Specific Case of an Internal Control-External Control (Bounded Domain)

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Abstract:- We try to study the controllability of bounded domain Ω consisting of several heated materials. For this, we will use the functionalJ(u), which optimizes the cost. The objective is to bring the excess heat inside the domain and also on the border of the domain and it's this excess heat that is a function called control.

I. INTRODUCTION

Let Ω be a bounded domain made up of heated materials, we give ourselves the conductivity k(x) at the point x ∈ Ω, the heat source f in $L^{2(\Omega)}$, the temperature T_D imposed on Γ_D by the system, the flux PN imposed on T_N. Let's consider the following problem:

$$(S)\begin{cases} -div\big(k(x)\nabla y(x)\big) = f \ in \ \Omega, \\ \gamma y = t_D \ on \ \Gamma_{D,} \\ k\gamma \dfrac{\partial y}{\partial n} = PN \ on \ \Gamma_{N.} \end{cases}$$

II. PROBLEM

Let y_D be a given temperature.

How to act on the system (S) so that y is close enough to y_D?

III. IDEA

It is a question of bringing a surplus of heat u so that y is as close as possible toy_D. This excess heat is a function called control. To leave a good choice of u, we must optimize the cost, which reflects what we want to achieve and the means at our disposal. What amounts to considering the functional J(u) defined by:

$$\mathrm{J}(u) = \frac{1}{2}\int_{\Omega} g(y - y_D)dx + \frac{\in}{2}\|u\|^2_{L^2(\Omega);}$$

Where g is a definite function of Ω a value in $\mathbb{R}$. $G(y - y_D)$ makes is possible to minimize the difference between $y \ and \ y_D$, ∈ being very small serves not only to prove the existence and uniqueness of the solution but also to minimize the cost.

IV. PROBLEM

Is there $\bar{u}$ ∈u such that:

$$\mathrm{J}(\bar{u}) = {}^{min}_{u\in \mathrm{u}}\mathrm{J}(u);$$

- u is in the set of admissible controls,
- $\bar{u}$ is the optimal control
- $y = y(\bar{u})$ is the optimal state
- The function J is the objective function

However, there are two types of controls:
- Internal control,
- Border control

NB: *we advise you to look at the course material on control for detailed proof.*

V. INTERNAL CONTROL

It is a question of bringing the excess heat inside the domain. Let us consider the following example of control:

Let f in $L^2(\Omega), y_D$ belonging to $L^2(\Omega)$ *and* u be a nonempty closed convex set of $L^2(\Omega)$. Let $u \in$ u,$y(u)$ be the solution of the following equation.

$$(S)\begin{cases} -\Delta y(u) = f + u \ in \ \ \Omega, \\ \gamma y(u) = 0 \qquad on \ \ \Gamma. \end{cases} (1)$$

Let's pose

$$\mathrm{J}(u) = \frac{1}{2}\int_{\Omega} (y - y_D)dx + \frac{\in}{2}\|u\|L^2(\Omega), (2)$$

$$= \|y - \|^2_{L^2(\Omega)} \quad + \quad \frac{\in}{2}\|u\|^2_{L^2(\Omega).}$$

So we get the following problem:

$$\begin{cases} Does \ \bar{u} \ exist \ in \ \mathrm{u} \\ \mathrm{J}(\bar{u}) = {}^{min}_{u\in U}\mathrm{J}(u) \end{cases} (3)$$

VI. RESOLUTION

$f \in L^2(\Omega), u \in L^2(\Omega) \Longrightarrow f + u \ \in L^2(\Omega),$

For any z in$L^2(\Omega)$,we know the problem below:

$$\begin{cases} -\Delta z = g \ \ in \ \Omega, \\ \gamma z = \ \ 0 \ \ \ on \ \ \Gamma. \end{cases} (4)$$

Has a unique solution in $H^1_0(\Omega)$, therefore $y(u)$ is the unique solution of problem (11).

We define the following operation:

$$A = L^2(\Omega) \longrightarrow L^2(\Omega)$$

$$g \ \longmapsto A(g) = z.$$

Were z is the solution of equation (4). A thus defined is linear and continuous.

We have

$$A(f+u)=y(u).$$

By replacing $y(u)$by its value in(2), we get:

$$\mathrm{J}(u)=\frac{1}{2}\|y(u)-y_D\|^2_{L^2(\Omega)}+\frac{\in}{2}\|u\|^2_{L^2(\Omega),}$$

$$=\frac{1}{2}\|\mathrm{A}(f+u)-y_D\|^2_{L^2(\Omega)}+\frac{\in}{2}\|u\|^2_{L^2(\Omega),}$$

$$=\frac{1}{2}\left\{\left\|\frac{1}{2}\|\mathrm{A}(u)\|^2_{L^2(\Omega)}+\in\|u\|^2_{L^2(\Omega)}\right\|\right\}+\langle A(f)-y_D,A(u)\rangle,$$

$$\mathrm{J}_1(\mathrm{u})+\langle\mathrm{A(f)}-\mathrm{y_D},\mathrm{A(u)}\rangle.$$

Where:

$$\mathrm{J}_1(u)=\frac{1}{2}\left\{\left\|\frac{1}{2}\|\mathrm{A}(u)\|^2_{L^2(\Omega)}+\in\|u\|^2_{L^2(\Omega)}\right\|\right\},$$

And $\langle A(f)-y_D,A(u)\rangle$ is a constant.

Minimizing the functional J amounts to minimizing the functional J_1. We put:

a: $L^2(\Omega)$x $L^2(\Omega)\longrightarrow\mathbb{R}$

$$(u,v)\longrightarrow<A(u),A(v)>L^2(\Omega)+<u,v>$$

$$l:\quad L^2(\Omega)\longrightarrow\mathbb{R}$$

$$(u,v)\longrightarrow<A(u)-y_D,A(v)>L^2(\Omega).$$

- a is bilinear, continous, coercive and symmetric,
- l is continous linear,
- u a nonempty closed convex set of $L^2(\Omega)$,

By stampacchia's theorem, there exists a unique $\bar{u}$ in usolution.

The solution $\bar{u}$ is characterized by:

$$\begin{cases}\bar{u}\in u\\ a(\bar{u},v-\bar{u})\geq l(v-\bar{u})\ \forall\ v\in u\end{cases}\quad(5)$$

Having obtained the existence and uniqueness of the solution, we are interested in giving the characteristics of $\bar{u}$. To do this, we introduce the optimality system. By interpreting, the reaction

$a(\bar{u},\ v-\bar{u})\geq\iota(v-\bar{u})\ \forall\ v\ \in u,$ *we obtain:*

$<\bar{y}-y_D,A(v-\bar{u})>L^2(\Omega)+\ \in<\bar{u},v-\bar{u}>L^2(\Omega)\geq 0.$

The Conjoint state $\bar{p}\in H^1_0(\Omega)$ *is the solution of the following* problem:

$$\begin{cases}-\Delta\bar{p}=\ \bar{y}-y_D\ in\ \ \Omega\\ \ \gamma\bar{p}=0\qquad\quad on\ \ \Gamma,\end{cases}$$

We therefore have

$$<\bar{p}+\epsilon\bar{u},v-\bar{u}>L^2(\Omega)\ \geq\ 0.$$

We get the following optimality system:

$$\begin{cases}\bar{u}\in u,\bar{p}\in H^1_0(\Omega),\ \ y(\bar{u})\ \in L^{2(\Omega)}\\ (SO1)\begin{cases}-\Delta y(u)=f+u\ \ in & \Omega,\\ \ \gamma y(u)=\ 0\quad on & \Gamma.\end{cases}\\ (SO2)\begin{cases}-\Delta\bar{P}=\ \bar{y}-y_D\ \ in & \Omega,\\ \gamma\bar{p}=0\qquad on & \Gamma,\end{cases}\\ (SO3)<\bar{P}+\epsilon\bar{u},v-\bar{u}>L^{2(\Omega)}\geq 0.\end{cases}$$

VII. PARTICULAR CASE

If $u=L^2(\Omega)$then $\bar{u}=\frac{1}{\epsilon}\bar{p}$,

If$u=\ L^2_2(\Omega)\ then\ \bar{u}=\frac{1}{\epsilon}(\bar{p})-$**NB**:

In the case of shareholder internal Control, we have

$$\begin{cases}-\Delta y(u)=\ \ f+\psi u\ \ \ in\ \ \ \ \Omega,\\ \ \gamma y(u)=\ \ 0\qquad\quad on\ \ \ \ \Gamma.\end{cases}$$

Where $u\ \in\mathbb{R}$, $\quad\psi\ \in\ L^\infty(\Omega)$

VIII. BORDER CONTROL

This is to bring the excess heat to the Boundary of the domain. Consider

The following boundary control example:

Let f in $L^2(\Omega),y_D$ belong to $L^2(\Omega),g$ a function of $L^2(T)$ andu nonempty closed convex of $L^2(T).Let\ u\ \epsilon$ u, $\mathrm{y}(u)$ be a solution of the following equation:

$$(S1)\begin{cases}-\Delta\mathrm{y}(u)=f in\Omega,\\ \gamma\mathrm{y}(u)=g+u\ \ on\Gamma.\end{cases}(6)$$

We pose:

$$\mathrm{J}(u)=\frac{1}{2}\int_\Omega(\mathrm{y}(u)-\mathrm{y}_D)dx+\frac{\in}{2}\|u\|L^2(\mathrm{T}),(7)$$

$$=\left\|\mathrm{y}(u)-\mathrm{y}_D\right\|^2_{L^2(\Omega)}\quad+\ \frac{\in}{2}\|u\|^2_{L^2(\mathrm{T}).}$$

IX. RESOLUTION

➢ **Simplification**

By setting:

$$\mathrm{y}(u)=\mathrm{y}_1(u)+z,$$

Where$\mathrm{y}_1(u)$ is the solution of the problem:

$$(S2)\begin{cases}-\Delta\mathrm{y}(u)=0\quad in\Omega,\\ \gamma\mathrm{y}(u)=g+u\ \ on\Gamma.\end{cases}\quad(8)$$

And z belongs to $H^1_0(\Omega)$ solution de :

$$(S1)\begin{cases}-\Delta\mathrm{z}=f in\Omega,\\ \gamma\mathrm{z}=0\quad on\Gamma.\end{cases}\qquad(9)$$

System S_1 becomes

$$(S1)\begin{cases}-\Delta\mathrm{y}(u)=0\quad in\Omega,\\ \gamma\mathrm{y}(u)=g+u\ \ on\Gamma.\end{cases}(10)$$

We pose :

$$J(u) = \frac{1}{2}\|y(u) - y_{D1}\| \quad + \quad \frac{\in}{2}\|u\|^2_{L^2(\Omega).}$$

Where

$$y_{D1} = y_d\text{-z.}$$

➢ **Transposition formula**

We associate to (S_2) the following transposition equation:

$$(S1)\begin{cases}-\Delta y(u) = 0 & in\Omega, \\ \gamma y(u) = g & on\Gamma.\end{cases} (11)$$

Let f be a function of $L^2(\Omega)$( ) *admitting a unique solution, we get from* ( )

$$\int_\Omega (\Delta y1(u))\, z\, dx = 0 \Rightarrow \int_\Omega fz\, dx = \int_\Omega (g\frac{\partial z}{\partial n} d\sigma.$$

Considering the following problem:

$$(PV)\begin{cases} find\ y\ in\ L^2(\Omega) such\ that \\ \forall\, f\, L^{2(\Omega)},\ \int_\Omega fydx = -\int_\Omega g\frac{\partial z}{\partial n}dx. \end{cases}$$

$$l\ L^{2(\Omega)} \longrightarrow \qquad \mathbb{R}$$

$$g \longmapsto -\int_\Gamma g\frac{\partial z}{\partial n}d\sigma.$$

$l$is linear and continuous application on $L^2(\Omega)$, by riesz's theorem, there exists a unique $y \in L^2(\Omega)$ such that

$$< f, g > L^2(\Omega) = l(f); \|y\|L^2(\Omega) = \|l\|\ (\mathscr{L}((L^2((\Omega), \mathbb{R})).$$

We define the operator

$$AL^2(\Gamma) \longrightarrow \quad L^2(\Omega)$$

$$g \longmapsto A(g) = y1$$

Replacing

$$A(f + u) = y(u)$$

In $J(u), We$ have:

$$J = J_1;$$

With

$$J_1(u) = \frac{1}{2}\left\{\|A(u)\|^2_{L^2(\Omega)} + \epsilon\|u\|L(\Omega)\right\} + < A, A\ \text{-}y_{D1} > L^2(\Omega)$$

We pose

$$a: \quad L^2(\Omega) x\, L^2(\Omega) \longrightarrow \mathbb{R}$$

$$(u, v) \longmapsto < A(u), A(v) > L^2(\Omega) + < u, v > L^2(\Gamma)$$

$$l: \quad L^2(\Omega) \longrightarrow \mathbb{R}$$

$$v \longmapsto -< A(u) - y_{D1}, A(v) > L^2(\Omega).$$

X. CONCLUSION

In short, to solve the problem we must optimize the cost according to the goal we are looking for and taking into account the means at our disposal. But the case here requires a good mastery of the solutions of optimal control and some demonstrations to prove the existence, uniqueness and stability of solutions. Also, we give some extensions to our investigation.

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