ISSN No:-2456-2165

Construction of a Positive Valued Scalar Function of Strictly Rectangular Complex Matrices using the Framework of Spacer Matrix Components and Related Matrices

Debopam Ghosh

Abstract:- The Research paper presents a mathematical method to construct a real, positive valued scalar function of matrices belonging to the strictly rectangular complex matrix spaces. The formulated method leads to association of a strictly rectangular complex matrix with a positive vector whose components sum to unity and belongs to the real co-ordinate space of dimensionality equal to that of the embedding dimension associated with the input degrees of freedom of the rectangular complex matrix space. These vector components act as the respective weights to the ordered eigenvalues (ordered from largest to smallest) of a real, hermitian and positive definite matrix, termed as the Reference matrix, which is uniquely determined by the Spacer component matrices associated with the strictly rectangular complex matrix space. The resulting weighted sum of these eigenvalues, expressed as an appropriate Inner product, is the numerical value which the scalar function assigns as the output for the input complex strictly rectangular matrix under consideration.

Keywords:- *Spacer matrix based generalized matrix multiplication, Spacer matrix components, Embedding dimension, Reference matrix associated with the Embedded Matrix space corresponding to a strictly rectangular complex matrix space.*

Notations

- N denotes the set of all Natural numbers
- C denotes the set of all Complex numbers
- R denotes the set of all Real numbers
- $I_{w\times w}$ denotes the Identity matrix of order 'w'
- C^{w} denotes the complex coordinate space of order 'w'
- $c^{\bullet}$ denotes the complex conjugate of the complex number ' c '
- $M_{x\times y}(C)$ denotes the complex Matrix space of order 'x' by 'y'
- ' s ' denotes the Embedding dimension
- $M_{s\times s}(C)$ denotes the Embedded Matrix Space

- $$|\omega\rangle \in C^{d},\ |\omega\rangle = \begin{bmatrix} \omega_1 \\ \omega_2 \\ . \\ . \\ \omega_d \end{bmatrix}_{d\times 1},\ \langle\omega| = \begin{bmatrix} \omega^{\bullet}_1 & \omega^{\bullet}_2 & . & . & \omega^{\bullet}_d \end{bmatrix}_{1\times d},\ \omega_1,\omega_2,....,\omega_d \in C$$

$$|\omega\rangle \in C^{d},\ |\mu\rangle \in C^{f},\ T_{d\times f} = \left[T_{ij}\right],\ T_{d\times f} \in M_{d\times f}(C),\ \text{therefore: } \langle\omega|T|\mu\rangle = \sum_{i=1}^{d}\sum_{j=1}^{f}\omega^{\bullet}_i T_{ij}\mu_j$$

$$|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m\times 1},\ |n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1},\ |s\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{s\times 1},\ |s-m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{(s-m)\times 1},\ |s-n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{(s-n)\times 1}$$

- $R^s(R)$ denotes the Real coordinate space of dimension ‘s’
- $R^s(R)(+) = \{|\pi\rangle \in R^s(R) \,|\, |\pi\rangle = \begin{bmatrix} \pi_1 \\ \pi_2 \\ . \\ . \\ \pi_s \end{bmatrix}_{s\times 1}, \pi_1, \pi_2, ..., \pi_s \in R, \pi_i > 0, i = 1, 2, ..., s\}$

$|\pi\rangle \in R^s(R)$, therefore: $\langle\pi| = \begin{bmatrix} \pi_1 & \pi_2 & . & . & \pi_s \end{bmatrix}_{1\times s}$

- A^H denotes the Hermitian conjugate of the matrix A
- $(X_{w\times w})^{-1}$ denote the Proper Inverse of the Invertible matrix $X_{w\times w}$, i.e. $(X_{w\times w})^{-1} X_{w\times w} = X_{w\times w}(X_{w\times w})^{-1} = I_{w\times w}$
- $X_{w\times w} \in M_{w\times w}(C)$, such that $X_{w\times w}$ is hermitian, positive definite , then $(X_{w\times w})^{-1/2} \in M_{w\times w}(C)$, $(X_{w\times w})^{-1/2}$ is hermitian, positive definite, such that: $(X_{w\times w})^{-1/2}(X_{w\times w})^{-1/2} = (X_{w\times w})^{-1}$, we also have: $(X_{w\times w})^{1/2} \in M_{w\times w}(C)$, $(X_{w\times w})^{1/2}$ is hermitian, positive definite, such that: $(X_{w\times w})^{1/2}(X_{w\times w})^{1/2} = X_{w\times w}$
- $\|A_{x\times y}\|_F$ denotes the Frobenius norm of the matrix $A_{x\times y}$
- $\max(a,b)$ denotes the maximum of the two inputs ‘a’ and ‘b’
- $|a-b|$ denotes the absolute value of the difference between the two inputs ‘a’ and ‘b’
- ‘EVD’ is the abbreviation for “ Eigen value decomposition ”
- $E.S(\hat{\Omega}_{s\times s})$ denotes the Eigenvalue spectrum of the matrix $\hat{\Omega}_{s\times s}$
- $diag(v_1, v_2, ..., v_w)$ denotes a diagonal matrix of order ‘w’, whose diagonal elements along the main diagonal, from top to bottom, are $v_1, v_2, ..., v_w$ respectively
- $X \subseteq Y$ denotes that the set X is a proper or an improper subset of the set Y

I. INTRODUCTION

The Spacer matrix and the Spacer component matrices allow many of the mathematical operations that are only possible on square matrices to be extended to strictly rectangular matrices. The mathematical framework based on Spacer matrix and its component matrices allows for a generalized matrix multiplication scheme involving the spacer units as the compensating and connecting units to be defined [6, 7, 11], subspaces of strictly rectangular complex matrix spaces in terms of the powers of the matrices (matrices connected by the spacer units) belonging to the complex matrix spaces to be constructed [13], correlation between co-ordinate vectors belonging to non-compatible co-ordinate spaces to be quantified [8, 12]

The present research article formulates a mathematical scheme to construct a positive valued scalar function of strictly rectangular complex matrices using the mathematical elements constructed from the spacer component matrices $P_{n\times s}$ and $Q_{s\times m}$. In the formulated method, the information on the input dimensions ‘m’ and ‘n’ is used to determine the embedding dimension ‘s’, which is followed by construction of the respective spacer matrix components and the corresponding Orthonormal matrices $W_{s\times n}$ and $G_{s\times m}$, the real, hermitian and positive semi-definite matrix $Z_{s\times s}$ and the Reference matrix $\hat{\Omega}_{s\times s}$.

The Input strictly rectangular matrix $A_{m\times n}$ is used to construct the hermitian, positive definite matrix $\hat{\rho}_{s\times s}$. Using the matrix $\hat{\rho}_{s\times s}$, the information on the eigenvalues and associated eigenvectors of the Reference matrix $\hat{\Omega}_{s\times s}$, the reference vector $|\delta\rangle$ and the distribution vector associated with the matrix $A_{m\times n}$, denoted by $|\nu(A_{m\times n})\rangle$, is formulated. The scalar function of the matrix $A_{m\times n}$, denoted as $\theta(A_{m\times n})$, is expressed as the inner product of the vectors $|\delta\rangle$ and $|\nu(A_{m\times n})\rangle$.

The article presents the mathematical framework and definitions of a set of associated mathematical elements. The methodology is numerically demonstrated using numerical samples from the complex matrix spaces of dimensions (m=2, n=1) and (m=2, n=3). The article concludes with a discussion of the numerical results obtained and insights about future directions of the follow up research studies.

II. MATHEMATICAL FRAMEWORK

- $m \in N, n \in N$ and $m \neq n$, $s = \max(m,n) + |m-n|$, therefore $s \in N$, $s > m$ and $s > n$

The Spacer component matrices [6, 7, 8, 9, 10, 11, 12, 13] , $P_{n\times s}$ and $Q_{s\times m}$, are defined as follows:

$$P_{n\times s} = \begin{bmatrix} I_{n\times n} & (\frac{1}{n})|n\rangle\langle s-n| \end{bmatrix}_{n\times s} = \left[\begin{array}{ccccc|ccccc} 1 & 0 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ 0 & 1 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 1/n & 1/n & . & . & 1/n \end{array}\right]_{n\times s}$$

$$Q_{s\times m} = \begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})|s-m\rangle\langle m| \end{bmatrix}_{s\times m} = \left[\begin{array}{ccccc} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \\ \hline 1/m & 1/m & . & . & 1/m \\ 1/m & 1/m & . & . & 1/m \\ . & . & . & . & . \\ . & . & . & . & . \\ 1/m & 1/m & . & . & 1/m \end{array}\right]_{s\times m}$$

$W_{s\times n} = (P_{n\times s})^H [(PP^H)^{-1/2}]_{n\times n}$, therefore $W^H W = I_{n\times n}$,

$G_{s\times m} = (Q_{s\times m})[(Q^H Q)^{-1/2}]_{m\times m}$, therefore $G^H G = I_{m\times m}$

- Construction of the Reference Matrix $\hat{\Omega}_{s\times s}$

$Z_{s\times s} = (\frac{1}{2})(P^H P)_{s\times s} + (\frac{1}{2})(QQ^H)_{s\times s}$, $\hat{\Omega}_{s\times s} = (\frac{1}{2})I_{s\times s} + (\frac{1}{2})Z_{s\times s} = (\frac{1}{2})I_{s\times s} + (\frac{1}{4})(P^H P)_{s\times s} + (\frac{1}{4})(QQ^H)_{s\times s}$,

$\hat{\Omega}_{s\times s} \in M_{s\times s}(C)$,

$\hat{\Omega}_{s\times s}$ is real, Hermitian and Positive definite

EVD of the matrix $\hat{\Omega}_{s\times s}$: $\hat{\Omega}_{s\times s} = U_{s\times s} D_{s\times s} (U_{s\times s})^H$ where $U_{s\times s} = \begin{bmatrix} |u_1\rangle & |u_2\rangle & . & . & |u_s\rangle \end{bmatrix}$,

$(U_{s\times s})^H = (U_{s\times s})^{-1}$, $D_{s\times s} = diag(\lambda_1, \lambda_2, ..., \lambda_s)$ such that $\lambda_1 \geq \lambda_2 \geq ....\lambda_s > 0$

We define: $E.S(\hat{\Omega}_{s\times s}) = \{\lambda_1, \lambda_2, ..., \lambda_s\}$, $Trace(\hat{\Omega}_{s\times s}) = \lambda^o = \sum_{j=1}^{s} \lambda_j$, $\hat{\Gamma} = \{\lambda \in R \mid \lambda_s \leq \lambda \leq \lambda_1\}$,

$\bar{\lambda} = (\frac{1}{s})\sum_{j=1}^{s} \lambda_j$, $\langle \lambda \rangle = \sum_{j=1}^{s} \lambda_j (\frac{\lambda_j}{\lambda^o})$ therefore we have $\bar{\lambda} \in \hat{\Gamma}$, $\langle \lambda \rangle \in \hat{\Gamma}$

- Construction of the Reference vector $|\delta\rangle$ and the Scaled Reference vector $|\varepsilon\rangle$

ISSN No:-2456-2165

$$\left|\delta\right\rangle=\begin{bmatrix}\lambda_1\\ \lambda_2\\ .\\ .\\ \lambda_s\end{bmatrix}_{s\times 1} \quad , \quad \left|\varepsilon\right\rangle=(\frac{1}{\lambda^o})\left|\delta\right\rangle=\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ .\\ .\\ \varepsilon_s\end{bmatrix}_{s\times 1} \quad \text{therefore: } \varepsilon_j=(\frac{\lambda_j}{\sum_{j=1}^{s}\lambda_j})=(\frac{\lambda_j}{\lambda^o}),\quad j=1,2,....,s$$

$$\left|\delta\right\rangle\in R^s(R)(+)\ ,\ \left|\varepsilon\right\rangle\in R^s(R)(+)\ ,\ \left\langle s|\varepsilon\right\rangle=1$$

- $A_{m\times n}\in M_{m\times n}(C)$, we define $B_{s\times s}\in M_{s\times s}(C)$ such that : $B_{s\times s}=G_{s\times m}A_{m\times n}(W_{s\times n})^H$

Thus, $trace(B^HB)=trace(BB^H)=(\|B_{s\times s}\|_F)^2=(\|A_{m\times n}\|_F)^2$

$$\rho_{s\times s}=(\frac{1}{2})I_{s\times s}+(\frac{1}{4})(B^HB)_{s\times s}+(\frac{1}{4})(BB^H)_{s\times s}\ ,\ \rho_{s\times s}\in M_{s\times s}(C),\quad trace(\rho_{s\times s})=(\frac{1}{2})(s+(\|A_{m\times n}\|_F)^2)$$

$$\hat{\rho}_{s\times s}\in M_{s\times s}(C)\ ,\ \hat{\rho}_{s\times s}=(\frac{1}{trace(\rho_{s\times s})})\rho_{s\times s}=(\frac{1}{s+(\|A_{m\times n}\|_F)^2})(I_{s\times s}+(\frac{1}{2})(B^HB)_{s\times s}+(\frac{1}{2})(BB^H)_{s\times s})$$

Thus, $\hat{\rho}_{s\times s}$ is Hermitian, Positive definite and has unit trace

$$\bar{\rho}_{s\times s}\in M_{s\times s}(C)\ ,\ \bar{\rho}_{s\times s}=(U_{s\times s})^H\hat{\rho}_{s\times s}U_{s\times s}=\begin{bmatrix}\left\langle u_1|\hat{\rho}|u_1\right\rangle & \left\langle u_1|\hat{\rho}|u_2\right\rangle & . & . & \left\langle u_1|\hat{\rho}|u_s\right\rangle\\ \left\langle u_2|\hat{\rho}|u_1\right\rangle & \left\langle u_2|\hat{\rho}|u_2\right\rangle & . & . & \left\langle u_2|\hat{\rho}|u_s\right\rangle\\ . & . & . & . & .\\ . & . & . & . & .\\ \left\langle u_s|\hat{\rho}|u_1\right\rangle & \left\langle u_s|\hat{\rho}|u_2\right\rangle & . & . & \left\langle u_s|\hat{\rho}|u_s\right\rangle\end{bmatrix}$$

Thus, $\bar{\rho}_{s\times s}$ is Hermitian, Positive definite and has unit trace

- Construction of the Distribution vector $\left|\nu(A_{m\times n})\right\rangle$

$$A_{m\times n}\in M_{m\times n}(C),\ \left|\nu(A_{m\times n})\right\rangle\in R^s(R)(+)\ ,\ \left|\nu(A_{m\times n})\right\rangle=\begin{bmatrix}\left\langle u_1|\hat{\rho}|u_1\right\rangle\\ \left\langle u_2|\hat{\rho}|u_2\right\rangle\\ .\\ .\\ \left\langle u_s|\hat{\rho}|u_s\right\rangle\end{bmatrix}_{s\times 1}\ ,\ \left\langle s|\nu(A_{m\times n})\right\rangle=1$$

- Analytical Expression of the scalar function $\theta(A_{m\times n})$

$$\theta():M_{m\times n}(C)\mapsto\hat{\Gamma}\ ,\quad \text{such that: } A_{m\times n}\in M_{m\times n}(C),\ \theta(A_{m\times n})=\left\langle\delta|\nu(A_{m\times n})\right\rangle=\sum_{j=1}^{s}\lambda_j\left\langle u_j|\hat{\rho}|u_j\right\rangle$$

We define the following subsets of the Matrix space $M_{m\times n}(C)$:

- $\hat{S}[M_{m\times n}(C)\,|\,\lambda=\bar{\lambda}]=\{A_{m\times n}\in M_{m\times n}(C)\,|\,\theta(A_{m\times n})=\bar{\lambda}\}$
- $\hat{S}[M_{m\times n}(C)\,|\,\lambda=\left\langle\lambda\right\rangle]=\{A_{m\times n}\in M_{m\times n}(C)\,|\,\theta(A_{m\times n})=\left\langle\lambda\right\rangle\}$
- $\hat{S}[M_{m\times n}(C)\,|\,\left|\nu(A_{m\times n})\right\rangle=(\frac{1}{s})\left|s\right\rangle]=\{A_{m\times n}\in M_{m\times n}(C)\,|\,\left|\nu(A_{m\times n})\right\rangle=(\frac{1}{s})\left|s\right\rangle\}$
- $\hat{S}[M_{m\times n}(C)\,|\,\left|\nu(A_{m\times n})\right\rangle=\left|\varepsilon\right\rangle]=\{A_{m\times n}\in M_{m\times n}(C)\,|\,\left|\nu(A_{m\times n})\right\rangle=\left|\varepsilon\right\rangle\}$

Therefore, we have the following:

ISSN No:-2456-2165

○ $\hat{S}[M_{m\times n}(C)\,|\,|\nu(A_{m\times n})\rangle = (\frac{1}{s})|s\rangle] \subseteq \hat{S}[M_{m\times n}(C)\,|\,\lambda = \bar{\lambda}]$ and $\hat{S}[M_{m\times n}(C)\,|\,|\nu(A_{m\times n})\rangle = |\varepsilon\rangle] \subseteq \hat{S}[M_{m\times n}(C)\,|\,\lambda = \langle\lambda\rangle]$

➢ The case of $A_{m\times n} = 0_{m\times n}$

$A_{m\times n} = 0_{m\times n}$, this implies: $\hat{\rho}_{s\times s} = \bar{\rho}_{s\times s} = (\frac{1}{s})I_{s\times s}$ therefore: $|\nu(A_{m\times n} = 0_{m\times n})\rangle = (\frac{1}{s})|s\rangle$ which implies: $0_{m\times n} \in \hat{S}[M_{m\times n}(C)\,|\,|\nu(A_{m\times n})\rangle = (\frac{1}{s})|s\rangle]$

Numerical Case Studies

❖ $m = 2, n = 1$ therefore $s = 3$

$$P_{1\times 3} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \;,\; Q_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \;,\; W_{3\times 1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;,\; G_{3\times 2} = \frac{1}{2\sqrt{3}}\begin{bmatrix} \sqrt{2}+\sqrt{3} & \sqrt{2}-\sqrt{3} \\ \sqrt{2}-\sqrt{3} & \sqrt{2}+\sqrt{3} \\ \sqrt{2} & \sqrt{2} \end{bmatrix},$$

$$Z_{3\times 3} = (\frac{1}{4})\begin{bmatrix} 4 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 3 & 3 \end{bmatrix} \;,\; \hat{\Omega}_{3\times 3} = (\frac{1}{8})\begin{bmatrix} 8 & 2 & 3 \\ 2 & 8 & 3 \\ 3 & 3 & 7 \end{bmatrix} \;,\; U_{3\times 3} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix},$$

$$|\delta\rangle = \begin{bmatrix} \frac{13}{8} \\ \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}_{3\times 1} \;,\; |\varepsilon\rangle = \begin{bmatrix} \frac{13}{23} \\ \frac{6}{23} \\ \frac{4}{23} \end{bmatrix}_{3\times 1} \;,\; E.S(\hat{\Omega}_{3\times 3}) = \{\lambda_1 = \frac{13}{8}, \lambda_2 = \frac{3}{4}, \lambda_3 = \frac{1}{2}\} \;,\; \hat{\Gamma} = \{\lambda \in R \,|\, \frac{1}{2} \le \lambda \le \frac{13}{8}\} \;,$$

$$\bar{\lambda} = \frac{23}{24}, \;\; \langle\lambda\rangle = \frac{221}{184}$$

<u>Example 1:</u>

$$A_{2\times 1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;,\; \text{therefore we have } (\|A_{2\times 1}\|_F)^2 = 1 \;,\; |\nu(A_{2\times 1})\rangle = \begin{bmatrix} \frac{7}{16} \\ \frac{5}{16} \\ \frac{1}{4} \end{bmatrix}_{3\times 1} \;,\; \theta(A_{2\times 1}) = \frac{137}{128}$$

ISSN No:-2456-2165

Example 2:

$$A_{2\times1}=\begin{bmatrix}0\\1\end{bmatrix}$$, therefore we have: $(\|A_{2\times1}\|_F)^2=1$, $|\nu(A_{2\times1})\rangle=\begin{bmatrix}\frac{7}{16}\\\frac{5}{16}\\\frac{1}{4}\end{bmatrix}_{3\times1}$, $\theta(A_{2\times1})=\frac{137}{128}$

Example 3:

$$A_{2\times1}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}\end{bmatrix}$$, therefore we have: $(\|A_{2\times1}\|_F)^2=1$, $|\nu(A_{2\times1})\rangle=\begin{bmatrix}\frac{7}{16}\\\frac{5}{16}\\\frac{1}{4}\end{bmatrix}_{3\times1}$, $\theta(A_{2\times1})=\frac{137}{128}$

Example 4:

$$A_{2\times1}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}\end{bmatrix}$$, therefore we have: $(\|A_{2\times1}\|_F)^2=1$, $|\nu(A_{2\times1})\rangle=\begin{bmatrix}\frac{7}{16}\\\frac{5}{16}\\\frac{1}{4}\end{bmatrix}_{3\times1}$, $\theta(A_{2\times1})=\frac{137}{128}$

Example 5:

$$A_{2\times1}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$, therefore we have: $(\|A_{2\times1}\|_F)^2=1$, $|\nu(A_{2\times1})\rangle=\begin{bmatrix}\frac{1}{2}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}_{3\times1}$, $\theta(A_{2\times1})=\frac{9}{8}$

Example 6:

$$A_{2\times1}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$$, therefore we have: $(\|A_{2\times1}\|_F)^2=1$, $|\nu(A_{2\times1})\rangle=\begin{bmatrix}\frac{3}{8}\\\frac{3}{8}\\\frac{1}{4}\end{bmatrix}_{3\times1}$, $\theta(A_{2\times1})=\frac{65}{64}$

- $m=2, n=3$ therefore $s=4$

$$P_{3\times4}=\begin{bmatrix}1&0&0&\frac{1}{3}\\0&1&0&\frac{1}{3}\\0&0&1&\frac{1}{3}\end{bmatrix}\ ,\ Q_{4\times2}=\begin{bmatrix}1&0\\0&1\\\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}\ ,\ W_{4\times3}=(\frac{1}{6})\begin{bmatrix}4+\sqrt{3}&-2+\sqrt{3}&-2+\sqrt{3}\\-2+\sqrt{3}&4+\sqrt{3}&-2+\sqrt{3}\\-2+\sqrt{3}&-2+\sqrt{3}&4+\sqrt{3}\\\sqrt{3}&\sqrt{3}&\sqrt{3}\end{bmatrix}$$

$$G_{4\times2}=(\frac{1}{2\sqrt{2}})\begin{bmatrix}1+\sqrt{2}&1-\sqrt{2}\\1-\sqrt{2}&1+\sqrt{2}\\1&1\\1&1\end{bmatrix}\ ,\ Z_{4\times4}=(\frac{1}{12})\begin{bmatrix}12&0&3&5\\0&12&3&5\\3&3&9&5\\5&5&5&5\end{bmatrix},\ \hat{\Omega}_{4\times4}=(\frac{1}{24})\begin{bmatrix}24&0&3&5\\0&24&3&5\\3&3&21&5\\5&5&5&17\end{bmatrix}$$

$$U_{4\times4}=\begin{bmatrix}\frac{1}{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{12}}\\\frac{1}{2}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{12}}\\\frac{1}{2}&0&-\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{12}}\\\frac{1}{2}&0&0&-\frac{3}{\sqrt{12}}\end{bmatrix}\ ,\ E.S(\hat{\Omega}_{4\times4})=\{\lambda_1=\frac{4}{3},\lambda_2=1,\lambda_3=\frac{3}{4},\lambda_4=\frac{1}{2}\}\ ,$$

$$|\delta\rangle=\begin{bmatrix}\frac{4}{3}\\1\\\frac{3}{4}\\\frac{1}{2}\end{bmatrix}_{4\times1}\ ,\quad |\varepsilon\rangle=\begin{bmatrix}\frac{16}{43}\\\frac{12}{43}\\\frac{9}{43}\\\frac{6}{43}\end{bmatrix}_{4\times1}\ ,\quad \hat{\Gamma}=\{\lambda\in R\,|\,\frac{1}{2}\leq\lambda\leq\frac{4}{3}\}\ ,\ \bar{\lambda}=\frac{43}{48}\ ,\ \langle\lambda\rangle=\frac{517}{516}$$

Example 1:

$A_{2\times3}=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}$, therefore we have: $(\|A_{2\times3}\|_F)^2=6$, $|\nu(A_{2\times3})\rangle=\begin{bmatrix}\frac{7}{10}\\\frac{1}{10}\\\frac{1}{10}\\\frac{1}{10}\end{bmatrix}_{4\times1}$, $\theta(A_{2\times3})=\frac{139}{120}$

Example 2:

$A_{2\times3} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, therefore we have: $(\|A_{2\times3}\|_F)^2 = 6$, $|\nu(A_{2\times3})\rangle = \begin{bmatrix} \frac{7}{30} \\ \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{10} \end{bmatrix}_{4\times1}$, $\theta(A_{2\times3}) = \frac{355}{360}$

Example 3:

$A_{2\times3} = \begin{bmatrix} 1 & -i & i \\ i & 1 & -i \end{bmatrix}$, therefore we have: $(\|A_{2\times3}\|_F)^2 = 6$, $|\nu(A_{2\times3})\rangle = \begin{bmatrix} \frac{7}{30} \\ \frac{2}{5} \\ \frac{4}{15} \\ \frac{1}{10} \end{bmatrix}_{4\times1}$, $\theta(A_{2\times3}) = \frac{173}{180}$

III. DISCUSSION AND CONCLUSION

The mathematical framework presented in the article essentially provides a method to construct a distribution vector $|\nu(A_{m\times n})\rangle$, which is a real positive valued vector, whose elements sum up to unity, for every matrix $A_{m\times n}$ belonging to the strictly rectangular complex matrix space $M_{m\times n}(C)$. The components of this vector are the diagonal elements of the matrix $\bar{\rho}_{s\times s}$, which incorporate information from the matrix $A_{m\times n}$ and the eigenvector information associated with the reference matrix $\hat{\Omega}_{s\times s}$. Thus, the scalar function $\theta(A_{m\times n})$ effectively provides a weighted average of the ordered eigenvalues of the reference matrix; these weights are components of the distribution vector associated with the matrix $A_{m\times n}$, and hence have implicit dependence on the matrix elements that constitute the matrix $A_{m\times n}$.

In the case of the numerical examples from the complex matrix space of dimensions (m=2, n=1), we can observe that the examples 1, 2, 3, 4 are mapped to the same distribution vector and hence, to the same numerical value for the scalar function. In case of the example 5 there is increased weightage along the eigen-space associated with the largest eigenvalue of the reference matrix and reduction in the weightage corresponding to the next largest eigenvalue of the reference matrix, in comparison to the respective values for the distribution vector corresponding to the examples 1,2,3,4. This leads to an overall larger value of the scalar function in the example 5 compared to that of the examples 1, 2, 3 and 4. In example 6, the weightage associated with the first two leading eigen-spaces become equal, which effectively leads to a lower value of the scalar function in example 6 as compared to that for the examples 1, 2, 3 and 4.

In the case of the numerical examples from the complex matrix space of dimensions (m=2, n=3), it can be observed that the value of the scalar function decreases successively from example 1 to 2 and from 2 to example 3. There is reduction in weightage of the leading eigen-space of the reference matrix and unequal increment in the weightage associated with the next two successive eigen-spaces, going from example 1 to example 2. However, on going from example 2 to example 3, there is no change in weightage associated with the eigen-spaces associated with the largest and the smallest eigenvalue of the reference matrix; only the weightage associated with the second leading eigen-space is reduced while that of the third leading eigen-space is increased while preserving their sum contribution in both the examples.

The presented mathematical framework allows any iterative evolution scheme defined on strictly rectangular complex matrix spaces to be mapped on to an evolution scheme involving positive vectors of unit sum belonging to the co-ordinate space $R^s(R)$ which can therefore, be visualized as being mediated by Markov type transition matrices. Follow up studies dedicated to understanding the interrelationship between the evolution schemes on the Input rectangular matrix space and the resulting mapped evolution sequences defined on $R^s(R)(+)$, in light of the mathematical framework presented in this article, is expected to produce a deeper understanding of the framework and its limitations and scope of applicability in domains of research problems arising in different areas of natural and applied sciences.

REFERENCES

[1]. Arfken, George B., and Weber, Hans J., *Mathematical Methods for Physicists*, 6th Edition, Academic Press

[2]. Brewer, J. W., *Kronecker Products and Matrix Calculus in System Theory*, IEEE Trans. on Circuits and Systems, 25, No.9, p 772-781 (1978)

[3]. Datta, B. N., *Numerical Linear Algebra and Applications*, SIAM

[4]. Dorai, Kavita, Mahesh, T. S., Arvind and Anil Kumar, *Quantum computation using NMR*, Current Science, Vol.79, No. 10, p 1447-1458 (2000)

[5]. Ghosh, Debopam, *A Tryst with Matrices: The Matrix Shell Model Formalism, 24by7 Publishing, India.*

[6]. Ghosh, Debopam, *A Generalized Matrix Multiplication Scheme based on the Concept of Embedding Dimension and Associated Spacer Matrices*, International Journal of Innovative Science and Research Technology, Volume 6, Issue 1, p. 1336 - 1343 (2021)

[7]. Ghosh, Debopam, *The Analytical Expressions for the Spacer Matrices associated with Complex Matrix spaces of order m by n, where m ≠ n, and other pertinent results* , (Article DOI: 10.13140/RG.2.2.23283.45603) (2021)

[8]. Ghosh, Debopam, *Construction of an analytical expression to quantify correlation between vectors belonging to non-compatible complex coordinates spaces, using the spacer matrix components and associated matrices* (Article DOI: 10.13140/RG.2.2.17857.68969) (2022)

[9]. Ghosh, Debopam, *Quantification of Intrinsic Overlap in matrices belonging to strictly rectangular complex matrix spaces, using the spacer matrix components and associated matrices* (Article DOI: 10.13140/RG.2.2.18405.06886) (2022)

[10]. Ghosh, Debopam , *A Mathematical Scheme defined on strictly rectangular complex matrix spaces involving Frobenius norm preservation under the possibility of matrix rank readjustment and internal redistribution of Variance using Spacer component matrices and a Completely Positive Trace Preserving transformation* , International Journal of Innovative Science and Research Technology, Volume 7, Issue 10, pp. 591 - 602 (2022)

[11]. Ghosh, Debopam, *A Compilation of the Analytical Expressions, Properties and related Results and formulation of some additional mathematical elements associated with the Spacer matrix components corresponding to strictly rectangular Complex Matrix Spaces* (Article DOI: 10.13140/RG.2.2.31777.89446/1) (2022)

[12]. Ghosh, Debopam, *Quantifying Correlation between vectors belonging to Non-Compatible Real Co-ordinate Spaces using a Mathematical scheme based on Spacer Matrix components*(Article DOI: 10.13140/RG.2.2.26336.76805) (2022)

[13]. Ghosh, Debopam, *The formulation of Critical subspace and Extended Critical subspace associated with matrices of Complex Matrix spaces of order m by n, where m≠n* (Article DOI: 10.13140/RG.2.2.13862.60488) (2022)

[14]. Graham, Alexander, *Kronecker Products & Matrix Calculus with Applications*, Dover Publications, Inc.

[15]. Hardy, Lucien, *Quantum Theory from Five Reasonable Axioms*, arXiv: quant-ph/0101012v4 (2001)

[16]. Hassani, Sadri, *Mathematics Physics A Modern Introduction to its Foundations*, Springer

[17]. Hogben, Leslie, (Editor), *Handbook of Linear Algebra*, Chapman and Hall/CRC, Taylor and Francis Group

[18]. Jordan, Thomas. F., *Quantum Mechanics in Simple Matrix Form*, Dover Publications, Inc.

[19]. Macklin, Philip A., *Normal matrices for physicists*, American Journal of Physics, 52, 513(1984)

[20]. Meyer, Carl. D., *Matrix Analysis and Applied Linear Algebra*, SIAM

[21]. Nakahara, Mikio, and Ohmi, Tetsuo, *Quantum Computing: From Linear Algebra to*

[22]. *Physical Realizations*, CRC Press

[23]. Neudecker, H., *Some Theorems on Matrix Differentiation with special reference to Kronecker Matrix Products*, J. Amer. Statist. Assoc., 64, p 953-963(1969)

[24]. Paris, Matteo G A, *The modern tools of Quantum Mechanics : A tutorial on quantum states, measurements and operations*, arXiv: 1110.6815v2 [quant-ph] (2012)

[25]. Roth, W. E., *On Direct Product Matrices*, Bull. Amer. Math. Soc., No. 40, p 461-468 (1944)

[26]. Sakurai, J. J., *Modern Quantum Mechanics*, Pearson Education, Inc

[27]. Steeb, Willi-Hans, and Hardy, Yorick, *Problems and Solutions in Quantum Computing and Quantum Information*, World Scientific

[28]. Strang, Gilbert, *Linear Algebra and its Applications*, Fourth Edition, Cengage Learning

[29]. Sundarapandian, V., *Numerical Linear Algebra*, PHI Learning Private Limited