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Exclusive Sum Labeling on Gear Graphs

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Abstract:- A graph with an injective mapping from vertex G to positive integers S such that two vertexes of G adjacent if and only if sum of labels can be an element of S. The aim of this research is to constructing exclusive sum labeling on gear graphs and obtaining isolated vertex sums for exclusive sum labeling on gear graphs. The data will be categorized in exclusive sum labeling on gear graph by Leonhard Euler. There are systematics in proving the theorem will be presented in the following steps: Defining labels for each vertex in a graph with adjacent sum of label is the label of isolated vertex. Proving the isolated vertex has obtained an optimal. This research conducted to find the exclusive sum labeling on gear graph by producing the optimal in exclusive sum labeling. In an exclusive addition of the integers of S which is the sum of two other integers of S can be labeling a set of isolated vertex which related to G graph. This article can show the construction of exclusive sum labeling on gear graphs.

Keywords:- *Gear Graph, Sum Labeling, Exclusive Sum Labeling.*

I. INTRODUCTION

Graph theory is a branch of mathematics by showing problems in forming graphs, so they can be explained more simply. Graph is a set pair, where is a finite set, whose elements are called knot, and is a set of pairs of elements called side (Hasmawati, 2020).

Labeling is only a mapping from the longitude graphs to positive integers, then it is called an edge labeling when labeling is a mapping from edges and vertex to positive integers, then they are called total labeling (Baĉa & Miller, 2009). Labeling is classified into several types, including magic labeling, anti-magic labeling, graceful labeling, sum labeling, and exclusive sum labeling (Gallian, 2010).

In 1990 Harary acquainted the concept of labeling sum graphs. The extension of the sum graph concept is growing and in acquainted exclusive sum labeling. Sum labeling of graph is called exclusive if there are knots worked on graph. In this type of labeling is sum number called the exclusive number (Miller et al., 2005; Sanjaya, 2011).

In graph addition, there is a vertex which is called a worked vertex if and only if there are labeling from vertex and vertex at that can be. A graph that has a sum labeling is called a sum graph. A sum graph can be together with the fewest isolated vertex is called a sum graph optimal (Gallian, 2020).

Sum labeling from graphs for a positive integer is said to be exclusive if all worked vertex at. Every graph can be made into an exclusive sum graph by adding some isolated vertex (Tuga et al., 2005). The minimal number of isolated vertex that need to be added to get a sum graph is called the denoted sum number and the minimum edge can be added in order to sum labeling that exclusive sum labeling is called exclusive sum number with notation (Miller, et al., 2017).

From the study of exclusive sum labeling that has been carried out by previous researchers, can be related to exclusive sum labeling on circle graphs and wheel graphs, the motivation arises to find labeling constructions, especially exclusive sum labeling on gear graphs which is one of the graphs formed by connecting a central vertex to a cycle vertex that ordered. In real life the concept of gear graph is modeled for patterning the channel determination of radio station.

II. OBJECTIVE OF THE STUDY

Based on the previous section, the objective of this study is formulated as follows: 1) Constructing exclusive sum labeling on gear graphs. 2) Obtaining isolated vertex sums for exclusive sum labeling on gear graphs. The data will be categorized in exclusive sum labeling on gear graph by Leonhard Euler.

III. LITERATURE REVIEW

Leonhard Euler was a Swiss mathematician that the first introduced graph theory in 1736. These are one of the discussions to be continuing on graph theory is labeling in graph. A labeling (rating) on a graph is mapping the elements of graph to sums (usually a positive integer). The choices have an original mapping is generally set with vertex and edge, set only vertex, or only that set of edges. The origin region is in the element combinations of other graphs that also can be possible (Addinnitya, 2012). Hasmawati (2020) explains graph structures are very diverse. Some graphs have their own characteristics so that they can be recognized and exist which have no special features (Ryan, 2009). Graph is a pair of set (V, E) with V is a discrete set whose members are called vertex, and E is the set of pair of member V is called an edge. Each graph has special characteristics so that the graphs are grouped in several types based on special characteristics. Exclusive sum labeling can be done on all types of graphs by adding a sum of isolated vertex. This research conducted to find the exclusive sum labeling on gear graph by producing the optimal in exclusive sum labeling.

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Exclusive sum labeling can be done on all types of graphs by adding of isolated vertex. In this research carried out to find the exclusive sum labeling on gear graph by an optimal generating in exclusive sum labeling (Wang & Li, 2010). Here are systematics in proving the theorem will be presented in the following steps: defining labels for each vertex in a graph with adjacent sum of label is the label of isolated vertex, proving the isolated vertex are obtained an optimal.

IV. METHOD

The type of research conducted a review theory (library research) with electronic books, handbooks, journals and web pages related to the research topic as references. The stages in this research in the following steps: graph definition, labeling for each vertex on graph, sum of vertex labels of adjacent, determining the label of isolated vertex, tried on the multiple of gear graphs (Grn), $n = 3,4, \dots , 8$, define patterns, checking if the pattern applies to gear graph (Grn) generally, proofing that there is none arcs additional between vertices who are not adjacent, proving that a lot of isolated vertex can be obtained an optimal.

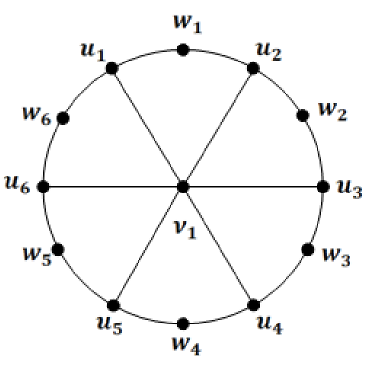
V. FINDING AND DISCUSSION

Exclusive sum labeling can be applied to all types of graphs by adding isolated vertex. In this research, the label value of each vertex on gear graph was determined. Next, determine the exclusive sum labeling constructions on gear graph producing the exclusive sum labeling that optimal. A graph is said to be optimal if.

Gear Graph have vertex on the outer cycle and 1 vertex on center vertex Gear Graph load wheel graph that having edges and there is an additional vertex among each pair of graph vertex that are directly connected to the outer cycle, there will be another edge, so.

The definition of vertex and edges on gear graph for the following:

$V(Gr_n) = \{v_1, u_i, w_i \mid i = 1, \dots .n\}$ with v_1 is central vertex, u_i and w_i is leaf vertex, and $E(Gr_n) = \{v_1u_i \mid i = 1, . . , n\} \cup \{u_iw_i \mid i = 1, \dots , 6\} \cup \{w_iu_{i+1} \mid i = 1, . . n - 1\} \cup \{u_1w_6\}$.C



Picture 1. Gear Graph Gr_n with $n = 6$

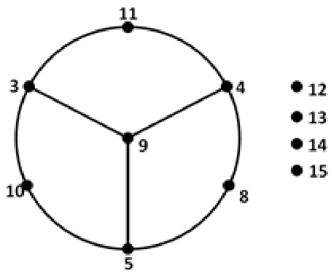
Before constructing exclusive sum labeling on graph, it is necessary to determine before limited of the exclusive sum number. The maximum degree of vertex on graph is notated $\Delta(Gr)$.

Lemma 4.1 $\boldsymbol{\varepsilon(Gr_n \odot \overline{K_4}) \geq n}$
Proof.

It has been previously proven that for any G graph it applies, $\varepsilon(G) \geq \Delta(G)$ (Three, et al., 2005). It is known that $\Delta(Gr_n \odot \overline{K_n}) \geq n$, then $\varepsilon(Gr_n \odot \overline{K_n}) \geq n$.

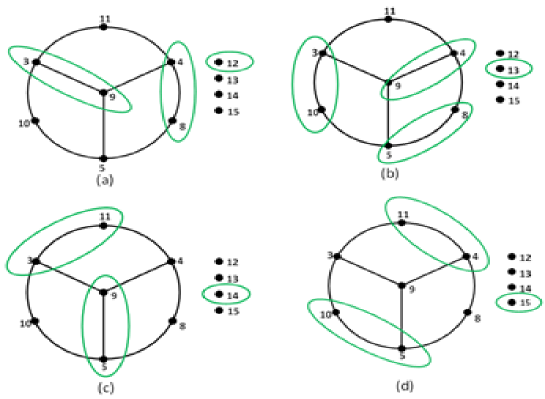
Case. $n = 3$

In determine exclusive sums on gear graph $Gr_n, n = 3$, can be known vertex sum 7 and edges sum 9 with $\Delta(Gr_3) = 3$ would be giving a label on each vertex then determining isolated vertex that can be additional each vertex adjacent Gr_3 graph with isolated vertex sum that minimum.



Picture 2. Exclusive Sum Labeling on $Gr_3 \odot \overline{K_4}$ Graph

Can be seen that exclusive sum labelling on graph Gr_3 with the isolated of vertex sum is 4, then ($\varepsilon(Gr_3) = 4$).



Picture 3 Label Sum of Vertex adjacent on $Gr_3 \odot \overline{K_4}$ Graph

In this picture can be seen the isolated vertex labels are 12, 13, 14 and 15. Part (a) shows that label sum of adjacent vertex that marked is worth 12, part (b) shows that label sum of adjacent vertex can be marked is worth 13, part (c) shows that label sum of adjacent vertex can be marked is worth 14 and in part (d) shows that label sum of adjacent vertex can be marked 15.

Based on the results in parts (a), (b), (c) and (d) it can be seen each label sum of adjacent vertex will be having the same isolated vertex labeling

Case 2. $n = 4$

It can be known that vertex sum 9 and edges sum 12 with $\Delta(Gr_4) = 4$. So that the exclusive sum number from the optimal of gear graph, the value of exclusive sum number must be $\varepsilon(Gr_4) = 4$

Theorem 4.2 $\boldsymbol{\varepsilon(Gr_4 \odot \overline{K_4}) = 4}$

Proof.

Definition $x_i, i = 1,2,\dots,n$ as isolated vertex. The systematic proving of this theorem will go through several stages of proof as mentioned at the beginning of this chapter

1. Defining labels for each vertex on gear graph.
Construction, for $n = 4$

- Central vertex $f(v_1) = 3n$
- Leaf vertex u_i with $i = 1,2,\dots,n$ that correlated with central vertex is labelled

$$f(u_i) = \begin{cases} n + (i-1) & Untuk\ \ i = 1\ dan\ 2 \\ n + 3 & Untuk\ \ i = n-1 \\ n + 2 & Untuk\ \ i = n \end{cases}$$

- Leaf vertex w_i with $i = 1,2,\dots,n$ cannot be related with the central vertex is labeled $f(w_i) =$

$$\begin{cases} \frac{n\cdot 7}{2} & Untuk\ \ i = 1 \\ n\cdot 3 - (i-1) & Untuk\ \ i = 2,\dots,n-1 \\ n\cdot 3 + 1 & Untuk\ \ i = n \end{cases}$$

- It is known the isolated vertex label is label sum of two vertex can be adjacent. Label sum of adjacent vertex is

Isolated vertex x_i with $i = 1, 2, \dots, n$ be given label

$$f(v_1) + f(u_i) = \begin{cases} 4n & Untuk\ i = 1 \\ 4n + i - 1 & Untuk\ i = 2,\dots,n \end{cases}$$

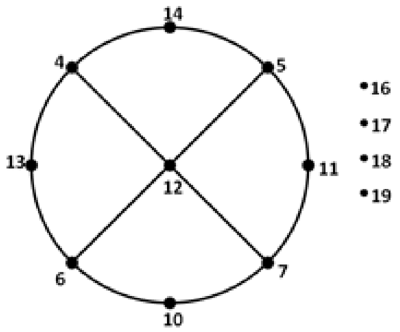
$$f(u_i) + f(w_i) = \begin{cases} 4n & Untuk\ i = 1 \\ 4n + i - 1 & Untuk\ i = 2,\dots,n \end{cases}$$

$$f(u_{i+1}) + f(w_i) = \begin{cases} 4n & Untuk\ i = 1 \\ 4n + i - 1 & Untuk\ i = 2,\dots,n-1 \end{cases}$$

$$f(u_1) + f(w_n) = 4n + 1$$

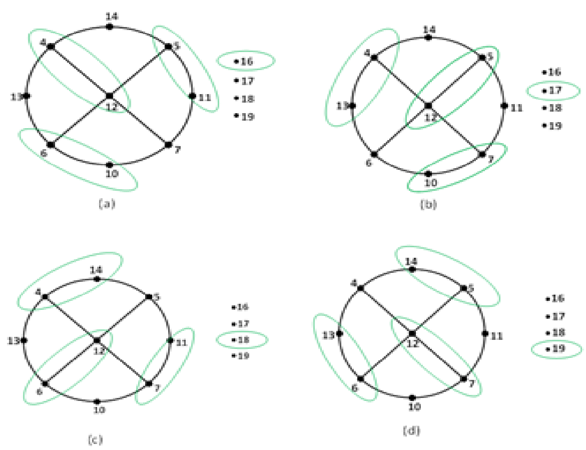
So, the label from isolated vertex on $Gr_4 \odot \overline{K_4}$ graph is $f(x_1) = 4n$, and $f(x_2) = 4n + i - 1$.

2. Showing isolated vertex that obtained is optimal.
Based on two steps of proof above, then $\varepsilon(Gr_4 \odot \overline{K_4}) \geq 4$, for $n = 4$ have proven.



Picture 3 Exclusive Sum Labeling on $Gr_4 \odot \overline{K_4}$ Graph.

By using the function of exclusive sum labeling have proven before. that picture also shows sum of isolated vertex produced is equal to the maximum degree by fulfilling the rule of exclusive sum labeling, that are, every sum vertex labels are adjacent. They are isolated vertex label.



Picture 5. The Label Sum of Vertex are adjacent to $Gr_4 \odot \overline{K_4}$ Graph

Isolated vertex labels are 16, 17, 18 and 19. Part (a) shows label sum of adjacent to vertex that marked 16, part (b) shows label sum of adjacent to vertex that marked 17, part (c) shows label sum of adjacent to vertex that marked 18 and in part (d) shows label sum of adjacent to vertex that marked 19. Based on the result in pictures of parts (a), (b), (c) and (d) they can be seen these each label sum of adjacent to vertex will be equal to isolated vertex label.

Thus it has been shown exclusive sum number on gear graph $\varepsilon(Gr_4) = 4$ and $\Delta(Gr_4) = 4$, then $\varepsilon(Gr_4) = \Delta(Gr_4) = 4$. Based on these, exclusive sum labeling on gear graph Gr_4 with using these function are exclusive sum labeling that optimal because $\varepsilon(Gr_4) = \Delta(Gr_4)$.

Case 5. $n \geq 5$

In determining exclusive sum on gear graph $Gr_n, n \geq 5$, the following construction is formed:

Theorem 4.3 $\boldsymbol{\varepsilon(Gr_n \odot \overline{K_n}) = n}$**, for** $\boldsymbol{n \geq 5}$

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Proof.

Definition $x_i, i = 1,2, \dots, n$ as isolated vertex. The systematic proving of this theorem will go through several stages of proof as mentioned at the beginning of this chapter

1. Defining labels for each vertex on gear graph.

Construction, for $n \geq 5$

- Central vertex $f(v_1) = 3n$
- Leaf vertex u_i with $i = 1,2, \dots, n$ that correlating with central vertex is labeled

$$f(u_i) = \begin{cases} n, & Untuk\ i = 1 \\ n+1, & Untuk\ i = 2 \\ n+i, & Untuk\ i = 3, \dots, n-1 \\ n+2, & Untuk\ i = n \end{cases}$$

- Leaf vertex w_i with $i = 1,2, \dots, n$ cannot be correlated with central vertex is labeled

a. For n Even

$$f(w_i) = \begin{cases} \frac{n \cdot 7}{2} - (i-1), & untuk\ i = 1, \dots, \frac{n}{2} - 1 \\ n \cdot i - (i-n), & untuk\ i = \frac{n}{2}, \dots, n-1 \\ 3n+1, & untuk\ i = n \end{cases}$$

b. For n Odd

$$f(w_i) = \begin{cases} \frac{n \cdot 7 + 1}{2} - (i-1), & Untuk\ i = 1, \dots, \frac{n-3}{2} \\ \frac{n \cdot 7 + 1}{2} - i, & Untuk\ i = \frac{n-1}{2} \\ n \cdot i - (n+1-i) & Untuk\ i = \frac{n+1}{2}, \dots, n-1 \\ 3n+2, & Untuk\ i = n \end{cases}$$

- It is known that isolated vertex label is label sum of two vertex are adjacent. Vertex label sum that adjacent.

Isolated vertex x_i with $i = 1, 2, \dots, n$ have labeled

$$f(v_1) + f(u_i) = \begin{cases} 4n & Untuk\ i = 1 \\ 4n + i - 1 & Untuk\ i = 2, \dots, n \end{cases}$$

$$f(u_i) + f(w_i) = \begin{cases} 4n & Untuk\ i = 1 \\ 4n + i - 1 & Untuk\ i = 2, \dots, n \end{cases}$$

$$f(u_{i+1}) + f(w_i) = \begin{cases} 4n & Untuk\ i = 1 \\ 4n + i - 1 & Untuk\ i = 2, \dots, n-1 \end{cases}$$

$$f(u_1) + f(w_n) = 4n + 1$$

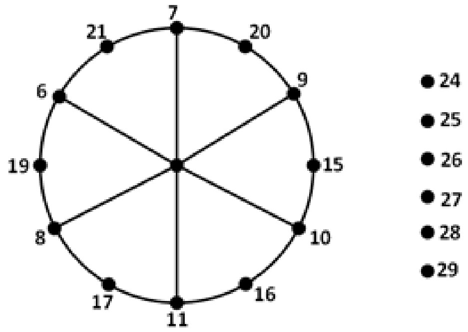
So, label of isolated vertex on $Gr_n \odot \overline{K_n}$ graph are $f(x_1) = 4n$, and $f(x_2) = 4n + i - 1$.

2. Showing isolated vertex that obtained is optimal.

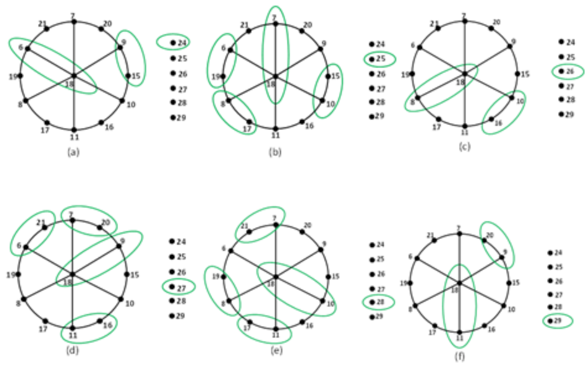
Exclusive sum labeling on gear graph with $n \geq 5$ above obtaining isolated vertex sum as much n, and have known $\Delta(Gr_n \odot \overline{K_n}) = n$.

Based on lemma 4.1, states that $\varepsilon(Gr_n \odot \overline{K_n}) \geq n$, then exclusive sum number obtained have optimal.

Based on two proof steps above, then $\varepsilon(Gr_n \odot \overline{K_n}) \geq n$, for $n = n$ have proven. Have given example of exclusive sum labeling on $Gr_6 \odot \overline{K_6}$ graph, on picture showing exclusive sum labeling on $Gr_6 \odot \overline{K_6}$ graph using exclusive sum labeling that have proven before. Vertex label that can be seen the injective result on gear graph with exclusive sum labeling that have proven before. Based on picture, it can also be seen that sum of isolated vertices obtained same as maximum degree on graph $Gr_6 \odot \overline{K_6}$ $(\varepsilon(Gr_6) = \Delta(Gr_6) = 6)$.



Picture 6. Exclusive Sum Labeling on Graph $Gr_6 \odot \overline{K_6}$



Picture 7. Sum of vertex label that adjacent to graph $Gr_6 \odot \overline{K_6}$

On picture 4.9 isolated vertex labels are 24, 25, 26, 27, 28, and 29. Part (a) shows sum of vertex label of adjacent marked is worth 24, part (b) shows sum of vertex label on adjacent to marked is worth 25, part (c) shows sum of vertex label of adjacent to marked is worth 26, part (d) shows sum of vertex label adjacent to marked is worth 27, part (e) shows sum of vertex label on adjacent to marked is worth 28 and part (f) indicates sum of vertax label on adjacent to marked is worth 29.

Based on the results in parts (a), (b), (c), (d), (e) and (f) they can be seen that each vertex label on adjacent to sum will have same with isolated vertex labelling. Thus, they have shown exclusive sum number on gear graph $\varepsilon(Gr_6) = 6$ and $\Delta(Gr_6) = 6$, so exclusive sum on graph $\varepsilon(Gr_6) = \Delta(Gr_6) = 6$. In these examples they can be seen the rules of exclusive sum labelling have fulfilled.

VI. CONCLUSION

Based on the existing research results the researcher can sum up that constructing the exclusive sum labeling on gear graphs so that can be obtained there are many isolated vertex on gear graphs as follows: 4, n =3, n, n>=4

This results can be obtained, be known for n= 3 you get a minimum isolated vertex and for n>= 4 so that can be concluded exclusive sum labeling on gear graph as an optimal. It has shown that optimal in exclusive sum of gear graphs with is as follows,

This research can still be continued to show the exclusive sum on Jahangir graphs.

$$\varepsilon(Gr_n) = \begin{cases} 4, & n = 3 \\ n, & n \geq 4 \end{cases}$$

This research can still be continued to show the exclusive sum on Jahangir graphs.

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