Some Classical Properties of the New Conformable Fractional Derivative

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Abstract:- Some research works dealing with fractional derivatives inspired us to work on this paper. In this paper, we prove some useful results using the definition of new Conformable Fractional Derivative given inAhmed Kajouni, Ahmed Chafiki, Khalid Hilal, and Mohamed Oukessou[7].

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I. INTRODUCTION

Non-integer order derivatives and integrals are studied and applied in fractional calculus, a new discipline of mathematics. It has more than a 320-years history. Its progress has primarily been centred on the pure mathematical discipline. Liouville, Riemann, Leibniz, and others appear to have conducted the first systematic research in the 19th century. Fractional differential equations (FDEs) have been utilized to explain a variety of stable physical phenomena with anomalous degradation throughout the last two decades. Many mathematical models of real-world problems that arise in engineering and research are either linear or non-linear systems. With the discovery of fractional calculus, it has been shown that differential systems may be used to describe the role of many systems. It's worth noting that the fractional calculus may be used to describe a variety of physical processes with memory and genetic features. In reality, fractional order systems make up the majority of real-world processes.

Fractional Calculus is as old as Calculus. L' Hospital wrote to Leibniz on September 30th, 1695, inquiring about a particular notation he had used in his writings for the mth-derivative of the linear function $f(x) = x, \frac{d^m x}{dx^m}$. L'Hopital's raised the question to Leibniz, what will happen if $m = \frac{1}{2}$? Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn"[19]. As a result, fractional calculus was born on that date, and it is now known as fractional calculus' birthday. Many academics have attempted to define fractional derivative since then. The Riemann-Liouville and Caputo definitions are the most important. We recommend the reader to [1-4] for the history and important results on fractional derivatives.

Definition 1.1 "Riemann-Liouville Definition": For $\beta \in [m-1, m)$, β-derivative of function g(u) is

$$D_a^\beta(g)(u) = \frac{1}{\lceil m-\beta} \frac{d^m}{dt^m} \int_a^u \frac{g(x)}{(u-x)^{\beta-m+1}} dx.$$

Definition 1.2 "Caputo Definition": For $\beta \in [m-1, m)$, β-derivative of function g(u) is

$$D_a^\beta(g)(u) = \frac{1}{\lceil m-\beta} \int_a^u \frac{g^m(x)}{(u-x)^{\beta-m+1}} dx.$$

All of the definitions thus far, including 1.1 and 1.2, satisfy the property that the fractional derivative is linear. By all definitions, this is the single property inherited from the first derivative. However, such definitions have some drawbacks:

- The R-L derivative does not satisfy $D_a^\beta(1) = 0$ ($D_a^\beta(1) = 0$ for "Caputo definition") if $\beta \notin \mathbb{N}$.
- All fractional derivatives do not satisfy the known formulas of the derivative of the product and quotient of two functions:

$$D_a^\beta(hg) = hD_a^\beta(g) + gD_a^\beta(h).$$

$$D_a^\beta\left(\frac{h}{g}\right) = \frac{gD_a^\beta(h) - hD_a^\beta(g)}{g^2}.$$

- All fractional derivatives do not satisfy the Chain Rule:

$$D_a^\beta(hog)(t) = h^{(\beta)}\big(g(t)\big)g^{(\beta)}(t).$$

- All fractional derivatives do not satisfy $D_a^\alpha D_a^\beta(g) = D_a^{\alpha+\beta}(g)$ in general.

II. NOTATIONS AND PRELIMNARIES

The conformable fractional derivative, defined by the authors in [5], [6], and [7], is a well-behaved simple fractional derivative that is based solely on the derivative's basic limit definition.

Khalil et. al. [5] have introduced a new derivative called "the conformable fractional derivative" of function $'g'$ of order $'\beta'$ and is defined by

$$T_\beta(g(u)) = \lim_{\epsilon\to 0}\frac{g(u+\epsilon u^{1-\beta})-g(u)}{\epsilon},$$

where $g:[0,\infty)\to\mathbb{R}$ and $0<\beta\le 1$, and the fractional derivative at 0 is defined as $g^{(\beta)}(0)=\lim_{t\to 0^+}T_\beta(g(u))$. Katungampola [6] introduced the new definition which is given by

$$D^\beta(g(u)) = \lim_{\epsilon\to 0}\frac{g\left(ue^{\epsilon u^{-\beta}}\right)-g(u)}{\epsilon},$$

for $u>0,\beta\in(0,1)$. If g is β-differentiable in some $(0,b), b>0$, and $\lim_{u\to 0^+}D^\beta(g(u))$ exists then, define $D^\beta\big(g(0)\big)=\lim_{u\to 0^+}D^\beta(g(u))$.

For the most natural extension of the standard limit definition of the derivative of a function g at a point, Ahmed Kajouni et al. [7] presented "a new conformable fractional derivative".

Definition [7]: Given a function $g:[0,\infty)\to\mathbb{R}$, and then the conformable fractional derivative of g of order β is defined by

$$D^\beta\big(g(u)\big)=\lim_{h\to 0}\frac{g(u+he^{(\beta-1)u})-g(u)}{h}, for\ u>0,\beta\in(0,1).$$

If g is β-differentiable in some $(0,b), b>0$, and $\lim_{u\to 0^+}D^\beta(g(u))$ exists then, define $D^\beta\big(g(0)\big)=\lim_{u\to 0^+}D^\beta(g(u))$.

As the consequences of above definitions, the authors in [5, 6, 7] showed that the β- derivatives obey the product rule, quotient rule, power rule, chain rule, Rolle's theorem and Lagrange's MVT in classical calculus.

Theorem 2.1 [5, 6, 7]: If a function $g:[0,\infty)\to\mathbb{R}$ is β-differentiable at $u_0>0$, then g is continuous at u_0.

Theorem 2.2 [5, 6, 7]: If $a>0$ and $g:[a,b]\to\mathbb{R}$ be a given function that satisfies
- g is continuous on $[a,b]$.
- g is β- differentiable in (a,b) for some $\beta\in(0,1)$.
- $g(a)=g(b)$.

Then, there exists $c\in(a,b)$, such that $g^{(\beta)}(c)=0$.

Theorem 2.3 [5, 6, 7]: If $a>0$ and $g:[a,b]\to\mathbb{R}$ be a given function that satisfies
- g is continuous on $[a,b]$.
- g is β- differentiable in (a,b) for some $\beta\in(0,1)$.

Then, there exists $c\in(a,b)$, such that

$$g^{(\beta)}(c)=\frac{g(b)-g(a)}{\left(\frac{1}{\beta}-1\right)e^{(\beta-1)b}-\left(\frac{1}{\beta}-1\right)e^{(\beta-1)a}}.$$

The objective of this paper is to prove some useful results of classical calculus that remain true in replacing the integer order derivative g' by the conformable fractional one $g^{(\beta)}$.

III. MAIN RESULTS

Theorem 3.1 "Cauchy Mean Value Theorem for Conformable Fractional Differentiable Functions": If $a>0$ and $h,g:[a,b]\to\mathbb{R}$ be two given functions that satisfy
- $h\ and\ g$ are continuous on [a, b].
- $h\ and\ g$ are β- differentiable in (a,b) for some $\beta\in(0,1)$.
- $g^{(\beta)}(x)\neq 0$ for any point of the open interval (a,b).

Then, there exists atleast one valued $of\ x$ of open interval (a,b), such that

$$\frac{h^{(\beta)}(d)}{g^{(\beta)}(d)}=\frac{h(b)-h(a)}{g(b)-g(a)}.$$

Proof: Firstly, we see that $g(b)-g(a)\neq 0$ for otherwise g would satisfy the conditions of Rolle's theorem and $g^{(\beta)}(x)$ would then vanish for atleast one value of x of open interval(a,b), thus contradicting the hypothesis that $g^{(\beta)}(x)\neq 0$ for any point of the open interval (a,b).

Now, we define the function $\emptyset(x)$ by the relation

$$\emptyset(x)=h(x)-h(a)-\left(\frac{h(b)-h(a)}{g(b)-g(a)}\right)(g(x)-g(a)).$$

Then, $\emptyset(a)=0$ and $\emptyset(b)=0$.

So, the function $\emptyset(x)$ satisfies the conditions of Rolle's theorem. So, there exists atleast one value $d\ of\ x$ of open interval (a,b), such that

$$\emptyset^{(\beta)}(d)=0$$

$$\emptyset^{(\beta)}(x)=h^{(\beta)}(x)-\left(\frac{h(b)-h(a)}{g(b)-g(a)}\right)g^{(\beta)}(x).$$

So, $\emptyset^{(\beta)}(d)=0$ implies that $h^{(\beta)}(d)-\left(\frac{h(b)-h(a)}{g(b)-g(a)}\right)g^{(\beta)}(d)=0$.

This implies,

$$\frac{h^{(\beta)}(d)}{g^{(\beta)}(d)}=\frac{h(b)-h(a)}{g(b)-g(a)}.$$

The following result [8] generalizes the L-Hospital Rule will be a particular case of which $\beta=1$.

Theorem 3.2: Let $a>0$ and $h,g:[a,b]\to\mathbb{R}$ be two given functions that satisfy
- $h\ and\ g$ are continuous on [a, b].
- $h\ and\ g$ are β- differentiable in (a,b) for some $\beta\in(0,1)$.
- $g^{(\beta)}(x)\neq 0$ for any point of the open interval $(a,b)\ and\ \lim_{x\to a^+}\frac{h^{(\beta)}(x)}{g^{(\beta)}(x)}=l$.

Then,

$$\lim_{x\to a^+}\frac{h(x)-h(a)}{g(x)-g(a)}=\lim_{x\to a^+}\frac{h^{(\beta)}(x)}{g^{(\beta)}(x)}=l.$$

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Proof: By the hypothesis (i), (ii) and (iii) we can apply theorem 3.1 for all $x \in [a,b]$ there exists $d_x \in (a,x)$ such that

$$\frac{h^{(\beta)}(d_x)}{g^{(\beta)}(d_x)} = \frac{h(x)-h(a)}{g(x)-g(a)}.$$

It is clear that if $x \to a$ so $d_x \to a$ then by (iii)

$\frac{h^{(\beta)}(d_x)}{g^{(\beta)}(d_x)} \to l.$

Finally, we have that

$$\lim_{x\to a^+} \frac{h(x)-h(a)}{g(x)-g(a)} = \lim_{x\to a^+} \frac{h^{(\beta)}(x)}{g^{(\beta)}(x)} = l.$$

Example 3.3: We calculate $\lim_{x\to 0} \frac{sinx}{x}$.

Let $h(x) = sinx, \qquad g(x) = x.$

We are exactly within the conditions for applying the result 3.2.

$$\lim_{x\to 0} \frac{h(x)}{g(x)} = \lim_{x\to 0} \frac{h(x)-h(0)}{g(x)-g(0)}$$
$$= \lim_{x\to 0} \frac{h^{(\beta)}(x)}{g^{(\beta)}(x)}$$
$$= \lim_{x\to 0} \frac{e^{(\beta-1)x} cosx}{e^{(\beta-1)x}}$$
$$= 1.$$

Example 3.4: We calculate $\lim_{x\to 0} \frac{1-cosx}{x^2}$.

Let $h(x) = 1 - cosx, \qquad g(x) = x^2.$

We are exactly within the conditions for applying the result 3.2.

$$\lim_{x\to 0} \frac{h(x)}{g(x)} = \lim_{x\to 0} \frac{h(x)-h(0)}{g(x)-g(0)}$$

$$= \lim_{x\to 0} \frac{h^{(\beta)}(x)}{g^{(\beta)}(x)}$$

$$= \lim_{x\to 0} \frac{e^{(\beta-1)x} sinx}{2e^{(\beta-1)x} x}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{sinx}{x}$$

$$= \frac{1}{2}.$$

IV. CONCLUSION

Cauchy Mean Value Theorem for New Conformable Fractional Derivative generalizes Lagrange's Mean Value Theorem for β- differentiable function in (a,b) for some $\beta \in (0,1)$. It may establish the relationship between the fractional derivatives of two functions and changes in these functions on a finite interval.

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