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Estimation of Cut Point in Burr III Sequence under Linear Exponential Loss

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Abstract:- This paper estimates the single cut-point in the mean of a Burr III Sequence and its scale parameters before and after the cut point. We introduce a strong estimator of the parameters with the help of Bayesian inference approach, by persevering these estimators in the criteria used to estimate the cut-point under Linear Exponential Loss Function. The simulation technique is used compare the estimators. Open-source R software is used in the simulation section. We have taken real data to estimate the parameters of the sequence and then hypothetical observations of the sequence to prove their robustness of the estimators.

Keywords:- *Cut Point, Burr III Sequence, Bayesian Inference, Linear Exponential Loss*

I. INTRODUCTION

The decision theory is based on uncertainty involve in order to estimate the parameters of any sequence in Statistics. It provides the rational frame work for dealing with such situations of uncertainty. The Bayesian method of estimation is a robust way in formulating and dealing with statistical decision problems. The Bayesian approach offers the method of amalgamating the prior belief with random observations of the sequence and convert it to posterior expectation. When the parameters of the sequence are integrated with respect to posterior distribution, one may get the bayes estimate of the parameter parameters. So this method is special type of decision making process in order to maximize the posterior expected utility values or to minimize the posterior expected loss.

II. LOSS FUNCTION

When the magnitude of loss are equal on the basis of positive and negative errors then it is said to be symmetric loss. But this situation not always exist, sometimes over estimation is more serious than underestimation and sometimes underestimation is more serious than overestimation. In such a situation the canfield(1970) pointed out that the use of symmetric loss function is not appropriate in order to estimate the reliability function. Overestimation of Reliability function or average life is much more serious in comparison to the underestimation of Reliability function or average life. Varion(1975) introduced an asymmetric loss function called as linear exponential loss function. Arnold Zellner & Geisel (1968) discussed many form of asymmetric loss functions. Aitchision & Dunsmore (1975) and Berger (1980) have considered the linear asymmetric loss function.

Varian (1975) introduced the following convex loss function known as Linear Exponential Loss Function given below as;

$$L(\delta) = be^{a\delta} - c\delta - b \; ; a, c \neq 0, b > 0 \qquad (1.2.1)$$

Where $\delta = \hat{\theta} - \theta$. It is clear that L(0) = 0 and the minimum occurs when ab=c, therefore , $L(\delta)$ can be written as

$$L(\delta) = b\left[e^{a\delta} - a\delta - 1\right], a \neq 0, b > 0 \qquad (1.2.2)$$

Where a and b are the parameters of the loss function may be defined as shape and scale respectively. The loss function has been considered by Zellner (1986), Basu and Ebrahimi (1991) considered the $L(\delta)$ as

$$L(\delta) = b\left[e^{a\delta} - a\delta - 1\right], \; a \neq 0, b > 0 \qquad (1.2.3)$$

Where, $\delta = \frac{\hat{\theta}}{\theta} - 1$

III. CUT POINTS

When a lifetime testing equipment is installed and start testing the life of items produced by the equipment, it often suffers with the random fluctuations, which gives at some point of time, the irregularity in the sequence of lifetimes of such items. Such irregularities are known as cut points or shift points. These cut points are very important in order to study the Statistical quality control process of the item produced by this system or equipment. These cut points can be single or multiple and can be estimated through classical technique of estimation or Bayesian technique of estimation. Broemeling and Tsurmi (1987) and Zack (1981) are useful references for studying such irregularities. Bayesian approach may play an important role in the study of such Cut or Shift point problem.

In this paper we estimate a single cut point in the sequence of Burr III distribution and the cut point and parameters of the distribution have been studied through Bayesian technique. Many authors Zellner(1986), Calibria and Pulcini(1994), Jani and Pandya(1999), JB Shah, MN Patel(2007), JB Shah(2011), Uma Srivastava(2012) and Uma Srivastava and Harish Kumar(2022) have been often proposed as a valid alternative in classical estimation procedure.

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IV. PRIOR DISTRIBUTION

In frequentist framework, sufficient statistic plays an important role in Bayesian inference in constructing a family of prior distributions known as Natural Conjugate Prior (NCP). The family of prior distributions $g(\theta)$, $\theta \in \Omega$, is called a natural conjugate family if the corresponding posterior distribution belongs to the same family as $g(\theta)$. De Groot (1970) has outlined a simple and elegant method of constructing a conjugate prior for a family of distributions $f(x|\theta)$ which admits a sufficient statistic. De Groot (1970) and Raffia & Schlaifer (1961) provide proof that when sufficient statistics exist a family of conjugate prior distributions exists.

The most widely used prior distribution of θ is the inverted Gamma distribution with the parameters 'a' and 'b' (> 0) with p.d.f. given by

$$g(\theta) = \begin{cases} \frac{b^a}{\Gamma a}\theta^{-(\alpha+1)}e^{-b/\theta}; & \theta > 0, (a,b) > 0 \\ 0 & ,\text{otherwise} \end{cases}$$

(1.4.1)

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted Gamma distribution is conjugate prior of θ Raffia & Schlaifer (1961), Bhattacharya (1967) and others have found that the inverted Gamma can also be used for practical reliability applications.

V. BURR III SEQUENCE

Burr III Sequence model was first introduced by Burr(1942), in the modelling of lifetime data named as Burr III distribution. The probability density function, cumulative density function, reliability function and hazard rate function of Burr III sequence is given below respectively.

$$f(x;\,\theta,\beta) = \theta\,\beta\,x^{-(\beta+1)}\left(1+x^{-\beta}\right)^{-(\theta+1)};\ x>0,\theta,\beta>0 \qquad (1.5.1)$$

And the Cumulative distribution function

$$F(x;\,\theta,\beta) = \left(1+x^{-\beta}\right)^{-\theta} \quad ;x>0,\theta>0,\beta>0 \qquad (1.5.2)$$

Reliability function is

$$R(t;\,\theta,\beta) = 1-\left(1+t^{-\beta}\right)^{-\theta};\ t>0,\theta>0,\beta>0 \qquad (1.5.3)$$

Hazard Rate Function is

$$H(t;\,\theta,\beta) = \frac{\theta\,\beta\,x^{-(\beta+1)}\left(1+x^{-\beta}\right)^{-(\theta+1)}}{1-\left(1+t^{-\beta}\right)^{-\theta}};\ t>0,\theta>0,\beta>0 \qquad (1.5.4)$$

Note that Burr XII sequence can be derived from Burr III sequence by replacing X with $\frac{1}{X}$. The usefulness and properties of Burr sequence are discussed by Burr and Cislak (1968). Abd-Elfattah and Alharbey (2012) considered a Bayesian estimation for Burr III sequence based on double censoring.

In this paper the Bayesian estimation of cut point 'm' and scale parameter 'θ' of Burr III distribution is obtained by using Linear Exponential Loss Function(L.L.F.) and Natural conjugate Prior distribution as Inverted Gamma prior. The comparison of Bayes estimators are done by R-programming.

VI. BAYESIAN ESTIMATION OF CUT POINT IN BURR III SEQUENCE UNDER LINEAR EXPONENTIAL LOSS FUNCTION (LLF)

An independent observation set of life times of n observations are recorded for $n \geq 3$ from Burr III Distribution with parameter θ, β . But it was found that there was a cut in the sequence at some point of time 'm' and it is reflected in the sequence after m^th^ observation which results cut in a sequence as well as parameter value θ. The Bayes estimate of θ and 'm' are obtained for Linear Exponential loss function under inverted Gamma prior.

1.6.1 Likelihood, Posterior and Marginal

Let $x_1, x_2, \ldots\ldots, x_n$, $(n \geq 3)$ be a sequence of observed discrete life times. First let observations $x_1, x_2, \ldots\ldots, x_n$ have come from Burr III sequence with probability density function as

$$f(x,\theta,\beta) = \theta\,\beta x^{-(\beta+1)}\left(1+x^{-\beta}\right)^{-(\theta+1)} \quad (x,\theta,\beta>0) \qquad (1.6.1.1)$$

Let 'm' is change point in the observation which breaks the distribution in two sequences as $(x_1, x_2, \ldots\ldots\ldots x_m)$ & $x_{(m+1)}, x_{(m+2)}, \ldots\ldots x_n$

The probability density functions of the above sequences are

$$f_1(x) = \theta_1\,\beta_1\,x^{-(\beta_1+1)}\left(1+x^{-\beta_1}\right)^{-(\theta_1+1)}; \qquad (1.6.1.2)$$

where $x_1,\dots,x_m > 0; \theta_1,\beta_1, > 0$

$$f_2(x) = \theta_2\,\beta_2\,x^{-(\beta_2+1)}\left(1+x^{-\beta_2}\right)^{-(\theta_2+1)}; \qquad (1.6.1.3)$$

where $x_{(m+1)}, x_{(m+2)},\dots,x_n;\ \theta_2,\beta_2 > 0$

The likelihood functions of probability density function of the sequence are

$$L_1(x|\theta_1,\beta_1) = \prod_{j=1}^{m} f\left(x_j|\theta_1,\beta_1\right)$$

$$L_1(x|\theta_1,\beta_1) = {\theta_1}^m {\beta_1}^m \prod_{j=1}^{m} \frac{{x_j}^{-(\beta_1+1)}}{(1+{x_j}^{-\beta_1})}\quad e^{-\theta_1 \sum_{j=1}^{m}\log\left(1+{x_j}^{-\beta_1}\right)}$$

$$L_1(x|\theta_1,\beta_1,) = (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{3m}} \qquad (1.6.1.4)$$

Where

$$U_1 = \prod_{j=1}^{m} \frac{{x_j}^{-(\beta_1+1)}}{\left(1+{x_j}^{-\beta_1}\right)}$$

$$T_{3m} = \sum_{j=1}^{m}\log\left(1+{x_j}^{-\beta_1}\right)$$

$$L_2(x|\theta_2,\beta_2) = \prod_{j=(m+1)}^{n} f\left(x_j|\theta_2,\beta_2\right)$$

$$L_2(x|\theta_2,\beta_2) = \theta_2^{(n-m)}\beta_2^{(n-m)} \prod_{j=(m+1)}^{n} \frac{{x_j}^{-(\beta_2+1)}}{(1+{x_j}^{-\beta_2})}\ e^{-\theta_2 \sum_{j=1}^{m}\log\left(1+{x_j}^{-\beta_2}\right)}$$

$$L_2(x|\theta_2,\beta_2) = (\theta_2\beta_2)^{(n-m)} U_2 e^{-\theta_2(T_{3n}-T_{3m})} \qquad (1.6.1.5)$$

where

$$U_2 = \prod_{j=m+1}^{n} \frac{{x_j}^{-(\beta_2+1)}}{(1+{x_j}^{-\beta_2})}$$

and $\quad T_{3n} - T_{3m} = \sum_{j=(m+1)}^{n}\log\left(1+{x_j}^{-\beta_2}\right)$

The joint likelihood function is given by

$$L\left(\theta_1,\theta_2|\underline{x}\right) \propto (\theta_1\beta_1)^m U_1\, e^{-\theta_1 T_{3m}}(\theta_2\beta_2)^{n-m} U_2\, e^{-\theta_2(T_{3n}-T_{3m})} \qquad (1.6.1.6)$$

1.6.2 Prior

Suppose the marginal prior distribution of θ_1 and θ_2 are natural conjugate prior

$$\pi_1\left(\theta_1,\underline{x}\right) = \frac{b_1^{a_1}}{\Gamma a_1}\theta_1^{(a_1-1)}e^{-b_1\theta_1};\qquad a_1,b_1 > 0, \theta_1 > 0 \qquad (1.6.2.1)$$

$$\pi_2\left(\theta_2,\underline{x}\right) = \frac{b_1^{a_2}}{\Gamma a_2}\theta_2^{(a_2-1)}e^{-b_2\theta_2};\qquad a_2,b_2 > 0, \theta_2 > 0 \qquad (1.6.2.2)$$

The joint prior distribution of θ_1 ,θ_2 and change point 'm' is

$$\pi(\theta_1,\theta_2,m) \propto \frac{b_1^{a_1}}{\Gamma a_1}\frac{b_2^{a_2}}{\Gamma a_2}\theta_1^{(a_1-1)}e^{-b_1\theta_1}\theta_2^{(a_2-1)}e^{-b_2\theta_2} \qquad (1.6.2.3)$$

Where $\theta_1,\theta_2 > 0\ \ \&\ \ m = 1,2,\dots\dots(n-1)$

1.6.3 Posterior

The joint posterior density of θ_1,θ_2 and m say $\rho\left(\theta_1,\theta_2,m/\underline{x}\right)$ is obtained by using equations (1.6.1.6) & (1.6.1.9)

$$\rho\left(\theta_1,\theta_2\,,m|\underline{x}\right) = \frac{\mathrm{L}(\theta_1\,,\theta_2/\underline{x})\pi(\theta_1\,,\theta_2,m)}{\sum_m \iint_{\theta_1\theta_2}\mathrm{L}(\theta_1\,,\theta_2/\underline{x})\pi(\theta_1,\theta_2,m)d\theta_1 d\theta_2} \qquad (1.6.3.1)$$

$$\rho\left(\theta_1,\theta_2\,,m|\underline{x}\right) = \frac{{\theta_1}^{(m+a_1-1)}\,e^{-\theta_1(T_{3m}+b_1)}\ {\theta_2}^{(n-m+a_2-1)}\ e^{-\theta_2(T_{3n}-T_{3m}+b_2)}}{\sum_m\int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,{\theta_1}^{(m+a_1-1)}\,d\theta_1\ \int_0^\infty {\theta_2}^{(n-m+a_2-1)}\ e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1)=x$ & $\theta_2(T_{3n}-T_{3m}+b_2)=y$

$$\theta_1=\frac{x}{(T_{3m}+b_1)} \quad \& \quad \theta_2=\frac{y}{T_{3n}-T_{3m}+b_2}$$

$$d\theta_1=\frac{dx}{(T_{3m}+b_1)} \quad \& \quad \mathrm{d}\theta_2=\frac{dy}{T_{3n}-T_{3m}+b_2}$$

$$\rho(\theta_1,\theta_2\,,m|\underline{x})=\frac{{\theta_1}^{(m+a_1-1)}\,e^{-\theta_1(T_{3m}+b_1)}\,{\theta_2}^{(n-m+a_2-1)}\,e^{-\theta_2(T_{3n}-T_{3m}+b_2)}}{\sum_m\int_0^\infty e^{-x}\frac{x^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dx}{(T_{3m}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{y}}\frac{\mathrm{y}^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dy}{(T_{3n}-T_{3m}+b_2)}}$$

$$\rho(\theta_1,\theta_2\,,m|\underline{x})=\frac{e^{-\theta_1(T_{3m}+b_1)}\,{\theta_1}^{(m+a_1-1)}\,e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,{\theta_2}^{(n-m+a_2-1)}}{\sum_m\frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_1,\theta_2\,,m|\underline{x})=\frac{e^{-\theta_1(T_{3m}+b_1)}\,{\theta_1}^{(m+a_1-1)}\,e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,{\theta_2}^{(n-m+a_2-1)}}{\xi(a_{1,}a_{2,}b_{1,}b_{2,}m,n)} \qquad (1.6.3.2)$$

Where $\xi(a_{1,}a_{2,}b_{1,}b_{2,}m,n)=\sum_{m=1}^{n-1}\left[\frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{m+a_1}}\,\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}\right]$

1.6.4 Marginal posterior

The Marginal posterior distribution of change point 'm' using the equations (1.6.1.6), (1.6.2.1) & (1.6.2.2)

$$\rho(m|\underline{x})=\frac{\mathrm{L}(\theta_1\,,\theta_2/\underline{x}\,)\,\pi(\theta_1)\,\pi(\theta_2)}{\sum_m \mathrm{L}(\theta_1\,,\theta_2/\underline{x}\,)\,\pi(\theta_1)\,\pi(\theta_2)} \qquad (1.6.4.1)$$

On solving which gives

$$\rho(m|\underline{x})=\frac{\int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,{\theta_1}^{(m+a_1-1)}\,d\theta_1\int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,{\theta_2}^{(n-m+a_2-1)}\,d\theta_2}{\sum_m\int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,{\theta_1}^{(m+a_1-1)}\,d\theta_1\int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,{\theta_2}^{(n-m+a_2-1)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1)=y$ & $\theta_2(T_{3n}-T_{3m}+b_2)=z$

$$\theta_1=\frac{y}{(T_{3m}+b_1)} \quad \& \quad \theta_2=\frac{z}{T_{3n}-T_{3m}+b_2}$$

$$d\theta_1=\frac{dy}{(T_{3m}+b_1)} \quad \& \quad d\theta_2=\frac{z}{T_{3n}-T_{3m}+b_2}$$

$$\rho(m|\underline{x})=\frac{\int_0^\infty e^{-y}\frac{y^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dy}{(T_{3m}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{z}}\frac{\mathrm{z}^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dz}{(T_{3n}-T_{3m}+b_2)}}{\sum_m\int_0^\infty e^{-y}\frac{y^{(m+a_1-1)}}{(T_{3m}+b_1)^{(m+a_1-1)}}\frac{dy}{(T_{3m}+b_1)}\int_0^\infty \mathrm{e}^{-\mathrm{z}}\frac{\mathrm{z}^{(n-m+a_2-1)}}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2-1)}}\frac{dz}{(T_{3n}-T_{3m}+b_2)}}$$

$$\rho(m|\underline{x})=\frac{\frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_{1,}a_{2,}b_{1,}b_{2,}m,n)} \qquad (1.6.4.2)$$

The marginal posterior distribution of θ_1, using equations (1.6.1.6) and (1.6.2.1)

$$\rho(\theta_1|\underline{x})=\frac{\mathrm{L}(\theta_1\,,\theta_2/\,\underline{\mathrm{x}})\,\pi(\theta_1)}{\int_0^\infty \mathrm{L}(\theta_1\,,\theta_2/\,\underline{\mathrm{x}})\,\pi(\theta_1)\mathrm{d}\theta_1}$$

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On solving which gives

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m e^{-\theta_1(T_{3m}+b_1)}\,\theta_1{}^{(m+a_1-1)}\int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,\theta_2{}^{(n-m+a_2-1)}\,d\theta_2}{\sum_m \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,\theta_1{}^{(m+a_1-1)}\,d\theta_1 \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,\theta_2{}^{(n-m+a_2-1)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1) = y$ & $\theta_2(T_{3n}-T_{3m}+b_2) = z$

$\theta_1 = \frac{y}{(T_{3m}+b_1)}$ & $\theta_2 = \frac{z}{T_{3n}-T_{3m}+b_2}$

$d\theta_1 = \frac{dy}{(T_{3m}+b_1)}$ & $d\theta_2 = \frac{dz}{T_{3n}-T_{3m}+b_2}$

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m e^{-\theta_1(T_{3m}+b_1)}\,\theta_1{}^{(m+a_1-1)}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_1|\underline{x}) = \frac{\sum_m e^{-\theta_1(T_{3m}+b_1)}\,\theta_1{}^{(m+a_1-1)}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)} \qquad (1.6.4.3)$$

The marginal posterior distribution of θ_2, using the equation (1.6.1.6) & (1.6.2.2) is

$$\rho(\theta_2|\underline{x}) = \frac{L(\theta_1,\theta_2/\underline{x})\,\pi(\theta_2)}{\int_0^\infty L(\theta_1,\theta_2/\underline{x})\,\pi(\theta_2)\,d\theta_2}$$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,\theta_2{}^{(n-m+a_2-1)}\int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,\theta_1{}^{(m+a_1-1)}\,d\theta_1}{\sum_m \int_0^\infty e^{-\theta_1(T_{3m}+b_1)}\,\theta_1{}^{(m+a_1-1)}\,d\theta_1 \int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,\theta_2{}^{(n-m+a_2-1)}\,d\theta_2}$$

Assuming $\theta_1(T_{3m}+b_1) = y$ & $\theta_1 = \frac{y}{(T_{3m}+b_1)}$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,\theta_2{}^{(n-m+a_2-1)}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}e^{-\theta_2(T_{3n}-T_{3m}+b_2)}\,\theta_2{}^{(n-m+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,m,n)} \qquad (1.6.4.4)$$

1.6.2 Bayes Estimators under Linear Exponential Loss Function (LLF)

The Bayes estimate $\hat{m}_{BL}$ of m under Linear Exponential Loss Function using marginal posterior of equation (1.6.4.2), is given as

$$\hat{m}_{BL} = -\frac{1}{k_1}\log\left[\frac{\sum_m e^{-k_1 m}\frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right] \qquad (1.6.2.1)$$

The Bayes estimate of $\hat{\theta}_{1BL}$ of θ_1 using marginal posterior of equation (1.6.4.3) under Linear Exponential Loss Function is given by

$$\hat{\theta}_{1BL} = -\frac{1}{k_1}\log E_\rho[\exp(-k_1\theta_1)]$$

$$\hat{\theta}_{1BL} = -\frac{1}{k_1}\log\left[\frac{\sum_m \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}\int_0^\infty e^{-\theta_1(T_{3m}+b_1+k_1)}\,{\theta_1}^{(m+a_1-1)}\,d\theta_1}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]$$

Assuming $\theta_1(T_{3m}+b_1+k_1) = y$ & $\theta_1 = \frac{y}{(T_{3m}+b_1+k_1)}$

$$\hat{\theta}_{1BL} = -\frac{1}{k_1}\log\left[\frac{\sum_m \frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2)^{(n-m+a_2)}}\frac{\Gamma(m+a_1)}{(T_{3m}+b_1+k_1)^{(m+a_1)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]$$

$$\hat{\theta}_{1BL} = -\frac{1}{k_1}\log\left[\frac{\xi[a_1,a_2,(b_1+k_1),b_2,m,n]}{\xi(a_1,a_2,b_1,b_2,m,n)}\right] \quad (1.6.2.2)$$

The Bayes estimate of $\hat{\theta}_{2BL}$ of θ_2 using marginal posterior of equation (1.6.4.4) under Linear Exponential Loss Function is given by

$$\hat{\theta}_{2BL} = -\frac{1}{k_2}\log E_\rho[\exp(-k_2\theta_2)]$$

$$\hat{\theta}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\int_0^\infty e^{-\theta_2(T_{3n}-T_{3m}+b_2+k_2)}\,{\theta_2}^{(n-m+a_2-1)}d\theta_2}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]$$

Assuming $\theta_2(T_{3n}-T_{3m}+b_2+k_2) = y$ & $\theta_2 = \frac{y}{(T_{3n}-T_{3m}+b_2+k_2)}$

Then

$$\hat{\theta}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{3m}+b_1)^{(m+a_1)}}\frac{\Gamma(n-m+a_2)}{(T_{3n}-T_{3m}+b_2+k_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]$$

$$\hat{\theta}_{2BL} = -\frac{1}{k_2}\log\left[\frac{\xi[a_1,a_2,b_1,(b_2+k_2),m,n]}{\xi(a_1,a_2,b_1,b_2,m,n)}\right] \quad (1.6.2.3)$$

- ***Numerical Comparison for Burr III Sequences***

Twenty observations are generated from Burr III sequence taking the parameters values as θ = 2 and $\beta = 0.5$. If the target value of θ_1 is unknown, its estimating $(\hat{\theta}_1.)$ is given by the mean of first m sample observation given m=11, $\theta = 0.827$.The observed sequence mean of Burr III sequence is 1.8829 and the observed sequence of Burr III sequence variance is 23.8886. If there is a cut in sequence on 11^{th} observation, then the means and variances of both sequences ($x_1, x_2, \ldots, x_m$) and ($x_{(m+1)}, x_{(m+2)}, \ldots, x_n$) are θ_1 = 0.8277, θ_2 = 3.2668 and ${\sigma_1}^2 = 0.7281$and ${\sigma_2}^2 = 51.8509$.

- ***Numerical Comparison of Bayes Estimates***

The numerical comparison of Bayes estimates before and after cut point and the parameters of prior distribution a_1, b_1, a_2 and b_2 is done by using R- programming. The calculated means and variances of the prior distribution are used as prior information in calculating these parameters. Then with these parameter values we have computed the Bayes estimates of m, θ_1 and θ_2 under Linear Exponential Loss Function (LLF) with considering different set of values of (a_1, b_1) and (a_2, b_2).We have taken different sample sizes from n=10(10)30. The Bayes estimates of the change point 'm' and the parameters θ_1 and θ_2 are calculated and discussed above under Linear Exponential Loss Function. Their respective mean squared errors(M.S.E's) are calculated by repeating this process 1000 times and presented in same table in small parenthesis under the estimated values of parameters. The greater values appear to be robust with respect to correct choice of prior parameter values and appropriate sample size. All the estimators perform better with sample size n=20.Similarly the Bayes estimates of Linear Exponential Loss Function are presented in table 1. appears to be sensitive with wrong choice of prior parameters and sample size. All the calculations are done by R- programming.

VII. CONCLUSION

The values of the estimates of Bayes estimators of the parameters θ_1 and θ_2 of Burr III sequence obtained under Linear Exponential Loss Function are different for different numerical values and exhibits its robustness as the sample size increases for increasing values of parameters. The respective M.S.E's shows that the Bayes estimates uniformly smaller for $\hat{\theta}_{1BL}$ and $\hat{\theta}_{2BL}$ under Linear Exponential Loss Function except of $\hat{m}_{BL}$. The Bayes estimates of the parameters are dominated uniformly for increasing values of prior parameters and increasing values of sample size. From the Table below we see that when parameter values for (a_1, b_1)= (1.25,1.50) and (a_2, b_2)= (1.50,1.60) for sample size n=10 the bayes estimate of cut point is nearly accurate and it has lower MSE than with large sample size, While the Bayes estimates before the cut $\hat{\theta}_{1BL}$ and after the cut $\hat{\theta}_{2BL}$ are nearly accurate with lower MSE's for large sample sizes. This position is similar for (a_1, b_1) = (2.50,2.75) and (a_2, b_2) = (2.50,2.60). It means for the whole interval of values of parameters a and b taken are appropriate choices of values of the parameters. In this interval with large sample sizes the Bayes estimates before the cut $\hat{\theta}_{1BL}$ and after the cut $\hat{\theta}_{2BL}$ show their dominance.

(a_1, b_1)	(a_2, b_2)	n	$\hat{m}_{BL}$	$\hat{\theta}_{1BL}$	$\hat{\theta}_{2BL}$
(1.25,1.50)	(1.50,1.60)	10	7.8948 (36.5171)	0.6413 (5.9861)	0.6152 (1.1503)
		20	17.3097 (250.3565)	0.5312 (0.8248)	0.8262 (0.5866)
		30	27.1469 (646.4207)	0.9763 (0.4552)	0.5862 (2.3298)
(1.50,1.75)	(1.70,1.80)	10	7.8690 (36.4594)	0.7627 (0.9434)	0.6014 (0.0222)
		20	17.6967 (247.5144)	1.3161 (1.9984)	0.7794 (0.9327)
		30	27.1218 (660.2824)	2.2996 (1.0895)	0.6672 (0.2426)
(1.75,2.0)	(1.90,2.0)	10	7.8521 (35.0195)	0.7155 (0.0042)	0.7345 (0.3444)
		20	17.3708 (251.8746)	1.4007 (1.5429)	0.5886 (0.0015)
		30	27.0210 (647.0855)	0.5083 (0.0059)	0.7076 (0.2569)
(2.0,2.25)	(2.10,2.20)	10	7.9550 (33.8673)	0.8548 (0.1671)	0.7306 (0.5313)
		20	17.5939 (247.5098)	1.1723 (1.3645)	1.3232 (0.0133)
		30	27.6032 (682.7767)	0.7353 (0.5945)	1.4175 (0.2827)
(2.25,2.50)	(2.30,2.40)	10	7.8909 (35.3837)	1.7819 (0.0316)	1.0418 (0.1047)
		20	17.6132 (247.8102)	0.7043 (0.1263)	0.7419 (1.2762)
		30	27.3636 (665.9606)	0.9789 (0.0913)	0.6826 (0.0061)
(2.50,2.75)	(2.50,2.60)	10	7.9322 (36.6559)	0.8778 (0.2733)	0.9283 (0.0524)
		20	17.5239 (244.2804)	0.7173 (0.0502)	0.5908 (0.0163)
		30	27.3548 (642.6078)	0.6623 (0.0058)	0.5610 (0.9069)

Table 1:- Bayes Estimates of m, θ_1 & θ_2 for Burr III and their respective M.S.E.'s Under Linear Exponential Loss

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