New Way to Solve the First Order Linear D.E. which Consist of Three Terms

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Abstract:- The new way depends on a differential equation Idifferentiated it from a functionI arranged it to solve all the types of the first order linear D.E.which consist of three terms in short time and less steps comparing with the known rules and methods .

By the new way any one can solve (Bernoulli , non exact, the homogeneous differential and Laplace transform by the same procedure and there is no need to know the type of the D.E.

I called my differential equation as Alsultani D.E. .

Keywords:- *New way of integration byAlsultani D.E. To solve any kind of the first order linear D.E.which consist of three terms .*

I. INTRODUCTION

The arranged function is :

$ax^m y^n - bx^s = f(x,y) = c$ $\quad eq.\ (1)$

Where (a,b,c,m,n and s) are arbitrary constants (x,y) are variables and the function exists and continous in the interval .

Differentiatieeq. (1) w.r. to x as avariable

$(anx^m y^{n-1})dy + (amx^{m-1}y^n)dx - bsx^{s-1}dx = 0 \quad$ eq. (2)

So $(\,am\,x^{m-1}y^n - bsx^{s-1})dx + (anx^m y^{n-1})dy = 0$

Let M = $amx^{m-1}y^n - bsx^{s-1}$ and $N = anx^m y^n$

Then $\frac{\partial M}{\partial y} = amnx^{m-1}y^{n-1}$(the partial direvative of M w.r. to y)

and $\frac{\partial N}{\partial x} = amny^{n-1}x^{m-1}$ (the partial direvative of N w.r. to x)

So equation (2) is an exact because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Divide eq. (2) by $(anx^m y^{n-1}dx)$ we get

$\frac{dy}{dx} + \frac{amx^{m-1}y^n}{anx^m y^{n-1}} = \frac{bsx^{s-1}}{anx^m y^{n-1}} = \frac{bsx^{s-1-m}}{any^{n-1}} = \frac{q(x)}{ny^{n-1}}$i.e q(x)=$\frac{bsx^{s-m-1}}{a}$

Now iet $\frac{q(x)}{n}$ = Q(x)

$$\frac{dy}{dx} + \frac{my}{nx} = \frac{Q(x)}{y^{n-1}} \times \frac{y^{1-n}}{y^{1-n}} = Q(x)y^{1-n}$$

$\frac{dy}{dx} + \frac{P(X)}{n}\,y = Q(x)y^{1-n}$ where P(x) =$\frac{m}{x}$ is a pure function of (x) only

Let 1- n = k then $\frac{dy}{dx} + \frac{P(X)}{n}\,y = Qy^k$ condition k=1-n eq.(3)I called eq. (3) Alsultani D.E. .

In eq. (3) the existing (n) remembers that P(x) /n isn't the pure function of x .

Important

In eq. (2) ;

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if m=n so the D.E. will be an exact because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

if m≠n then the D.E. is non exact because $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

3. even the homogeneous .

4. Bernoullidifferential equation which is

$$y' + p(x)y = q(x)y$$

$$we\ see\ it\ is\ not\ equivalent\ only\ when$$

n=1 so $\frac{dy}{dx} + p(x)y = Q(x)y\ and\ it`s\ solution\ is$

$$\int \frac{dy}{y} = \int [Q(x) - p(x)\backslash dx + c$$

5.if k=0 then the solution is Laplace transform except when the differential equation is in the form $\frac{dy}{dx} \pm Ax^u y = Bx^u$

$where\ (A, B\ and\ u\ )\ are\ arbitrary\ constants$ so the solution will be by the separated parameters i.e $\frac{dy}{dx} = Bx^u \pm Ax^u y$ and

$\int \frac{dy}{B \pm Ay} = \int x^u dx + c$ or by the new way .

So all of the above five kinds of equations can be solved by the hew way (Alsultani D.E.).

When we find the magmitude of (n) one must multiply eq. (3) by n to get

n$\frac{dy}{dx} + P(x)y = nQ(x)$ eq. (4)

then the integrating factor I(x) = $e^{\int P(x)dx}$

soat once one find n and I(x) he can integrate the right side of eq. (3) directly to finish the integration of the given differential equation .

Any way when we want to solve any question we must compare k with 1-n to find the magnitude of n

I(x)$y^n = \int nI(x)Q(x)dx + c$ solution

By the way that the above function was appeared when I had took the answer and revrsed the procedure of the solutlon to find it`s question and I repeated that for many problems .

Main Results

Now I will solve (5) questions one of each type of the non exact , Bernoulli , homogeneous and Laplace transform by my new way and by the known ways to see the differnces between them .

1 . solve the following Bernoulli differential equation :

x$y' + y = \frac{1}{y^2}$ eq. (5)

a .solution by the new way

Divide eq. (5) by x

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{xy^2}$eq. (6)

so according to the new way $\frac{dy}{dx} + \frac{P(X)}{n} y = Q(x)y^k$

.k = -2 = 1-n then n = 3

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Multiply eq. (6) by (3)

$3\ \frac{dy}{dx}+\frac{3}{x}y=\ \frac{3}{xy^2}$ so P(x) =$\frac{3}{x}$ and I(x) = $e^{\int\frac{3dx}{x}}=\ e^{3lnx}$ =x^3

Then x^3y^3=3$\int\frac{x^3}{x}\ dx=\ x^3+c\quad so\ \ y^3=1+cx^{-3}$ solution

6 steps only

b. solution by Bernoulli D.E.

x$y'+y=\frac{1}{y^2}$eq. (7)

divide eq. (7) by x we $\frac{dy}{dx}+\frac{y}{x}=\ \frac{1}{xy^2}$eq. (8)

so n= -2

Let v=$y^{1-n}=\ y^{1-(-2)}=\ y^3$ then $y=v^{\frac{1}{3}}$

So $y'=\frac{1}{3}v^{\frac{-2}{3}}\frac{dv}{dx}$

$\frac{1}{3}v^{\frac{-2}{3}}+\frac{v^{\frac{1}{3}}}{x}=\frac{v^{\frac{-2}{3}}}{x}$ $divide\ by\ \ v^{\frac{-2}{3}}$ so$\frac{1}{3}\frac{dv}{dx}+\frac{v}{x}=\frac{1}{x}$ OLE with v

Multiply by 3

$\frac{dv}{dx}+\frac{3v}{x}=\frac{3}{x}$ so P(x) = $\frac{3}{x}$ then I(x) = $e^{\int\frac{3}{x}dx}=e^{3lnx}=\ x^3$ now multiply by x^3

$$x^3\frac{dv}{dx}+\ 3x^2v=3x^2$$

So $\int\frac{d}{dx}(vx^3)=\int 3x^2dx+c$

Here we said that n=-2 but we see now that y^3 i.e. n=3

14 steps

2 . Solve the following non exact D.E.

x$y^3dx+(x^2y^2-1)dy=0$eq. (9)

a . solving by the new way

Divide eq. (9) byxy^3dy

$\frac{dx}{dy}+\frac{x}{y}=\ \frac{1}{xy^3}$eq. (10)

So k= -1 = 1-m then m=2

Multiply eq. (10) by 2

$2\frac{dx}{dy}+\frac{2x}{y}=\ \frac{2}{xy^3}$ so P(y) =$\frac{2}{y}$ $and\ I(y)=\ \ e^{\int\frac{2}{y}dy}=e^{2lny}=\ y^2$

$$x^2y^2=\ \int\frac{2y^2}{y^3}dy=2lny+c$$

8 steps

b . solving by the old method

x$y^3dx+(x^2y^2-1)dy=0$eq. (11)

M = xy^3 so $\frac{\partial M}{\partial y}=\ 3xy^2$

N = $x^2y^2 - 1$ then $\frac{\partial N}{\partial x} = 2xy^2$ so they are not equal and the equation is not an exact

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^3}(2xy^2 - 3xy^2) = \frac{-1}{y}$

I(x,y) = $e^{\int\frac{-dy}{y}} = e^{-lny} = \frac{1}{y}$

M_1 = IM and N_1 = IN

$M_1 = \frac{xy^3}{y} = xy^2$ and $N^1 = \frac{x^2y^2}{y} = xy^2 - \frac{1}{y}$ so

$$\frac{\partial M_1}{\partial y} = 2xy \text{ and } \frac{\partial N_1}{\partial x} = 2xy \text{ then an exact}$$

F(x,y) = $\int M_1dx + \int N_1dy = \int xy^2dx + \int \frac{-1}{y}dy = \frac{x^2y^2}{2} - lny + c$

16 steps

3 . solve the following homogeneous D.E.

$$x^2dy + y(x+y)dx = 0 \text{eq. } (12)$$

a , solving by the new way

Divide eq.(12) by x^2dx

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \qquad \text{eq. } (13)$$

$$\text{so } k = 2 = 1 - n \text{ then } n = -1$$

Multiply eq. (13) by -1

$-\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$ then $P(x) = \frac{-1}{x}$ so $I(x) = e^{\int\frac{-dx}{x}} = e^{-lnx} = \frac{1}{x}$

Then $\frac{1}{xy} = \int \frac{dx}{x(x^2)} + c = \frac{-1}{2x^2} + c = \frac{-1+2cx^2}{2x^2}$

So $xy = \frac{2x^2}{cx^2-1}$ where c_1=2c and $y = \frac{2x}{cx^2-1}$ solution

9 steps

b . solving the old way

$x^2dy + y(x+y)dx = 0$ eq. (14)

Let y=vx then and differentiate w.r. to x

$$x^2(vdx + xdv) + vx(x+vx)dx = 0 \quad \text{so} \quad vdx + xdv + vdx + v^2dx = 0$$

$$-\frac{dx}{x} = \frac{dv}{v^2+2v} = \frac{dv}{v(v+2)}$$

Let $\frac{1}{v(v+2)} = \frac{A}{v} + \frac{B}{v+2}$ then $A(v+2) + Bv = 1$

2A=1 so $A=\frac{1}{2}$ and $Bv + Av = 0$ $B = -\frac{1}{2}$

$$\int\frac{-dx}{x} = \int\frac{dv}{2v} - \int\frac{dv}{2(v+2)} \quad \text{so } -\ln x = 2\ln v - 2\ln(v+2) + \ln c \quad \text{and} \quad -2\ln x = \ln\frac{cv}{v+2}$$

$$\frac{1}{x^2} = \frac{c\frac{y}{x}}{\frac{y}{x}+2} = \frac{cy}{y+2x}$$

$$\text{so} \quad y = 2x/(cx^2 - 1)$$

17 steps

4 . solve the following D.E. by Laplace transform

$$x' + 4x = \text{cost} \quad , \ x(0) = 0 \qquad \text{eq. (15)}$$

Solution

a . by the new way

$$x' + 4x = x^0 \cos(t)$$

$x^0 = 1$ so k= 0 = 1-m then m=1

I(t) = $e^{\int 4dt} = e^{4t}$ so multiply eq.(15) by e^{4t}

$$x'e^{4t} + 4xe^{4t} = e^{4t}\cos(t)$$

Then $xe^{4t} = \int e^{4t}\cos(t)\,dt + c$

Let $e^{4t} = v$ so $dv = 4e^{4t}dt$ and $du = \cos(t)\,dt$ so $u = \sin(t)$

xe^{4t} = uv-$\int$ udv $= e^{4t}\sin(t) - \int 4\,e^{4t}\sin(t)dt$

let $4e^{4t} = v$ so $dv = 16\,e^{4t}dt$ and let $du = \sin(t)du$ then

u= -cos(t)

uv-$\int$ udv $= e^{4t}\sin(t) + 4e^{4t}\cos(t) - \int 16e^{4t}\cos(t)dt$

$\int e^{4t} cost\,(t)$dt = $e^{4t}\sin(t)$+ $4e^{4t}\cos(t) - 16\int e^{4t}$co(s)dt +c

$17\int e^{4t}\cos(t)dt = e^{4t}\sin(t) + 4e^{4t}\cos(t) + c$ divide 17 we get ;

$$\int e^{4t}\cos(t)\,dt = \frac{1}{17}e^{4t}\sin(t) + \frac{4}{17}e^{4t}\cos(t) + \frac{c}{17}$$

.x(0)=0

$xe^{4t} = \frac{1}{17}e^{4t}\sin(t) + \frac{4}{17}e^{4t}\cos(t) + \frac{c}{17}$ let $\frac{c}{17}$=c_1

0=0+$\frac{4}{17}$ ++c_1 then c_1=$\frac{-4}{17}$

$xe^{4t} = \frac{1}{17}e^{4t}\sin(t) + \frac{4}{17}e^{4t}\cos(t) - \frac{4}{17}$ now divide by e^{4t} we get ;

x(t)=$\frac{\sin(t)}{17} + \frac{4\cos(t)}{17} - \frac{4e^{-4t}}{17}$ solution

b . by Laplace transform

$$x' + 4x = cost \quad , \ x(0) = 0\text{eq. (16)}$$

Solution

By using the tables of Laplace transform we find ;

L{x}=x(s)

L{x'}=sx(s)-x(0)

L{cost}=$\frac{s}{s^2+1}$

.x(s)-x(0)+4x(s)= $\frac{s}{s^2+1}$ and x(0)=0

.x(s)[s+4]= $\frac{s}{s^2+1}$

.x(s)=$\frac{s}{(s+4)(s^2+1)}$

$\frac{s}{(s+4)(s^2+1)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+1}$

.s=$A(s^2+1)+(Bs+C)(s+4)$

.s=-4 → -4=17A+0 , A=$\frac{-4}{17}$

.s=$As^2+A+Bs^2+4Bs+Cs+4C$

$0s^2+1s+0=(A+B)s^2+(4B+C)s+4C$

A+B=0 → $B=\frac{4}{17}$

4B+C=1

A+4C=0 → $C=\frac{1}{17}$

.x(s))=$\frac{-4}{17(S+4)}+\frac{4s}{17(S^2+1)}+\frac{1}{17}-\frac{1}{S^2+1}$

.x(t)=$\frac{-4e^{-4t}}{17}+\frac{4\cos t}{17}+\frac{\sin t}{17}$

5 . solve the following question ;

$\frac{dy}{dx}+2xy=x$eq. (17)

Solution

a . by the new way

$$\frac{dy}{dx}+2xy=x$$

Here $y^0=1$ so $k=0=1-n$ and $n=1$

Then P(x)= 2x and I(x)=$e^{\int 2xdx}=e^{x^2}$

$I(x)\frac{dy}{dx}+2xyI(x)=xI(x)$

$ye^{x^2}=\int xe^{x^2}dx+c$

so $ye^{x^2}=\frac{e^{x^2}}{2}$+c multiply by $e^{x^{-2}}$

we get y=$\frac{1}{2}$+c $e^{x^{-2}}$ solution

b . by sabrable methode ;

$\frac{dy}{dx}+2xy=x$ eq. (18)

then $\frac{dy}{dx}=x-2xy=x(1-2y)$ by separating the parameters we get ;

$\frac{dy}{1-2y}=xdx$eq. (19)

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So $\int \frac{dy}{1-2y} = \int xdx$+c₁

$\frac{-1}{2}$ln(1-2y)=$\frac{x^2}{2} + c_1$ multiply (-2)

Then ln(1-2y)=-$x^2 - 2c_1$ so $e^{ln(1-2y)}$=$e^{x^{-2}-2c_1} = -ce^{x^{-2}}$

(1-2y)=c $e^{x^{-2}}$ and y=$\frac{1}{2}$+c $e^{x^{-2}}$ solution

II. CONCLUSIONS

- The new way of solving the differential equations is faster than the old rules and methods .
- There is no need to know the type of the D.E. to solve it .
- The new way will participate in the advance .
- By the new way one can see that Bernoulli D.E. is inquivalent equation because P(x) and$y^n$are not true such that there is needing to the substitution ($v = y^{1-n}$) $and\ its\ complications\ as$ because of the fractional powers of some variables . Also with Bernoulli solutions we begin with y^n $and\ finish\ with\ y^{1-n}$.
- There is needing to the Laplace transform tables to solve the problems but with the new way there is no need to them .

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