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Automorphisms of Some $\boldsymbol{p}-$ Groups of Order $\boldsymbol{p^4}$

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Abstract:- There are fifteen groups of order p^4, Out of which five are abelian and the rest are non-abelian. In this paper, we compute the automorphisms of some non-abelian groups of order p^4, where p is an odd prime and the verification of number of automorphisms has been made through GAP (Group Algorithm Programming) software.

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I. INTRODUCTION

Out of available research papers related to automorphisms of p-group for instance [1], [5], [4] etc. We take G be a p-group of order p^4, p - odd prime and $Aut(G)$ be the group of all automorphisms of a group G. In [1], the automorphisms of groups of order p^3 are computed along with the automorphisms of abelian groups of order p^4. This paper will serve as an extension of the research work in [1].There are 10 non-abelian groups of order p^4, In this paper, we shall compute the automorphisms of two groups of order p^4. W. Burnside [2] classified all groups of order p^4. Burnside's [2] classification will be used to compute the automorphisms of groups of order p^4. Two sections will constitute this paper . The first section is devoted to the classification of p-groups of order p^4 and the second section will investigate the number of automorphisms of some non-abelian groups of order p^4.

II. CLASSIFICATION OF p-GROUPS OF ORDER p^4

As per the classification provided by the W. Burnside [2], if p is an odd prime number, then there are 15 groups of order p^4. Five of which are abelian and rest are non-abelian which are listed below:

A. Abelian Groups:

$G_1 =< w_1 : w_1^{p^4} = 1 > \; \cong Z_{p^4}$

$G_2 =< w_1, w_2 : w_1^{p^3} = w_2^p = 1, w_1w_2 = w_2w_1 > \; \cong \; Z_{p^3} \times Z_p$

$G_3 =< w_1, w_2 : w_1^{p^2} = w_2^{p^2} = 1, w_1w_2 = w_2w_1 > \; \cong Z_{p^2} \times Z_{p^2}$

$G_4 =< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^p = 1, w_1w_2 = w_2w_1, w_2w_3 = w_3w_2, w_1w_3 = w_3w_1 > \; \cong Z_{p^2} \times Z_p \times Z_p$

$$G_5 =< w_1, w_2, w_3, w_4 : w_1^p = w_2^p = w_3^p = w_4^p = 1, w_1w_2 = w_2w_1, w_1w_3 = w_3w_1, w_1w_4 = w_4w_1, w_2w_3 = w_3w_2, w_2w_4 = w_4w_2, w_3w_4 = w_4w_3 > \; \cong Z_p \times Z_p \times Z_p \times Z_p$$

B. Non-Abelian Groups:

$$G_6 =< w_1, w_2 : w_1^{p^3} = w_2^p = 1, w_2w_1 = w_1^{1+p^2}w_2 > \cong Z_{p^3} \rtimes_\phi Z_p, \qquad \phi(y) \leftrightarrow (1+p^2)^y$$

$$G_7 =< w_1, w_2 : w_1^{p^2} = w_2^{p^2} = 1, w_2w_1 = w_1^{1+p}w_2 > \cong Z_{p^2} \rtimes_\phi Z_{p^2}, \qquad \phi(y) \leftrightarrow (1+p)^y$$

$$G_8 =< w_1, w_2, w_3 : w_1^p = w_2^p = w_3^{p^2} = 1, w_1w_2 = w_2w_1, w_2w_3 = w_3w_2, w_3w_1 = w_1w_2w_3 > \; \cong (Z_p \times Z_p) \rtimes Z_{p^2}$$

$$G_9 =< w_1, w_2, w_3, w_4 : w_1^p = w_2^p = w_3^p = w_4^p = 1, w_4w_3 = w_3w_4, w_2w_4 = w_4w_2, w_1w_4 = w_4w_1, w_2w_3 = w_3w_2, w_2w_1w_3 = w_3w_1, w_1w_2 = w_2w_1 > \; \cong ((Z_p \times Z_p) \rtimes Z_p) \times Z_p$$

For $p > 3, G_{10} =< w_1, w_2, w_3, w_4 : w_1^p = w_2^p = w_3^p = w_4^p = 1, w_4w_3 = w_2w_3w_4, w_4w_2 = w_1w_2w_4, w_1w_4 = w_4w_1, w_2w_3 = w_3w_2, w_1w_3 = w_3w_1, w_1w_2 = w_2w_1 > \cong (Z_p \times Z_p \times Z_p) \rtimes Z_p$

if $p = 3$, then $< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^p = 1, w_1 w_2 = w_2 w_1, w_1 w_3 = w_3 w_1 w_2, w_2 w_3 = w_3 w_1^{-p} w_2 >$

$$G_{11} =< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^p = 1, w_2 w_1 = w_1^{1+p} w_2, w_1 w_3 = w_3 w_1, w_2 w_3 = w_3 w_2 > \cong (Z_{p^2} \rtimes Z_p) \times Z_p$$

$$G_{12} =< w_1^{p^2} = w_2^p = w_3^p = 1, w_2 w_1 = w_1 w_2, w_1 w_3 = w_3 w_1, w_3 w_2 = w_1^p w_2 w_3 > \cong (Z_{p^2} \times Z_p) \rtimes Z_p$$

$$G_{13} =< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^p = 1, w_2 w_1 = w_1^{1+p} w_2, w_3 w_1 = w_1 w_2 w_3, w_2 w_3 = w_3 w_2 > \cong (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p, \phi_1(z) \leftrightarrow (1,1,0)^z$$

For $p > 3, G_{14} =< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^p = 1, w_2 w_1 = w_1^{1+p} w_2, w_3 w_1 = w_1^{1+p} w_2 w_3, w_3 w_2 = w_1^p w_2 w_3 > \cong (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_2} Z_p, \phi_2(z) \leftrightarrow (1,1,1)^z$

if $p = 3$, then $< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = 1, w_3^p = w_1^p, w_1 w_2 = w_2 w_1^{1+p}, w_1 w_3 = w_3 w_1 w_2^{-1}, w_3 w_2 = w_2 w_3 >$

For $p > 3, G_{15} =< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^p = 1, w_2 w_1 = w_1^{1+p} w_2, w_3 w_1 = w_1^{1+dp} w_2 w_3, w_3 w_2 = w_1^{dp} w_2 w_3 > \cong (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_3} Z_p, \phi_3(z) \leftrightarrow (1,1,d)^z$ where $d \not\equiv 0,1 (mod\ \ p)$,

For $p = 3$, $< w_1, w_2, w_3 ; w_1^{p^2} = w_2^p = 1, w_3^p = w_1^{-p}, w_1 w_2 = w_2 w_1^{1+p}, w_1 w_3 = w_3 w_1 w_2^{-1}, w_3 w_2 = w_2 w_3 >$

III. NUMBER OF AUTOMORPHISMS OF SOME NON-ABELIAN GROUPS OF ORDER p^4

The automorphisms of five abelian groups namely G_1, G_2, G_3, G_4 and G_5 of order p^4 are already computed in the paper [1]. In this paper, we compute the automorphisms of some non-abelian $p - groups$ of order p^4 particularly G_{12} and G_{13} denoted above.

A. First we are going to compute automorphisms of G_{12}.

$$G_{12} =< w_1, w_2, w_3 ; w_1^{p^2} = w_2^p = w_3^p = 1, w_3 w_2 = w_1^p w_2 w_3, w_2 w_1 = w_1 w_2, w_1 w_3 = w_3 w_1 >= (Z_{p^2} \times Z_p) \rtimes Z_p$$

From this it is straight forward to find the formula for following expressions:

$$w_3^i w_2^j = w_1^{ijp} w_2^j w_3^i$$
$$(w_2 w_3)^n = w_1^{\frac{n(n-1)}{2} p} w_2^n w_3^n$$
$$(w_2^i w_3^j)^n = w_1^{\frac{n(n-1)}{2} ijp} w_2^{ni} w_3^{nj}$$

With these we begin our study of automorphism group of $(Z_{p^2} \times Z_p) \rtimes Z_p$.

Let $\phi \in (Z_{p^2} \times Z_p) \rtimes Z_p$ be defined as

$$\phi : \begin{cases} w_1 \to w_1^i w_2^j w_3^k, & i \in Z_{p^2}, j, k \in Z_p. \\ w_2 \to w_1^l w_2^m w_3^n, & l \in Z_{p^2}, m, n \in Z_p. \\ w_3 \to w_1^q w_2^r w_3^s, & q \in Z_{p^2}, r, s \in Z_p. \end{cases}$$

If order of $w_1 \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is p^2, then the order of $\phi(w_1)$ is also p^2, So

$$(w_1^i w_2^j w_3^k)^p \neq 1$$
$$\Rightarrow w_1^{pi} \neq 1$$
$$\Rightarrow p \nmid i$$

Also, order of $w_2 \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is p, so order of $\phi(w_2)$ is also p,

$$(w_1^l w_2^m w_3^n)^p = 1$$
$$\Rightarrow w_1^{lp} = 1$$
$$\Rightarrow p / l$$
$$\Rightarrow l = pt; \text{forsome} \quad t \in Z_p$$

And order of $w_3 \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is p, therefore order of $\phi(w_3)$ is p

$$\Rightarrow (w_1^q w_2^r w_3^s)^p = 1$$
$$\Rightarrow w_1^{qp} = 1$$
$$\Rightarrow p / q$$
$$\Rightarrow q = pu; \quad \text{for some} \quad u \in Z_p$$

Since this is a non-abelian group, we also must have that if $w_3 w_2 = w_1^p w_2 w_3$ then

$$\phi(w_3)\phi(w_2) = \phi(w_1)^p \phi(w_2)\phi(w_3)$$
$$\Rightarrow w_1^q w_2^r w_3^s w_1^l w_2^m w_3^n = (w_1^i w_2^j w_3^k)^p w_1^l w_2^m w_3^n w_1^q w_2^r w_3^s$$

$$\Rightarrow w_1^q w_1^l w_2^r w_3^s w_2^m w_3^n = w_1^{pi} w_1^l w_1^q w_2^m w_3^n w_2^r w_3^s$$
$$\Rightarrow w_1^{q+l} w_2^r w_1^{smp} w_2^m w_3^s w_3^n = w_1^{pi+l+q} w_2^m w_1^{nrp} w_2^r w_3^n w_3^s$$
$$\Rightarrow w_1^{q+l} w_1^{smp} w_2^r w_2^m w_3^{s+n} = w_1^{pi+l+q} w_1^{nrp} w_2^m w_2^r w_3^{n+s}$$
$$\Rightarrow w_1^{q+l+smp} w_2^{r+m} w_3^{s+n} = w_1^{pi+l+q+nrp} w_2^{m+r} w_3^{n+s}$$
$$\Rightarrow w_1^{smp} = w_1^{pi+nrp}$$
$$\Rightarrow smp \equiv pi + nrp (mod \quad p^2)$$
$$\Rightarrow sm \equiv i + nr (mod \quad p)$$
$$\Rightarrow i \equiv sm - nr (mod \quad p)$$

From abelian relation, $w_1 w_2 = w_2 w_1$, we get

$$\phi(w_1)\phi(w_2) = \phi(w_2)\phi(w_1)$$
$$\Rightarrow w_1^i w_2^j w_3^k w_1^l w_2^m w_3^n = w_1^l w_2^m w_3^n w_1^i w_2^j w_3^k$$
$$\Rightarrow w_1^i w_1^l w_2^j w_3^k w_2^m w_3^n = w_1^l w_1^i w_2^m w_3^n w_2^j w_3^k$$
$$\Rightarrow w_1^{i+l} w_2^j w_1^{kmp} w_2^m w_3^k w_3^n = w_1^{l+i} w_2^m w_1^{njp} w_2^j w_3^n w_3^k$$
$$\Rightarrow w_1^{i+l} w_1^{kmp} w_2^j w_2^m w_3^{k+n} = w_1^{l+i} w_1^{njp} w_2^m w_2^j w_3^{n+k}$$
$$\Rightarrow w_1^{i+l+kmp} w_2^{j+m} w_3^{k+n} = w_1^{l+i+njp} w_2^{m+j} w_3^{n+k}$$
$$\Rightarrow w_1^{kmp} = w_1^{njp}$$
$$\Rightarrow kmp \equiv njp (mod \quad p^2)$$
$$\Rightarrow km \equiv nj (mod \quad p)$$

$$\Rightarrow km - nj \equiv 0 (mod \quad p) \tag{1}$$

Also, $w_1 w_3 = w_3 w_1$, then

$$\phi(w_1)\phi(w_3) = \phi(w_3)\phi(w_1)$$
$$w_1^i w_2^j w_3^k w_1^q w_2^r w_3^s = w_1^q w_2^r w_3^s w_1^i w_2^j w_3^k$$
$$\Rightarrow w_1^i w_1^q w_2^j w_3^k w_2^r w_3^s = w_1^q w_1^i w_2^r w_3^s w_2^j w_3^k$$
$$\Rightarrow w_1^{i+q} w_2^j w_1^{krp} w_2^r w_3^k w_3^s = w_1^{q+i} w_2^r w_1^{sjp} w_2^j w_3^s w_3^k$$
$$\Rightarrow w_1^{i+q} w_1^{krp} w_2^j w_2^r w_3^{k+s} = w_1^{q+i} w_1^{sjp} w_2^r w_2^j w_3^{s+k}$$
$$\Rightarrow w_1^{q+i+krp} w_2^{j+r} w_3^{k+s} = w_1^{q+i+sjp} w_2^{r+j} w_3^{k+s}$$
$$\Rightarrow w_1^{krp} = w_1^{sjp}$$
$$\Rightarrow krp \equiv sjp (mod \quad p^2)$$
$$\Rightarrow kr \equiv sj (mod \quad p)$$

$$\Rightarrow kr - sj \equiv 0 (mod \quad p) \tag{2}$$

(1) and (2) are two homogeneous equations with $sm - nr \neq 0$ determinant. So they have only trivial solution.

$$\Rightarrow j = k = 0.$$

Therefore we have that $j = k = 0$, $i = sm - nr \neq 0$, p/l and p/q are the only constraints. we must place on the homomorphism to satisfy the relations of the group . Now, we only have to see when it is bijective. Since we are talking about finite group, so, it is enough to show that it is injective and so we can show that the kernel is trivial. Let $a^x w_2^y w_3^z \in (Z_{p^2} \times Z_p) \rtimes Z_p$ and is mapped to identity of group.

i.e. $\phi(w_1^x w_2^y w_3^z) = 1$

$$\Rightarrow \phi(w_1)^x \phi(w_2)^y \phi(w_3)^z = 1$$
$$\Rightarrow (w_1^i)^x (w_1^l w_2^m w_3^n)^y (w_1^q w_2^r w_3^s)^z = 1$$
$$\Rightarrow w_1^{ix} w_1^{ly+\frac{y(y-1)}{2}mnp} w_2^{my} w_3^{ny} w_1^{qz+\frac{z(z-1)}{2}rsp} w_2^{rz} w_3^{sz} = 1$$
$$\Rightarrow w_1^{ix+ly+\frac{y(y-1)}{2}mnp} w_1^{qz+\frac{z(z-1)}{2}rsp} w_2^{my} w_3^{ny} w_2^{rz} w_3^{sz} = 1$$
$$\Rightarrow w_1^{ix+ly+\frac{y(y-1)}{2}mnp} w_1^{qz+\frac{z(z-1)}{2}rsp} w_2^{my} w_1^{nyrzp} w_2^{rz} w_3^{ny} w_3^{sz} = 1$$
$$\Rightarrow w_1^{ix+ly+\frac{y(y-1)}{2}mnp+qz+\frac{z(z-1)}{2}rsp} w_1^{nryzp} w_2^{my} w_2^{rz} w_3^{ny+sz} = 1$$
$$\Rightarrow w_1^{ix+ly+\frac{y(y-1)}{2}mnp+qz+\frac{z(z-1)}{2}rsp+nryzp} w_2^{my+rz} w_3^{ny+sz} = 1$$

$\Rightarrow ix + ly + \frac{y(y-1)}{2}mnp + qz + \frac{z(z-1)}{2}rsp + nryzp \equiv 0 (mod \quad p^2)$, (3)

$my + rz \equiv 0 (mod \quad p)$, $\quad and$ (4)

$ny + sz \equiv 0(mod \quad p) \quad (5)$

(4) and (5) are two homogeneous equations with $sm - nr \neq 0$ determinant. Hence these equations have only trivial solution.

$$\Rightarrow y = z = 0.$$

by using the values of y and z in (3), we get

$$ix \equiv 0(mod \quad p^2)$$

But $i \not\equiv 0(mod \quad p) \Rightarrow i \not\equiv 0(mod \quad p^2)$. So $x = 0(modp^2)$

So, if $g \in (Z_{p^2} \times Z_p) \rtimes Z_p$ and $\phi(g) = 1$ it must be that $g = 1$. Hence the kernel is trivial and ϕ is an automorphism with the constraints we have already deduced. Now, we can calculate the order of automorphism group. There are p choices for both l, q and m, n, r, s will in some sense be equivalent to a matrix in $GL_2(F_p)$, so that will give as $(p^2 - 1)(p^2 - p)$ choices for those elements. Hence $\phi \in (Z_{p^2} \times Z_p) \rtimes Z_p$ is really defined as:

$$\phi: \begin{cases} w_1 \to w_1^{sm-nr}, & sm - nr \neq 0, \\ w_2 \to w_1^{pt} w_2^m w_3^n, & t \in Z_p, \\ w_3 \to w_1^{pu} w_2^r w_3^s, & u \in Z_p. \end{cases}$$

and

$$\left|Aut\left((Z_{p^2} \times Z_p) \rtimes Z_p\right)\right| = p^3(p-1)(p^2-1).$$

B. Now we are going to compute automorphisms of G_{13}

$$G_{13} =< w_1, w_2, w_3 : w_1^{p^2} = w_2^p = w_3^P = 1, w_2 w_1 = w_1^{1+p} w_2, w_3 w_1 = w_1 w_2 w_3, w_2 w_3 = w_3 w_2 >= (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p, \phi_1(z) \leftrightarrow (1,1,0)^z$$

By using structure description of group G_{13} and elementary calculations, we can find some useful relations as:

$$w_2^i w_1^j = w_1^{j+ijp} w_2^i$$

$$w_3^i w_1^j = w_1^{j+\frac{ij(j-1)}{2}p} w_2^{ij} w_3^i$$

$$(w_1^i w_2^j)^n = w_1^{ni+\frac{n(n-1)}{2}ijp} w_2^{nj}$$

$$(w_1^i w_3^j)^n = w_1^{ni+\frac{pin(n-1)}{2}(\frac{(i-1)j}{2}+\frac{(n-2)ij}{3})} w_2^{\frac{n(n-1)}{2}ij} w_3^{nj}$$

$$(w_1^i w_2^j w_3^k)^n = w_1^{ni+\frac{pn(n-1)i}{2}(\frac{(i-1)k}{2}+j+\frac{(n-2)ik}{3})} w_2^{nj+\frac{n(n-1)}{2}ik} w_3^{nk}$$

With these we begin our study of automorphism group of $(Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p$.

Let $\phi \in Aut((Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p)$ be defined as

$$\phi: \begin{cases} w_1 \to w_1^i w_2^j w_3^k, & i \in Z_{p^2}, j, k \in Z_p. \\ w_2 \to w_1^l w_2^m w_3^n, & l \in Z_{p^2}, m, n \in Z_p. \\ w_3 \to w_1^q w_2^r w_3^s, & q \in Z_{p^2}, r, s \in Z_p. \end{cases}$$

As order of $w_1 \in (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p$ is p^2, therefore the order of $\phi(w_1)$ is also p^2, So

$$(w_1^i w_2^j w_3^k)^p \neq 1$$
$$\Rightarrow w_1^{pi} \neq 1$$
$$\Rightarrow p \nmid i$$

Also, order of $w_2 \in (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p$ is p, so order of $\phi(w_2)$ is also p, hence

$$(w_1^l w_2^m w_3^n)^p = 1$$
$$\Rightarrow w_1^{lp} = 1$$
$$\Rightarrow p/l$$
$$\Rightarrow l = pd; \text{for some} \quad d \in Z_p$$

And ,order of $w_3 \in (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p$ is p, hence order of $\phi(w_3)$ is p

$$\Rightarrow (w_1^q w_2^r w_3^s)^p = 1$$
$$\Rightarrow w_1^{qp} = 1$$
$$\Rightarrow p/q$$

$\Rightarrow q = pf$; for some $f \in Z_p$

As the group G_{13} is a non-abelian group, we also must have that if $w_3 w_1 = w_1 w_2 w_3$ then

$$\phi(w_3)\phi(w_1) = \phi(w_1)\phi(w_2)\phi(w_3)$$
$$\Rightarrow w_1^q w_2^r w_3^s w_1^i w_2^j w_3^k = w_1^i w_2^j w_3^k w_1^l w_2^m w_3^n w_1^q w_2^r w_3^s$$
$$\Rightarrow w_1^{pf} w_2^r w_3^s w_1^i w_2^j w_3^k = w_1^i w_2 w_3^k w_1^{pd} w_2^m w_3^n w_1^{pf} w_2^r w_3^s$$
$$\Rightarrow w_1^{pf} w_2^r w_1^{i+\frac{si(i-1)}{2}p} w_2^{si} w_3^s w_2^j w_3^k = w_1^i w_1^{pd} w_1^{pf} w_2^j w_3^k w_2^m w_3^n w_2^r w_3^s$$
$$\Rightarrow w_1^{pf} w_1^{\frac{si(i-1)}{2}p} w_2^r w_1^i w_2^{si} w_2^j w_3^s w_3^k = w_1^{i+pd+pf} w_2^j w_2^m w_2^r w_3^k w_3^n w_3^s$$
$$\Rightarrow w_1^{pf+\frac{si(i-1)}{2}p} w_1^{i+rip} w_2^r w_2^{si+j} w_3^{s+k} = w_1^{i+pd+pf} w_2^{j+m+r} w_3^{k+n+s}$$
$$\Rightarrow w_1^{pf+\frac{si(i-1)}{2}p+i+rip} w_2^{r+si+j} w_3^{s+k} = w_1^{i+pd+pf} w_2^{j+m+r} w_3^{k+n+s}$$
$$\Rightarrow w_1^{\frac{si(i-1)}{2}p+rip} w_2^{si} = w_1^{pd} w_2^m w_3^n$$

$\Rightarrow \frac{si(i-1)}{2}p + rip \equiv pd(mod \quad p^2)$ or

$\Rightarrow \frac{si(i-1)}{2} + ri \equiv d(mod \quad p),$ (6)

And $si \equiv m(mod \quad p)$,(7)

and $n \equiv 0(mod \ \ p)$ (8)

Also $w_2 w_1 = w_1^{1+p} w_2$

$$\Rightarrow \phi(w_2)\phi(w_1) = \phi(w_1)^{1+p}\phi(w_2)$$
$$\Rightarrow w_1^l w_2^m w_3^n w_1^i w_2^j w_3^k = (w_1^i w_2^j w_3^k)^{1+p} w_1^l w_2^m w_3^n$$
$$\Rightarrow w_1^{pd} w_2^m w_1^i w_2^j w_3^k = (w_1^i w_2^j w_3^k)^{1+p} w_1^{pd} w_2^m$$
$$\Rightarrow w_1^{pd} w_1^{i+imp} w_2^m w_2^j w_3^k = w_1^i w_2^j w_3^k (w_1^i w_2^j w_3^k)^p w_1^{pd} w_2^m$$
$$\Rightarrow w_1^{pd+i+imp} w_2^{m+j} w_3^k = w_1^i w_2^j w_3^k w_1^{pi} w_1^{pd} w_2^m$$
$$\Rightarrow w_1^{pd+i+imp} w_2^{m+j} w_3^k = w_1^i w_1^{pi} w_1^{pd} w_2^m w_2^j w_3^k$$
$$\Rightarrow w_1^{pd+i+imp} w_2^{m+j} w_3^k = w_1^{i+pi+pd} w_2^{m+j} w_3^k$$
$$\Rightarrow w_1^{imp} = w_1^{pi}$$
$$\Rightarrow imp \equiv pi(mod \ \ p^2)$$
$$\Rightarrow im \equiv i(mod \ \ p)$$

But $p \nmid i$

$\Rightarrow m \equiv 1(mod \ \ p)$ (9)

Abelian relations gives unnecessary information.
Use (7) in (9), we get

$$si \equiv 1(mod \ \ p)$$

From (6),

$$ri \equiv d - \frac{i-1}{2}(mod \ \ p)$$
$$\Rightarrow p/ri - d + \frac{i-1}{2}$$

$\Rightarrow ri - d + \frac{i-1}{2} = px$ for some $x \in Z_p$

$$\Rightarrow ri = px + d - \frac{i-1}{2}$$
$$\Rightarrow r = (px + d - \frac{i-1}{2})/i$$

We must place on the homomorphism to satisfy the relations of group. Now we only have to see when it is bijective. Since we are talking about finite group, it is sufficient to show that it is injective or to show that kernel is trivial. Let $w_1^x w_2^y w_3^z$ be any element of $(Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p$ such that it is mapped to identity element of group.

$$\phi(w_1^x w_2^y w_3^z) = 1$$
$$\Rightarrow \phi(w_1)^x \phi(w_2)^y \phi(w_3)^z = 1$$
$$\Rightarrow (w_1^i w_2^j w_3^k)^x (w_1^{pd} w_2)^y (w_1^{pf} w_2^r w_3^s)^z = 1$$
$$\Rightarrow w_1^{xi+\frac{px(x-1)i}{2}(\frac{(i-1)k}{2}+j+\frac{(x-2)ik}{3})} w_2^{xj+\frac{x(x-1)}{2}ik} w_3^{xk} w_1^{pdy} w_2^y w_1^{pfz} w_2^{rz} w_3^{sz} = 1$$
$$\Rightarrow w_1^{xi+\frac{px(x-1)i}{2}(\frac{(i-1)k}{2}+j+\frac{(x-2)ik}{3})} w_1^{pdy} w_1^{pfz} w_2^{xj+\frac{x(x-1)}{2}ik} w_2^y w_2^{rz} w_3^{xk} w_3^{sz} = 1$$
$$\Rightarrow w_1^{xi+\frac{px(x-1)i}{2}(\frac{(i-1)k}{2}+j+\frac{(x-2)ik}{3})+pdy+pfz} w_2^{xj+\frac{x(x-1)}{2}ik+y+rz} w_3^{xk+sz} = 1$$

$$\Rightarrow xi + \frac{px(x-1)i}{2}(\frac{(i-1)k}{2} + j + \frac{(x-2)ik}{3}) + pdy + pfz \equiv 0(mod \;\; p^2), \qquad (10)$$

$$xj + \frac{x(x-1)}{2}ik + y + rz \equiv 0(mod \;\; p), \qquad (11)$$

$$xk + sz \equiv 0(mod \;\; p) \qquad (12)$$

From (10),

$$p^2/xi + \frac{px(x-1)i}{2}(\frac{(i-1)k}{2} + j + \frac{(x-2)ik}{3}) + pdy + pfz$$
$$\Rightarrow p/p^2/xi + \frac{px(x-1)i}{2}(\frac{(i-1)k}{2} + j + \frac{(x-2)ik}{3}) + pdy + pfz$$
$$\Rightarrow p/xi + \frac{px(x-1)i}{2}(\frac{(i-1)k}{2} + j + \frac{(x-2)ik}{3}) + pdy + pfz$$
$$\Rightarrow p/xi$$

But $p \nmid i$

$$\Rightarrow p/x$$
$$\Rightarrow x = 0.$$

Using this in (12), we get

$$sz \equiv 0(mod \;\; p)$$

But $p \nmid s$

$$\Rightarrow z = 0.$$

Using these in (11), we get

$$y = 0.$$

So if $h \in (Z_{p^2} \rtimes Z_p) \rtimes_{\phi_1} Z_p$ and $\phi(h) = 1$ it must be that $h = 1$, hence the kernel is trivial and ϕ is an automorphism with the constraints we have already deduced. Now we can calculate the order of an automorphism group. there are $p^2 - p$ choices for i and p choices for all j, k, l, q. Therefore

$$\phi: \begin{cases} w_1 \to w_1^i w_2^j w_3^k, & p \nmid i, \quad j,\; k \in Z_p, \\ w_2 \to w_1^{pd} w_2, & d \in Z_p, \\ w_3 \to w_1^{pf} w_2^r w_3^s, & f \in Z_p,\; p \nmid s. \end{cases}$$

And

$$\left|Aut\left(Z_{p^2} \rtimes Z_p\right) \rtimes_{\phi_1} Z_p\right| = p^5(p-1).$$

IV. CONCLUSION

In this research, we calculate the automorphism group of two groups $(Z_{p^2} \times Z_p) \rtimes Z_p$ and $\left(Z_{p^2} \rtimes Z_p\right) \rtimes_{\phi_1} Z_p$, $\quad \phi_1(z) \leftrightarrow (1,1,0)^z$. The automorphism group for $(Z_{p^2} \times Z_p) \rtimes Z_p$ be

$$\phi: \begin{cases} w_1 \to w_1^{sm-nr}, & sm - nr \neq 0, \\ w_2 \to w_1^{pt} w_2^m w_3^n, & t \in Z_p, \\ w_3 \to w_1^{pu} w_2^r w_3^s, & u \in Z_p. \end{cases}$$

and

$$\left|Aut\left(\left(Z_{p^2} \times Z_p\right) \rtimes Z_p\right)\right| = p^3(p-1)(p^2-1).$$

And also for $\left(Z_{p^2} \rtimes Z_p\right) \rtimes_{\phi_1} Z_p$, $\quad \phi_1(z) \leftrightarrow (1,1,0)^z$, the automorphism group is

$$\phi: \begin{cases} w_1 \to w_1^i w_2^j w_3^k, & p \nmid i, \quad j, \quad k \in Z_p, \\ w_2 \to w_1^{pd} w_2, & d \in Z_p, \\ w_3 \to w_1^{pf} w_2^r w_3^s, & f \in Z_p, \quad p \nmid s. \end{cases}$$

And

$$\left|Aut\left(Z_{p^2} \rtimes Z_p\right) \rtimes_{\phi_1} Z_p\right| = p^5(p-1).$$

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