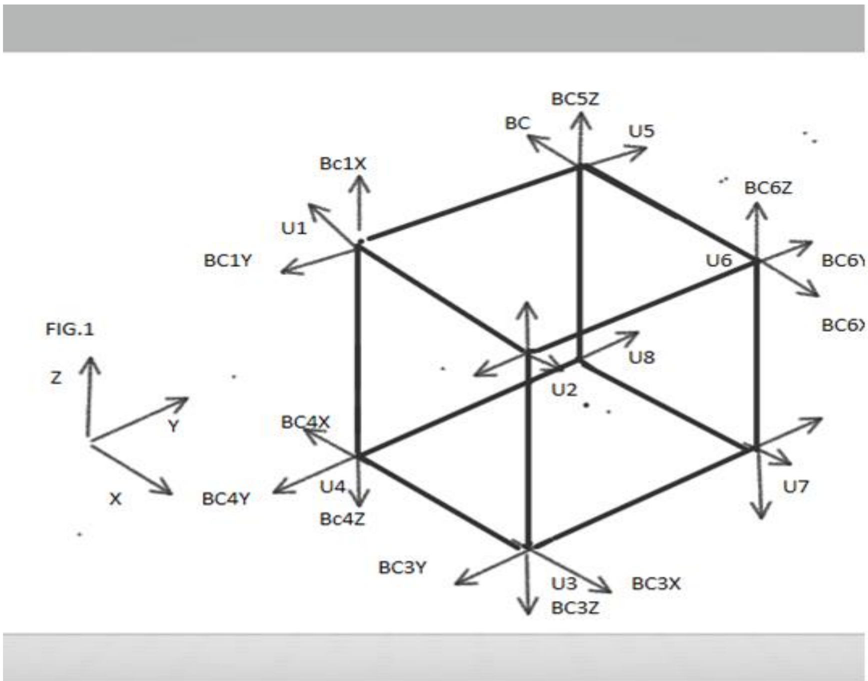

Fig. 2. Transient heat diffusion in a 3D cube of 8 free nodes with 8 modified Dirichlet source/sink BC.

Here, the input entries of the symmetric 8X8 matrix B are expressed in terms of their four defining conditions explained in Ref [3,8],

B=

RO 1/6-RO/6 1/6-RO/6 0.00000000 1/6-RO/6 0.0000000 0 0.00000000 0.00000000

1/6-RO/6 RO 0.00000000 1/6-RO/6 0.00000000 1/6-RO/6 0.00000000 0.00000000
1/6-RO/6 0.00000000 RO 1/6-RO/6 0.00000000 0.00000000 1/6-RO/6 0.00000000
0.00000000 1/6-RO/6 1/6-RO/6 RO 0.00000000 0.00000000 0.00000000 1/6-RO/6
1/6-RO/6 0.00000000 0.00000000 0.00000000 RO 1/6-RO/6 1/6-RO/6 0.0000000
0.00000000 1/6-RO/6 0.00000000 0.00000000 1/6-RO/6 RO 0.00000000 1/6-RO/6
0.00000000 0.00000000 1/6-RO/6 0.00000000 1/6-RO/6 0.00000000 RO 1/6-RO/6
0.00000000 0.00000000 0.00000000 1/6-RO/6 0.00000000 0. 1/6-RO/6 1/6-RO/6 RO

Where RO the main diagonal input element is an element of the closed interval [0,1].

The matrix B and consequently its derived transfer matrices D and E are always symmetric and therefore conform to the physical principle of detail equilibrium

In effect the choice of the numerical value of the element RO in the interval [0,1] is crucial since it determines the value of the thermal diffusion coefficient which is a physical thermal property of the material.

We assume that the time dependence of energy density on temperature energy density is exponential with the exponent Alpha which is a numerically proven rule.

Therefore, we run Equation 3 for different values of RO in the interval [0, 1] and the corresponding temperature-time curves are obtained.

From these curves, the (dT/dt}/T ratio is calculated for different values of RO and the numerical results are presented in Table I where the (dT/dt)/T ratio is conveniently named Alpha.

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Table I. Numerical results for RO vs alpha up to equation 3 for the 3D cube 8 free nodes Fig2.

RO	0.	0.2	0.4	0.6	0.8	1.0
Alpha	0.693	0.511	0.357	0.223	0.105	0.00

Alpha expressed in logarithmic form,

Log 2 Log 1.667 Log 1.429 Log1.25 Log1.111 Log1.0

The conclusion obtained from Table I is that it accurately predicts an important relationship between 3D Alpha and RO as a logarithmic relationship,

Alpha = Log {1 / (1/2 + RO / 2)} . . . Relationship . . . (1)

Now consider another more complicated case of 3D geometric configuration. A 3D rectangular cuboid of 27 equidistant free nodes and 52 BC which can be reduced to 27 boundary conditions, as shown in Figure 3.

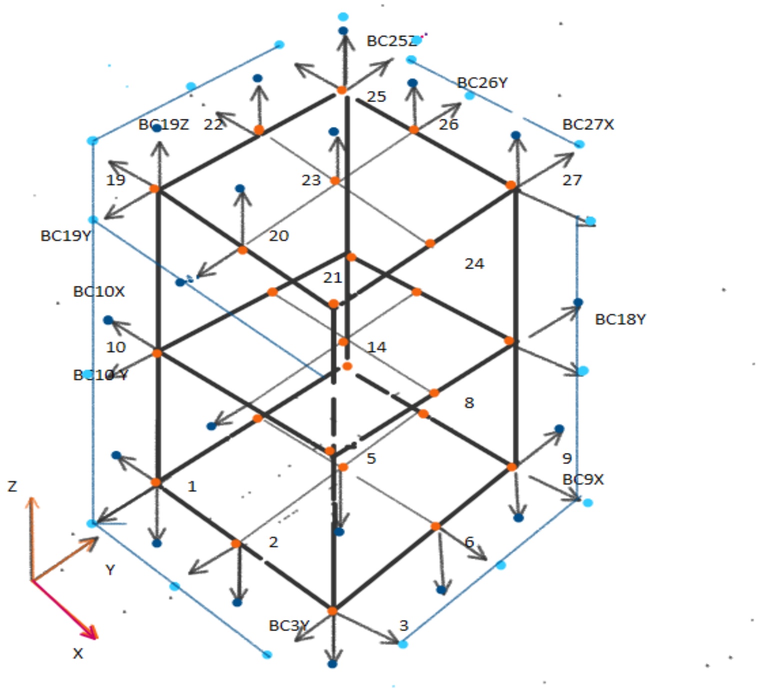

Fig3.Transient heat diffusion in a 3D rectangular cuboid of 27 free nodes with 57 BC reducible to 27 modified Dirichlet BC technique.

Similar procedure to that followed in figure 1, the entries of the matrix B 27X27 are expressed in the form,27X27 B-Matrix inputs,

Line 1
RO 1/6-RO/6 0.0000 1/6- RO 1/6-RO/6 0.0000 0.0000 0.0000 0.0000
1/6-RO/6 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
Line 2
1/6-RO/6 RO 1/6-RO/6 0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.000
0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Line 3
0.0000 1/6-RO/6 RO 0.0000 0.0000 1/6RO/6 0.0000 0.0000 0.0000

0.0000 0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

. . . .

Lin14

0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000	0.0000	0.0000	0.0000
0.0000	1/6-RO/6	0.0000	1/6-RO/6	RO	1/6-RO/6	0.0000	1/6-RO/6	0.0000
0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000	0.0000	0.0000	0.0000

. . . .

. . . .

Line 25

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000	0.0000
0.0000	0.0000	0.0000	1/6-RO/6	0.0000	0.0000	RO	1/6-RO/6	0.0000

Line 26

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000
0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000	1/6-RO/6	RO	1/6-RO/6

Line 27

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1/6-RO/6
0.0000	0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000	1/6-RO/6	RO

We again repeat the procedure followed in Figure 2 for different values of RO in the interval [0, 1].
The numerical results are presented in Table II.

Table II. Numerical results for Alpha vs RO up to equation 3 for the 3D cube 27 free nodes Fig 3.

RO	0.	0.2	0.4	0.6	0.8	1.
ALPHA	0.376	0.282	0.206	0.129	0.0636	0.

Note that the numerical values for Alpha in table II are proportional to those in in Table I where the constant of proportionality is 16/9 as expected.

The 16/9 ratio comes from the ratio of h^2 of Tables I and II where h=L/3 to that of Table I and h=L/4 of Table II.
In short, Tables I and II confirm relationship 1.

Relationship 1 is striking in itself. It predicts that the exponent of the exponential time evolution of thermal energy density is given by well-defined Alpha,

U(t)=U0 EXP (-ALPHA. t) . . . Continuous time cooling curve,

Alpha=D/L 2
Where L is the characteristic length equals the side of cuboide for rectangular shapes.

And

U(t)=UF (1. -EXP(-Log 1./(.5+.5*RO)N)
For discrete time steps N dt in the heating curve.

Similarly, we can show,

RO=D dt /L^2.

What is amazing is that it predicts that Alpha has a minimum value of zero (thermal insulator RO = 0) and a maximum value of Log 2 = 0.697 for thermal super conductor (RO = 1).

Finally, we present an elegant example of the proposed technique shown in Figure 4. Consider another slightly more advanced case of non-cubic 3D geometry configuration,

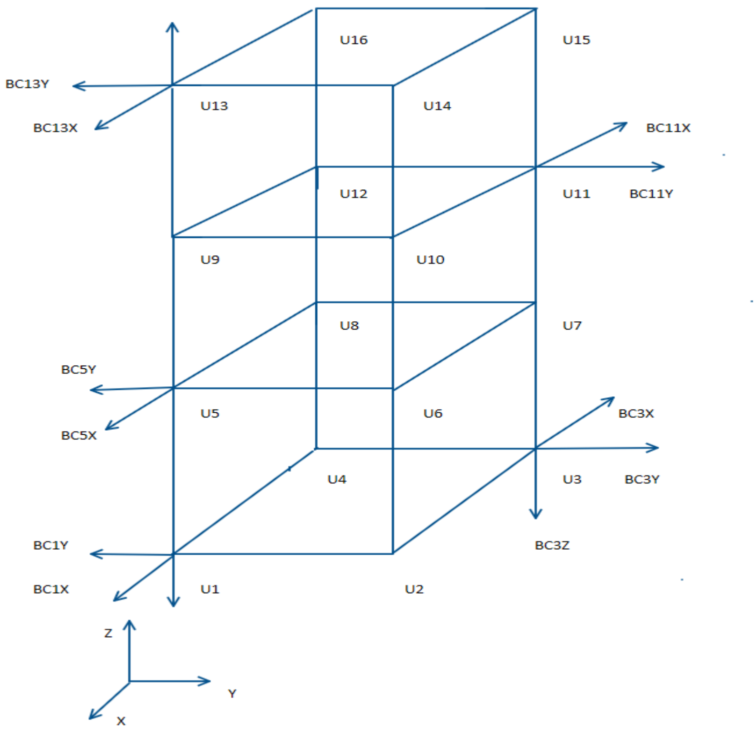

Fig. 4. Transient heat diffusion in a 3D rectangular cuboid of 16 free nodes with 16 modified Dirichlet BC.

Here the entries of the 16x16 matrix B which contain all the necessary information for the transient and stationary solutions are constructed for example RO=0 and the transfer matrix Enxn with N large enough (N=30) is calculated by two different methods,

i-Matrix Power series . . . Eq 5
ii-Expression E=(I-B)^-1 . . Eq 6

Both methods gave exactly the same value for the matrix E as:
E=
1.1070 0.211 .0780 .211 .220 0.083 0.044 .083 .047 .026 0.017.026 .010 .007 0.005 0.007
.211 1.1070.2110.0780.0830.2200.0830.0440.0260.0470.0260.0170.0070.0100
.007 0.005
0.078.2111.1070.2110.0440.0830.220.0830.0170.0260.0470.0260.0050.0070.0100.007
0.2110.0780.2111.1070.0830.0440.0830.2200.0260.0170.0260.0470.0070.0050.0070.010
0.2200.0830.0440.0831.1540.2370.0940.2370.2300.0900.0490.0900.0470.0260.0170.026
0.0830.2200.0830.0440.2371.1540.2370.0940.0900.2300.0900.0490.0260.0470.0260.017
0.0440.0830.2200.0830.0940.2371.1540.2370.0490.0900.2300.0900.0170.0260.0470.026

0.0830.0440.0830.2200.2370.0940.2371.1540.0900.0490.0900.2300.0260.0170.0260.047
0.0470.0260.0170.0260.2300.0900.0490.0901.1540.2370.0940.2370.2200.0830.0440.083
0.0260.0470.0260.0170.0900.2300.0900.0490.2371.1540.2370.0940.0830.2200.0830.044
0.0170.0260.0470.0260.0490.0900.2300.0900.0940.2371.1540.2370.0440.0830.2200.083
0.0260.0170.0260.0470.0900.0490.0900.2300.2370.0940.2371.1540.0830.0440.0830.220
0.0100.0070.0050.0070.0470.0260.0170.0260.2200.0830.0440.0831.1070.2110.0780.211
0.0070.0100.0070.0050.0260.0470.0260.0170.0830.2200.0830.0440.2111.1070.2110.078
0.0050.0070.0100.0070.0170.0260.0470.0260.0440.0830.2200.0830.0780.211
1.107 0.211
0.0070.0050.0070.0100.0260.0170.0260.0470.0830.0440.0830.2200.2110.0780

.211 1.107

IV. STEADY STATE SOLUTIONS

The steady-state equilibrium solution is defined mathematically for Equation 1 as,
dU / dt) partial = 0

Again, as explained in Section II, the steady-state numerical solution of the matrix E is obtained by one of two methods, namely,

1-matrix power series summation ,

E = B ^ 0 + B + B ^ 2 ...... + B ^ N .... (5)

for N sufficiently large.

2-Using the equivalence relation,

E = (I-B) ^ - 1. . . . . (6)

Finally the solution in steady state of equilibrium is given by the matrix D = E-I, [3,8] ,

U = D. (b + S) .... (7)

Equation 7 shows that the distribution of the equilibrium temperature field reached after a sufficiently long time t=Ndt is a function of D, b and S.

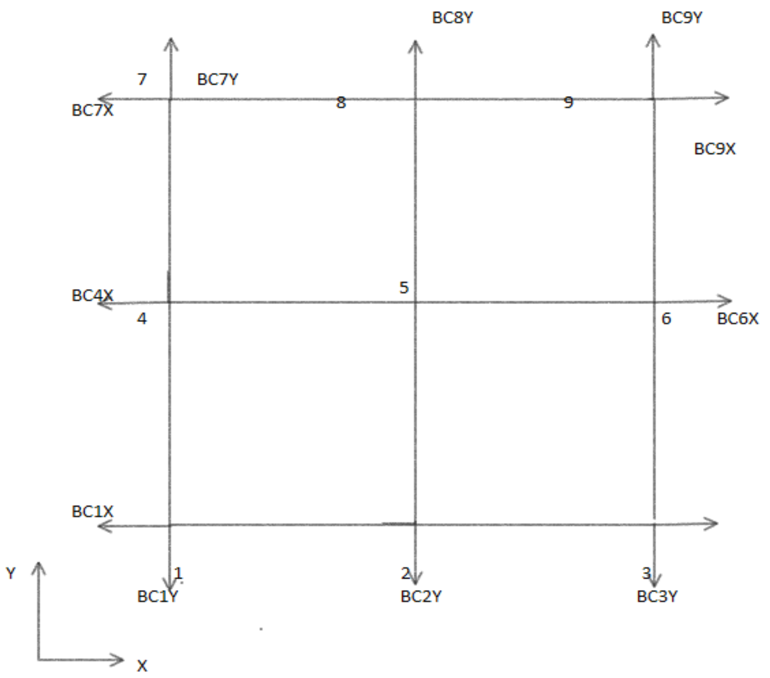

Fig.5 A 2D rectangular domain with 9 equidistant free nodes.

Consider the simple case of a rectangular domain with 9 equidistant free nodes, u1, u2, u3, ... u9 and 12 Dirichlet modified boundary conditions BC1 to BC12 reduced to 9 BC as illustrated in Figure5.

The 12 boundary conditions in Figure 5 can be reduced to 9 modified BCs for the 9 boundary nodes as follows,

BC1 = BC1X + BC1Y

. . . .

BC9 = BC9X + BC9Y

The transition matrix B 9x9 is built to satisfy the conditions i-iv [3,8]for arbitrary RO and is given by,
1- RO 1/4-RO/4 0.000 1/4-RO/4 0.000 0.000 0.000 0.000 0.000
2- 1/4-RO/4 RO 1/4-RO/4 0.000 1/4-RO/4 0.000 0.000 0.000 0.000

3- 0.000 1/4-RO/4 RO 0.000 1/4-RO/4 0.000 0.000 0.000 0.000
4-1/4-RO/4 0.000 0.000 RO 1/4-RO/4 0.000 1/4-RO/4 0.000 0.000
5- 0.000 1/4-RO/4 0.000 1/4-RO/4 RO 1/4-RO/4 0.000 1/4-RO/4 0.000
6- 0.000 0.000 1/4-RO/4 0.000 1/4-RO/4 RO 0.000 0.000 1/4-RO/4
7- 0.000 0.000 0.000 1/4-RO/4 0.000 0.000 RO 1/4-RO/4 0.000
8- 0.000 0.000 0.000 1/4-RO/4 0.000 1/4-RO/4 0.000 RO 1/4-RO/4
9- 0.000 0.000 0.000 0.000 0.000 1/4-RO/4 0.000 1/4-RO/4 RO

And the transfer matrix E calculated by equation (5) converges rapidly for a large N. Here are the numerical entries of the matrix E for N=30 and RO=0,

1-
1.1964247567313038 0.39284951346260755 0.124996185302
0.39284951346260755 0.24999237060546875 0.1071352277483
0.12499618530273438 0.10713522774832995 5.356761387416E-02

2- 0.39284951346260755 1.3214209420340381 0.392849513462
0.24999237060546875 0.49998474121093750 0.2499923706055 0.10713522774832995 0.17856379917689935
0.1071352277483

3-0.12499618530273438 0.39284951346260755 1.19642475673130 0.10713522774832995 0.24999237060546875
0.3928495134626 5.3567613874165E-02 0.10713522774832995 0.12499618530273

4- 0.39284951346260755 0.24999237060546875 0.1071352277483
1.3214209420340381 0.49998474121093750 0.17856379917689
0.39284951346260755 0.24999237060546875 0.1071352277483

5- 0.24999237060546875 0.49998474121093750 0.24999237060
0.49998474121093750 1.4999847412109375 0.4999847412109
0.24999237060546875 0.49998474121093750 0.2499923706055

6 -0.10713522774832995 0.24999237060546875 0.3928495134626 0.17856379917689935 0.49998474121093750
1.321420942034 0.10713522774832995 0.24999237060546875 0.3928495134626

7 – 0.12499618530273 0.10713522774832995 5.35676138742 E-02
0.39284951346260 0.24999237060546875 0.1071352277483299
1.1964247567313038 0.39284951346260755 0.12499618530273

8 - 0.10713522774832995 0.17856379917689935 0.107135227748
0.24999237060546875 0.49998474121093750 0.2499923706055

0.39284951346260755 1.3214209420340381 0.3928495134626

9- 5.3567613874164977E-02 0.10713522774833 0.12499618530
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Note that the general relation that we can call Relationship 2, between the matrix B and the matrix E,
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Mathews [4] classically solved the system resulting from 9 linear algebraic equations to find steady state temperature distribution using Gaussian elimination method in a more efficient scheme by extend the tridiagonal algorithm to the more sophisticated pentadiagonal algorithm for its arbitrarily chosen vector BC,
b = [100,20,20,80,0,0,260,180,180] T.. . . . . (8)

and arrived at the solution vector:
U=[55,7143,43,2143,27,1429,79,6429,70,0000,45,3571,112,357,111,786,84,2857] T.. . . (9)

Now, in the modified Dirichlet boundary condition vector for the proposed statistical solution corresponding to Figure 5, the vector b of equation 8 is simply rewritten,
[100/4, 20/4, 20/4, 20/4, 80/4, 0. , 0.260/4, 180/4, 180/4] T.. . . . . (10)

The calculated transfer matrix D for RO=0 can be multiplied by the vector (b) of formula 10 to obtain the numerical solution of the proposed technique, we obtain,
U=[55.7132187 43.2126846 27.1417885 79.6412506 69.9978638 45.3555412 112.856079 111.784111 84.2846451]T. . . . ..(11)

If we compare the statistical solution (11) proposed by the Cairo technique with that of Mathews (9), we find a striking precision. The agreement between the results of Mathews and the results of the chain B of the matrix is excellent. Even in the stationary solution of the heat diffusion equation, the superiority of the proposed technique to that of FDM and FEM is evident.

We introduce and explain the numerical theory of the so-called Cairo technique where the space-time PDE such as the heat equation can be discarded and the 2D/3D physical situation is translated directly into a fast-converging stable algorithm.

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The proposed procedure operates on a 4D space-time fabric as a unit and the classically defined scalar thermal diffusion coefficient as K/Roh S is reformulated accordingly

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$UN = D(N)\,(b+S) + B^N\,(IC)$ . . . (3)

For any number of time steps iterations N.

Where B is the transition matrix nxn , b is the boundary condition n vector arranged in a proper order and IC is the initial conditions n vector at the considered n points.

Note that in equation 3, B^N tends to zero for large N, which means that the initial IC conditions dampen out and we are left with the forced BC solution.

Again, the stochastic transition matrix B nxn with all its entries ($b_{i,j}$) is well defined as well as its consequent transfer matrices D(N) and E(N) [3,8,9].

The transfer matrix D(N) is given by,

$D(N)=B+B^2+\ldots+B^N$ . . . . (4)

while the transfer matrix E(N) is simply defined as the D-matrix plus the unit matrix I.

$E(N)=D(N)+I$

In the matrix B power series notations,

$E(N)=B^0+B+B^2+\ldots+B^N$ . . . . (5)

.Obviously $B^0=I$

For sufficiently large number of iteration N steps it can be shown that,

$E(N)=(I-B)^{-1}$ . . . . (6)

Equation 7 below describes the stationary or steady state solution for a sufficiently large number of iterations N,

$U(x,y,z)=D.(b+S)$ . . . . (7)

For any arbitrary boundary conditions vector b and any arbitrary source term S.

Equation 3 is efficient and of great importance because it gives the time-dependent solution of U(x,y,z,t) as a function of BC boundary conditions and IC initial conditions simultaneously in one formula.

It also shows that the boundary condition vector b is treated as the source/sink term vector S.

For nature's solution, the two BC&S are indistinguishable and can be added together.

In order not to worry too much about the details of the theory, we go directly to the numerical examples and their verification in section III.

IIIA. Transient Solutions

Consider the simplest case of 3D geometric configuration. A 3D cube of 8 equidistant free nodes and 8 boundary conditions as shown in figure2.

Fig. 2. Transient heat diffusion in a 3D cube of 8 free nodes with 8 modified Dirichlet source/sink BC.

Here, the input entries of the symmetric 8X8 matrix B are expressed in terms of their four defining conditions explained in Ref [3,8], B=

RO 1/6-RO/6 1/6-RO/6 0.00000000 1/6-RO/6 0.0000000 0 0.00000000 0.00000000

1/6-RO/6 RO 0.00000000 1/6-RO/6 0.00000000 1/6-RO/6 0.00000000 0.00000000

1/6-RO/6 0.00000000 RO 1/6-RO/6 0.00000000 0.00000000 1/6-RO/6 0.00000000

0.00000000 1/6-RO/6 1/6-RO/6 RO 0.00000000 0.00000000 0.00000000 1/6-RO/6

1/6-RO/6 0.00000000 0.00000000 0.00000000 RO 1/6-RO/6 1/6-RO/6 0.00000000 0.00000000 1/6-RO/6 0.00000000 0.00000000 1/6-RO/6 RO 0.00000000 1/6-RO/6

0.00000000 0.00000000 1/6-RO/6 0.00000000 1/6-RO/6 0.00000000 RO 1/6-RO/6

0.00000000 0.00000000 0.00000000 1/6-RO/6 0.00000000 0. 1/6-RO/6 1/6-RO/6 RO

Where RO the main diagonal input element is an element of the closed interval [0,1].

The matrix B and consequently its derived transfer matrices D and E are always symmetric and therefore conform to the physical principle of detail equilibrium

Indeed the choice of the numerical value of the element RO in the interval [0,1] is crucial since it determines the value of the thermal diffusion coefficient which is a physical thermal property of the material.

We assume that the time dependence of energy density on temperature energy density is exponential with the exponent Alpha which is a numerically proven rule.

Therefore, we run Equation 3 for different values of RO in the interval [0, 1] and the corresponding temperature-time curves are obtained.

From these curves, the (dT/dt}/T ratio is calculated for different values of RO and the numerical results are presented in Table I where the (dT/dt)/T ratio is conveniently named Alpha.

Table I. Numerical results for RO vs alpha up to equation 3 for the 3D cube 8 free nodes Fig2.

RO	0.	0.2	0.4	0.6	0.8	1.0
Alpha	0.693	0.511	0.357	0.223	0.105	0.00

Alpha expressed in logarithmic form,

Log 2 Log 1.667 Log 1.429 Log1.25 Log1.111 Log1.0

The conclusion obtained from Table I is that it accurately predicts an important relationship between 3D Alpha and RO as a logarithmic relationship,

Alpha = Log {1 / (1/2 + RO / 2)} . . . Relationship . . . (1)

Now consider another more complicated case of 3D geometric configuration. A 3D rectangular cuboid of 27 equidistant free nodes and 52 BC which can be reduced to 27 boundary conditions, as shown in Figure 3.

Fig. 3. Transient heat diffusion in a 3D rectangular cuboid of 27 free nodes with 57 BC reducible to 27 modified Dirichlet BC technique.

Similar procedure to that followed in figure 1, the entries of the matrix B 27X27 are expressed in the form,27X27 B-Matrix inputs, Line 1

RO 1/6-RO/6 0.0000 1/6- RO 1/6-RO/6 0.0000 0.0000 0.0000 0.0000

1/6-RO/6 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Line 2

1/6-RO/6 RO 1/6-RO/6 0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.000

0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Line 3

0.0000 1/6-RO/6 RO 0.0000 0.0000 1/6RO/6 0.0000 0.0000 0.0000

0.0000 0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

. . . .

Lin14

0.0000 0.0000 0.0000 0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.0000

0.0000 1/6-RO/6 0.0000 1/6-RO/6 RO 1/6-RO/6 0.0000 1/6-RO/6 0.0000

0.0000 0.0000 0.0000 0.0000 1/6-RO/6 0.0000 0.0000 0.0000 0.0000

. . . .

. . . .

Line 25

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
0.0000 0.0000 0.0000 1/6-RO/6 0.0000 0.0000

0.0000 0.0000 0.0000 1/6-RO/6 0.0000 0.0000 RO 1/6-RO/6 0.0000

Line 26

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 1/6-RO/6 0.0000

0.0000 0.0000 0.0000 0.0000 1/6-RO/6 0.0000 1/6-RO/6 RO 1/6-RO/6

Line 27

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

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0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1/6-RO/6
0.0000	0.0000	0.0000	0.0000	0.0000	1/6-RO/6	0.0000	1/6-RO/6	RO

We again repeat the procedure followed in Figure 2 for different values of RO in the interval [0, 1].

The numerical results are presented in Table II.

Table II. Numerical results for Alpha vs RO up to equation 3 for the 3D cube 27 free nodes Fig 3.

RO	0.	0.2	0.4	0.6	0.8	1.
ALPHA	0.376	0.282	0.206	0.129	0.0636	0.

Note that the numerical values for Alpha in table II are proportional to those in in Table I where the constant of proportionality is 16/9 as expected.

The 16/9 ratio comes from the ratio of h^2 of Tables I and II where h=L/3 to that of Table I and h=L/4 of Table II.

In short, Tables I and II confirm relationship 1.

Relationship 1 is striking in itself. It predicts that the exponent of the exponential time evolution of thermal energy density is given by well-defined Alpha,

U(t)=U0 EXP (-ALPHA. t) . . . Continuous time cooling curve,

Alpha=D/L 2

Where L is the characteristic length equals the side of cuboide for rectangular shapes.

And

U(t)=UF (1. -EXP(-Log 1./(.5+.5*RO)N) For discrete time steps N dt in the heating curve.

Similarly, we can show,

RO=D dt /L^2.

What is amazing is that it predicts that Alpha has a minimum value of zero (thermal insulator RO = 0) and a maximum value of Log 2 = 0.697 for thermal super conductor (RO = 1).

Finally, we present an elegant example of the proposed technique shown in Figure 4. Consider another slightly more advanced case of non-cubic 3D geometry configuration,

A 3D rectangular cuboid of 16 equidistant free nodes and 16 BC

Fig. 4. Transient heat diffusion in a 3D rectangular cuboid of 16 free nodes with 16 modified Dirichlet BC.

Here the entries of the 16x16 matrix B which contain all the necessary information for the transient and stationary solutions are constructed for example RO=0 and the transfer matrix Enxn with N large enough (N=30) is calculated by two different methods,

i-Matrix Power series Eq 5

ii-Expression E=(I-B)^-1Eq 6

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Both methods gave exactly the same value for the matrix E as:

E=

1.1070 211 .0780 .211 .220 0.083 0.044 .083 .047 .026 0.017.026 .010 .007 0.005 0.007

0.2111.1070.2110.0780.0830.2200.0830.0440.0260.0470.0260.0170.0070.0100.0070.005

0.0780.2111.1070.2110.0440.0830.2200.0830.0170.0260.0470.0260.0050.0070.0100.007

0.2110.0780.2111.1070.0830.0440.0830.2200.0260.0170.0260.0470.0070.0050.0070.010

0.2200.0830.0440.0831.1540.2370.0940.2370.2300.0900.0490.0900.0470.0260.0170.026

0.0830.2200.0830.0440.2371.1540.2370.0940.0900.2300.0900.0490.0260.0470.0260.017

0.0440.0830.2200.0830.0940.2371.1540.2370.0490.0900.2300.0900.0170.0260.0470.026

0.0830.0440.0830.2200.2370.0940.2371.1540.0900.0490.0900.2300.0260.0170.0260.047

0.0470.0260.0170.0260.2300.0900.0490.0901.1540.2370.0940.2370.2200.0830.0440.083

0.0260.0470.0260.0170.0900.2300.0900.0490.2371.1540.2370.0940.0830.2200.0830.044

0.0170.0260.0470.0260.0490.0900.2300.0900.0940.2371.1540.2370.0440.0830.2200.083

0.0260.0170.0260.0470.0900.0490.0900.2300.2370.0940.2371.1540.0830.0440.0830.220

0.0100.0070.0050.0070.0470.0260.0170.0260.2200.0830.0440.0831.1070.2110.0780.211

0.0070.0100.0070.0050.0260.0470.0260.0170.0830.2200.0830.0440.2111.1070.2110.078

0.0050.0070.0100.0070.0170.0260.0470.0260.0440.0830.2200.0830.0780.2111.1070.211

0.0070.0050.0070.0100.0260.0170.0260.0470.0830.0440.0830.2200.2110.0780.2111.107

IIIB. Steady state solutions

The steady-state equilibrium solution is defined mathematically for Equation 1 as,

dU / dt) partial = 0

Again, as explained in Figure 4, the steady-state numerical solution of the matrix E is obtained by one of two methods, namely,

1-matrix power series summation ,

E = B ^ 0 + B + B ^ 2 ...... + B ^ N .... (5)

for N sufficiently large.

2-Using the equivalence relation,

E = (I-B) ^ - 1. . . . . (6)

Finally the solution in steady state of equilibrium is given by the matrix D = E-I, [3,8] ,

U = D. (b + S) .... (7)

Equation 7 shows that the distribution of the equilibrium temperature field reached after a sufficiently long time t=Ndt is a function of D, b and S.

Fig.5 A 2D rectangular domain with 9 equidistant free nodes.

Consider the simple case of a rectangular domain with 9 equidistant free nodes, u1, u2, u3, ... u9 and 12 Dirichlet modified boundary conditions BC1 to BC12 reduced to 9 BC as illustrated in Figure5.

The 12 boundary conditions in Figure 5 can be reduced to 9 modified BCs for the 9 boundary nodes as follows,

BC1 = BC1X + BC1Y

. . . .

BC9 = BC9X + BC9Y

The transition matrix B 9x9 is built to satisfy the conditions i-iv [3,8]for arbitrary RO and is given by,

1- RO 1/4-RO/4 0.000 1/4-RO/4 0.000 0.000 0.000 0.000 0.000

2- 1/4-RO/4 RO 1/4-RO/4 0.000 1/4-RO/4 0.000 0.000 0.000 0.000

3- 0.000 1/4-RO/4 RO 0.000 1/4-RO/4 0.000 0.000 0.000 0.000

4-1/4-RO/4 0.000 0.000 RO 1/4-RO/4 0.000 1/4-RO/4 0.000 0.000

5- 0.000 1/4-RO/4 0.000 1/4-RO/4 RO 1/4-RO/4 0.000 1/4-RO/4 0.000

6- 0.000 0.000 1/4-RO/4 0.000 1/4-RO/4 RO 0.000 0.000 1/4-RO/4

7- 0.000 0.000 0.000 1/4-RO/4 0.000 0.000 RO 1/4-RO/4 0.000

8- 0.000 0.000 0.000 1/4-RO/4 0.000 1/4-RO/4 0.000 RO 1/4-RO/4

9- 0.000 0.000 0.000 0.000 0.000 1/4-RO/4 0.000 1/4-RO/4 RO

And the transfer matrix E calculated by equation (5) converges

rapidly for a large N. Here are the numerical entries of the matrix E for N=30 and RO=0,

1-

1.1964247567313038 0.39284951346260755 0.124996185302

0.39284951346260755 0.24999237060546875 0.1071352277483

0.12499618530273438 0.10713522774832995 5.356761387416E-02

2- 0.39284951346260755 1.3214209420340381 0.392849513462

0.24999237060546875 0.49998474121093750 0.2499923706055 0.10713522774832995 0.17856379917689935 0.1071352277483 3-0.12499618530273438 0.39284951346260755 1.19642475673130 0.10713522774832995 0.24999237060546875 0.3928495134626 5.3567613874165E-02 0.10713522774832995 0.12499618530273

4- 0.39284951346260755 0.24999237060546875 0.1071352277483

1.3214209420340381 0.49998474121093750 0.17856379917689

0.39284951346260755 0.24999237060546875 0.1071352277483

5- 0.24999237060546875 0.49998474121093750 0.24999237060

0.49998474121093750 1.4999847412109375 0.4999847412109

0.24999237060546875 0.49998474121093750 0.2499923706055

6 -0.10713522774832995 0.24999237060546875 0.3928495134626 0.17856379917689935 0.49998474121093750 1.321420942034 0.10713522774832995 0.24999237060546875 0.3928495134626

7 –

0.12499618530273 0.10713522774832995 5.35676138742 E-02

0.39284951346260 0.24999237060546875 0.1071352277483299

1.1964247567313038 0.39284951346260755 0.12499618530273

8 - 0.10713522774832995 0.17856379917689935 0.107135227748

0.24999237060546875 0.49998474121093750 0.2499923706055

0.39284951346260755 1.3214209420340381 0.3928495134626

9- 5.3567613874164977E-02 0.10713522774833 0.12499618530

0.10713522774833 0.24999237060546875 0.39284951346260755 0.12499618530273438 0.39284951346260755 1.1964247567313

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V. CONCLUSION

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