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Union of 3-Total Difference Cordial Graph with the Bistar $B_{n,n}$

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Abstract:- Let G be any graph with p vertices and q edges. We construct a map f on the set of all vertices of G to the set $\{0, 1, 2, \ldots, k-1\}$ where $k \in \mathbb{N}$ and $k > 1$. If the edges are la- belled by the absolute difference of the labels of their end vertices then f is said to be a k-total difference cordial labeling of G if $|td_f(i) - td_f(j)| \leq 1$ for all $i, j \in \{0, 1, 2, \cdots, k-1\}$ where $td_f(r)$ denotes the total number of vertices and the edges labeled by r. If a graph has a above mentioned labeling then it will be called as k-total difference cordial graph. In this paper we discuss the nature of 3-total difference cordial labeling of union of bistar with a 3-total difference cordial graph.

Keywords:- *Union of Graphs, Corona Graphs, Bistar and Comb.*

I. INTRODUCTION

In this manuscript we discuss only simple undirected graphs. In graph theoretic terminology, labeling is just to assign numbers, mostly,integers to the graph elements with some constraints. Graph labeling plays a vital role on many areas of computer science engineering and other fields of science [2]. Rosa introduced this concept with graceful labeling [6] in 1967. After that several labeling notions were introduced by numuourous authors, see [2]. After two decades of Rosa's notion, Cahit [1] initiated another important notion, namely cordial labeling. Ponraj et al. introduced k-total difference cordial labeling in [4]. In[4, 5], they studied 3-total difference cordiality of some graphs and arepath, cycle, complete graph, star, bistar, comb, crown, armed crown, wheel, etc. and also they proved that every graph is a subgraph ofa connected k-total difference cordial graphs. A graph whose vertex set V can be partitioned into two non-empty disjoint subsets (V_1, V_2) with the property that an edge of G joins a vertex of a V_1 to a vertexof V_2. If each vertex of the former is adjacent with that of the later then we say that such a graph is called a complete bipartite graph. Generally, it is denoted by $K_{m,n}$, where the cardinality of the first set m and that of the second is n. If $m = 1$ then such a graph is called a star. A bi-star $B_{m,n}$ is a graph derived from two stars $K_{1,m}$ and $K_{1,n}$by making adjacency with a vertex of degree m in $K_{1,m}$ with a vertexof degree n in $K_{1,n}$. The corona of two graphs is obtained by takinga graph to together with p copies of another graph, where p denote the number vertices in the former. Now, join each vertex of the formergraph with all the vertices of its corresponding copy of the later graph.Symbiotically if G and H be two graphs then their corona is denoted by $G \odot H$. Corona of a path with a trivial graph is called a comb graph. The disjoint union of two graphs is just taking union of their corresponding graph elements. In this paper we discuss the 3-total difference cordial labeling of union of a 3-total difference cordial graphwith a bistar $B_{n,n}$. For more graph theoretic notions relevant to this manuscript, see [3].

II. PRELIMINARIES

Definition 2.1. Consider a map f defined on the set of all vertices ofa graph G to the set $\{0, 1, 2, \ldots, k-1\}$ where $k \in \mathbb{N}$ and $k > 1$. Forthe edges of G, we put the absolute difference of the labels of their endvertices then the map f is known as k-total difference cordial labeling of G if the total number of vertices and edges labelled by the number$r$ and not by r will differ by at most 1. We denote $td_f(r)$ as numberof vertices and edges labelled by r. A graph consisting of the above labeling property is known as a k-total difference cordial graph and themap f is known as a k-total difference cordial labeling.

Here we recollect a result on bistar [4] which admits a new 3-total difference cordial labeling if and only if $n \equiv 1, 2 \pmod 3$. Consider the bistar $B_{n,n}$. Let u, v be the central vertices of the two copies of the star. Let u_i, $1 \leq i \leq n$ be the vertices which are adjacent to u and v_i,$1 \leq i \leq n$ be the vertices which are adjacent to v.

Theorem 2.1. [4] The bistar $B_{n,n}$ is 3-total difference cordial if and only if $n \equiv 1, 2 \pmod 3$.

Proof. It is noteworthy that the order and the size of $B_{n,n}$ are $2n + 2$ and $2n + 1$ respectively. According to the nature of the n, the proof is divided into three possible cases.

Case 1. $n \equiv 0 \pmod 3$.
Let $n = 3t$ where $t \in \mathbb{N}$. Suppose there exists a 3-total difference cordial labeling φ, then $td_\varphi(0) = td_\varphi(1) = td_\varphi(2) = 4t + 1$.
Subcase 1(a). $\varphi(u) = \varphi(v) = 0$.

If we assign the label 2 to any one pendent vertex then it will contribute two to $td_\varphi(2)$. It follows that $td_\varphi(2)$ must be an even number, an impossibility.

Subcase 1(b). $\varphi(u) = 0$ and $\varphi(v) = 1$.

To get $td_\varphi(0) = 4t+1$, a contradiction. Suppose $3t$ number of vertices $v_i$are labeeled by 1 then $td_\varphi(2)$ must be an even number, a contradiction.In a similar argument, we can show that u and v cannot be labeled by 0 and 1 respectively in all the possible cases.

Subcase 1(c). $\varphi(u) = 0$ and $\varphi(v) = 2$.

As in subcase1(b), we get $td_\varphi(1) < 4t + 1$ (or) $td_\varphi(1)$ must be an odd number. In either case, the map φ fails to be a 3-total difference cordiallabeling.

Subcase 1(d). $\varphi(u) = \varphi(v) = 1$.
To get $td_\varphi(2) = 4t + 1$, two must be label of $4t + 1$ pendent vertices. Then $td_\varphi(1) > 4t + 1$, a contradiction.
Subcase 1(e). $\varphi(u) = 1$ and $\varphi(v) = 2$.

To get $td_\varphi(2) = 4t + 1$, the vertices v_i, $(1 \le i \le n)$ are labeled by zero and $t + 1$ vertices of u_i are labeled by 2. If it is so, $td_\varphi(0) < 4t + 1$. Similarly we can prove for all the possible cases that $td_\varphi(2)$ can never be equal to $4t + 1$, a contradiction.
Subcase 1(f). $\varphi(u) = \varphi(v) = 2$.Similar to subcase 1(d).

Case 2. $n \equiv 1 \pmod 3$.
Let $n = 3t + 1$. Define a function f from $V(B_{n,n})$ to the set $\{0, 1, 2\}$by $\varphi(u) = \varphi(v) = 0$,

$$\varphi(u_i) = \begin{cases} 1 & \text{if } 1 \le i \le 2t+1 \\ 0 & \text{if } 2t+2 \le i \le 3t+1 \end{cases}$$

and

$$\varphi(v_i) = \begin{cases} 2 & \text{if } 1 \le i \le 2t+1 \\ 0 & \text{if } 2t+2 \le i \le 3t+1 \end{cases}$$

One can easily check that $td_\varphi(0) = 4t + 3$ and $td_\varphi(1) = td_\varphi(2) = 4t + 2$. This shows that φ is a 3-total difference cordial labeling.

Case 3. $n \equiv 2 \pmod 3$.
Let $n = 3t + 2$. As in case 2, assign the label to the vertices u, v,u_i $(1 \le i \le n)$, v_i, $(1 \le i \le n)$. Then put the labels 1 and 2 to the vertices u_n and v_n respectively. It is easy to show that $td_\varphi(0) = 4t + 3$ and $td_\varphi(1) = td_\varphi(2) = 4t + 4$. Thus φ is a 3-total difference ccordial labeling of $B_{n,n}$.

III. MAIN RESULTS

In this section, we create a new 3-total difference cordial graph froma graph G which has alredy permits this labeling. This new graph is obtained by taking union of G with a bistar $B_{n,n}$. For, we utilize the Theorem 2.1.

Theorem 3.1. Let G be a (p, q)-3-total difference cordial graph. Then $G \cup B_{n,n}$ is 3-total difference cordial.

Proof. Let ψ be a 3-total difference cordial labeling of G. we define a map $f : V(G \cup B_{n,n}) \to \{0, 1, 2\}$ by

$$f(x) = \begin{cases} \psi(x) & \text{if } x \in V(G) \\ \varphi(x) & \text{if } x \in V(B_{n,n}) \end{cases}$$

Now, we check whether the above mentioned labeling h is the required 3-total difference cordial labeling. For, we split up the proof into three cases.

Case 1. $p + q \equiv 0 \pmod 3$.
Let $p + q = 3r$. In this case $td_\psi(0) = td_\psi(1) = td_\psi(2) = r$. The Table

1 given below ensures that h satisfies the requirements of a 3-total difference cordial labeling if $n \equiv 1, 2 \pmod 3$.

Nature of n	$td_f(0)$	$td_f(1)$	$td_f(2)$
$n \equiv 1 \pmod 3$	$r + 4t + 3$	$r + 4t + 2$	$r + 4t + 2$
$n \equiv 2 \pmod 3$	$r + 4t + 3$	$r + 4t + 4$	$r + 4t + 4$

TABLE 1. Constraints of h

Case 2. $p + q \equiv 1 \pmod 3$.

Let $p + q = 3r + 1$. In this case we have the folowing possibilities:
a) $td_\psi(0) = td_\psi(1) = r$ and $td_\psi(2) = r + 1$.
b) $td_\psi(0) = td_\psi(2) = r$ and $td_\psi(1) = r + 1$.
c) $td_\psi(1) = td_\psi(2) = r$ and $td_\psi(0) = r + 1$.Now, we divide this case into two subcases. Subcase 2(a). $n \equiv 1 \pmod 3$.

If either (a) or (b) happens then the Table 2 shows that f is a 3-total difference cordial labeling.

Suppose (c) is true then we relabel the vertex v_n by 1. It is easy tocheck that f is a 3-total difference cordial labeling.
Subcase 2(b). $n \equiv 2 \pmod 3$.

Let $n = 3t+2$. If (c) is true then $td_f(0) = td_f(1) = td_f(2) = r+4t+4$.
If (a) is true then we relabel f as follows: $f(u) = 0$, $f(v) = 2$, $f(v_i) = 0$where $(1 \le i \le n)$ and $f(x) = \psi(x)$ for all $x \in V(G)$, and

Options	$td_f(0)$	$td_f(1)$	$td_f(2)$
(a)	$r+4t+3$	$r+4t+2$	$r+4t+3$
(b)	$r+4t+3$	$r+4t+3$	$r+4t+2$

TABLE 2. Constraints of f

$$f(u_i)=\begin{cases} 2 & \text{if } 1\le i\le t\\ 0 & \text{if } t+1\le i\le 2t+1\\ 1 & \text{if } 2t+2\le i\le 6t+5\end{cases}$$

where $n\equiv 5 \pmod 6$. Suppose $n\equiv 2 \pmod 6$, then assign the labels as in the case that $n\equiv 5 \pmod 6$. Then the remaining six vertices u_{n-2}, u_{n-1}, u_n, v_{n-2}, v_{n-1} and v_n are labeled by 1, 1, 2, 0, 0 and 0 respectively. The Table 3 provide vertex and edge conditions of f.

Nature of n	$td_f(0)$	$td_f(1)$	$td_f(2)$
$n\equiv 5 \pmod 6$	$r+8t+8$	$r+8t+2$	$r+8t+8$
$n\equiv 2 \pmod 6$	$r+8t+8$	$r+8t+4$	$r+8t+4$

TABLE 3. Conditions of f

Hence f is a 3-total difference cordial labeling.
Case 3. $p+q\equiv 2 \pmod 3$.
Let $p+q = 3r+2$. Then ψ must satisfy any one of the following conditions:

a) $td_\psi(0) = td_\psi(1) = r+1$ and $td_\psi(2) = r$.
b) $td_\psi(0) = td_\psi(2) = r+1$ and $td_\psi(1) = r$.
c) $td_\psi(1) = td_\psi(2) = r+1$ and $td_\psi(0) = r$.

We divide this case into two subcases. Subcase 3(a). $n\equiv 1 \pmod 3$.
Let $n = 3t+1$. If (c) is true then $td_f(0) = td_f(1) = td_f(2) = r+4t+3$.
If (a) is true then we reconstruct f as follows $f(x) = \psi(x)$ for all $x\in V(G)$, $f(u) = 0$, $f(v) = 2$ and

$$f(u_i)=\begin{cases} 0 & \text{if } 1\le i\le 3t+2\\ 2 & \text{if } 3t+3\le i\le 6t+4\end{cases}$$

$$f(v_i)=\begin{cases} 0 & \text{if } 1\le i\le 2t+1\\ 1 & \text{if } 2t+2\le i\le 6t+4\end{cases}$$

It is easy to check that $td_f(0) = td_f(1) = td_f(2) = r+8t+7$ andhence f is a 3-total difference cordial labeling, where $n\equiv 4 \pmod 6$.

Suppose $n\equiv 1 \pmod 6$ then take $n = 6t+1$. Here we redefine f as

$f(x) = \psi(x)$ for all $x\in V(G)$, $f(u) = 0$, $f(v) = 2$ and

$$f(u_i)=\begin{cases} 0 & \text{if } 1\le i\le 3t\\ 2 & \text{if } 3t+1\le i\le 6t\\ 1 & \text{if } i = 6t+1\end{cases}$$

$$f(v_i)=\begin{cases} 0 & \text{if } 1\le i\le 2t+1\\ 1 & \text{if } 2t+2\le i\le 6t+1\end{cases}$$

In this case $td_f(0) = td_f(1) = td_f(2) = r+8t+3$. Hence f is a 3-totaldifference cordial labeling. Suppose (b) is true then we relabel as in the previous case. It is easy to verify that f satisfies the vertex and edge condition of a 3-total difference cordial labeling.

Subcase 3(b). $n\equiv 2 \pmod 3$.
Let $n = 3t+2$. If either (a) or (b) is true then f is the required 3-total difference cordial labeling, see Table 4

Options	$td_f(0)$	$td_f(1)$	$td_f(2)$
(a)	$r+4t'+4$	$r+4t'+5$	$r+4t'+4$
(b)	$r+4t'+4$	$r+4t'+4$	$r+4t'+5$

TABLE 4. Vertex and edge conditions of f

For the case (c) $n\equiv 5 \pmod 6$ or $n\equiv 2 \pmod 6$ assign the labels to the vertices as in subcase 3(a). Then assign the labels 0 and 1 respec- tively to the vertices u_n and v_n. The Table 5 provides the existence of a 3-total difference cordial labeling f.

Nature of n	$td_f(0)$	$td_f(1)$	$td_f(2)$
$n\equiv 5 \pmod 6$	$r+8t+8$	$r+8t+9$	$r+8t+8$
$n\equiv 2 \pmod 6$	$r+8t+4$	$r+8t+5$	$r+8t+4$

TABLE 5. Constrains of f

Hence, if $n\equiv 2 \pmod 3$ then $G\cup B_{n,n}$ is a 3-total difference cordial graph where G is any 3-total difference cordial graph.

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Illustration 1. A 3-total difference cordial labeling of $(P_6 \odot K_1)\cup B_{7,7}$, where $P_n\odot K_1$ is a comb, is given in Figure 3.

Conclusion. In this study, we provide an idea that if we have a 3-total difference cordial graph then we can find another one by taking disjointunion with a bistar.

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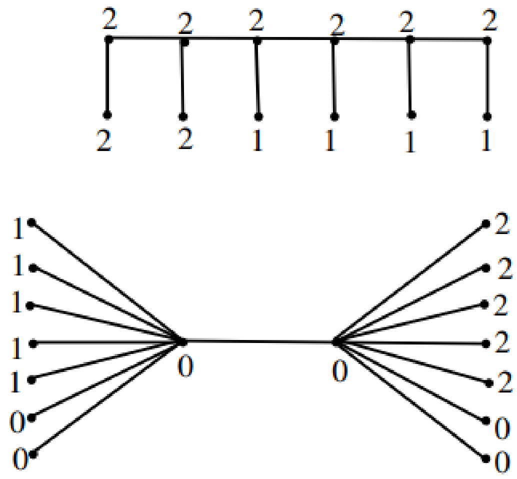

Fig 1. A 3-total difference cordial labeling of $(P_6 \odot K_1) \cup B_{7,7}$

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