A Study of Caristi’s Fixed Point Theorem on Normed Space

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Abstract:- This paper aims at treating a study of Caristi’s fixed point theorem for mapping results that introduced in the setting of normed space. The classical Caristi’s fixed point theorem is a generalization of this work.

Keywords: - Norm Space, Complete Norm Space, Operator, Transfinite induction, Zorn’s lemma and Banach Fixed Point Theorem.

I. INTRODUCTION AND PRELIMINARIES

It is conventional to this work motivated by some recent work on Caristi’s fixed point theorem for mappings defined on metric spaces with a partial order or a graph. One of the most important theorems is the Caristi fixed point theorem and it’s related to a complete normed space with the Banach fixed point theorem. Here, Banach contraction principle and Caristi’s fixed point theorem [3] is important for nonlinear analysis. It’s a modification of the ε-variational principle of Ekeland [1, 2]. Then the different directions of the Caristi’s fixed point theorem have been investigated by several authors; see, for example, [4–17] and references therein. Finally, we study of Caristi’s fixed point theorem (Theorem-3) were given as mention above.

In this paper, we will discuss a proof of Caristi’s fixed point theorem using mapping results which is introduced in the setting of normed spaces.

Normed Spaces 1.1: A normed on X is a real function \( \| \cdot \| : X \rightarrow \mathbb{R} \) defined on X such that for any \( x, y \in X \) and for all \( \lambda \in \mathbb{K} \).

i. \( \| x \| \geq 0 \)

ii. \( \| x \| = 0 \) if and only if \( x = 0 \).

iii. \( \| \lambda x \| = |\lambda| \| x \| \)

iv. \( \| x + y \| \leq \| x \| + \| y \| \) \( (\text{Triangle inequality}) \)

A norm on X defines a metric d on X which is given by \( d(x, y) = \| x - y \| \); \( x, y \in X \) and is called the metric induced by the norm.

The normed space is denoted by \((X, \| \cdot \| )\) or simply by X

II. A STUDY OF CARISTI’S FIXED POINT THEOREM ON NORMED SPACE

Here, we present a study of Caristi’s fixed point theorem for mapping results which is introduced in setting of normed spaces such as.

Theorem 2.1 (Caristi [3]). Suppose that \((X, \| \cdot \| )\) is a complete normed spaces and let \( T : X \rightarrow X \) be a mapping such that \( \| a - Ta \| \leq \phi(a) - \phi(Ta) \ \forall \ a \in X \ldots \ldots \ldots \ldots \ (i) \)

Where \( \phi : X \rightarrow [0, \infty) \) is a lower semi-continuous mapping. Then T has at least fixed point.
**Proof:** We consider that $Ta \neq a$ $\forall a \in X$.

A map $N(a): X \to 2^X$ (Power of X) by

$$N(a) = \{b \in X : \|a - b\| \leq \phi(a) - \phi(b)\}.$$  

From equation (i) we know that $Ta \in N(a)$ and hence $N(a) \neq \emptyset$ $\forall a \in X$.

We observe that for each $b \in N(a)$,

Here $\phi(b) \leq \phi(a)$ and $N(b) \subseteq N(a)$.

Let $b \in N(a)$ be given.

We have $\|a - b\| \leq \phi(a) - \phi(b)$.

So we have $\phi(b) \leq \phi(a)$ since $N(b) \neq \emptyset$.

Let $t \in N(b)$.

Then $t \neq b$ and $\|b - t\| \leq \phi(b) - \phi(t)$.

It follows that $\phi(t) \leq \phi(b) \leq \phi(a)$ and hence $\|a - t\| \leq \|a - b\| + \|b - t\| \leq \phi(a) - \phi(t)$.

Also we have $t \neq a$. Indeed if $t = a$. Then $\phi(t) = \phi(a)$.

So,

$$\|a - b\| = 0$$

$$\Rightarrow a - b = 0$$

$$\Rightarrow a = b$$

This would imply $b = t$, a contradiction.

Hence $t \in N(a)$. Therefore we prove $N(b) \subseteq N(a)$.

We construct a sequence $<a_n >$.

The induction for any point $a_i \in X$.

Let $a_n \in X$.

Then for $a_{n+1} \in N(a)$ such that $N(a_{n+1}) \leq \inf_{i \in N(a_n)} \phi(t) + \frac{1}{n}$, $n \in N$. ..................(ii)

For any $n \in N$, since $a_{n+1} \in N(a)$, we have

$$\|a_n - a_{n+1}\| \leq \phi(a_n) - \phi(a_{n+1})$$

..............(iii)

So, $\phi(a_{n+1}) \leq \phi(a_n)$, $\forall$ $n \in N$.

If $\phi$ is bounded below, then

$$\alpha = \lim_{n \to \infty} \phi(a_n) = \inf_{n \in N} \phi(a_n)$$

exist .

..............(iv)

From (iii) and (iv) then we get

$$\|a_n - a_{n+1}\| \leq \sum_{j=n}^{m-1} \|a_j - a_{j+1}\| \leq \phi(a_n) - \alpha$$

[:: m > n and $m, n \in N$]

Since $\lim_{n \to \infty} \phi(a_n) = \alpha$ we get, $\lim_{n \to \infty} \sup \{ \|a_n - a_m\| \} = 0$ $\forall m > n$.

Therefore $<a_n>$ is a Cauchy sequence in $X$.

Then $\exists \nu \in X$ s.t. $a_n \to \nu$ as $n \to \infty$.

i.e $\phi$ is a lower semi-continuous.
From (iv) then we get
\[ \phi(v) \leq \lim_{n \to \infty} \inf_{a_n} \phi(a_n) = \inf_{a_n} \phi(a_n) \leq \phi(a_j) \quad \forall \ j \in \mathbb{N} \]
\[ \text{..............................(v)} \]
Then we show that \( \bigcap_{n=1}^{\infty} N(a_n) = \{v\} \)

For \( m > n \) with \( m, n \in \mathbb{N} \)
From (iii) and (iv) then we get
\[ \|a_n - a_m\| \leq \sum_{j=1}^{m-1} \|a_j - a_{j+1}\| \leq \phi(a_n) - \phi(v) \]
\[ \text{..............................(vi)} \]
Since \( a_m \to v \) as \( m \to \infty \) the inequality (vi) implies
\[ \|a_n - v\| \leq \phi(a_n) - \phi(v) \quad \forall n \in \mathbb{N} \]
\[ \text{..............................(vii)} \]
By (vii) we know \( v \in \bigcap_{n=1}^{\infty} N(a_n) \)

Hence \( \bigcap_{n=1}^{\infty} N(a_n) \neq \phi \) and \( N(v) \subseteq \bigcap_{n=1}^{\infty} N(a_n) \)

From (ii) then we have
\[ \|a_n - w\| \leq \phi(a_n) - \phi(w) \leq \inf_{t \in (a_n)} \phi(t) \leq \phi(a_n) - \phi(a_{n+1}) + \frac{1}{n} \]
\[ \Rightarrow \|a_n - w\| \leq \phi(a_n) - \phi(a_{n+1}) + \frac{1}{n} \quad \forall n \in \mathbb{N} \]
Hence, \( \lim_{n \to \infty} \|a_n - w\| = 0 \Rightarrow a_n - w = 0 \Rightarrow a_n = w \).

Then, \( w = v \).

Therefore, \( \bigcap_{n=1}^{\infty} N(a_n) = \{v\} \)

Since \( N(v) \neq \phi \) and \( n(v) \subseteq \bigcap_{n=1}^{\infty} N(a_n) = \{v\} \). We obtain \( N(v) = \{v\} \)

On the contrary, from (i) we know \( Tv = v \).

Hence \( T \) has a fixed point \( v \) in \( X \).

Q. E. D

Remark 2.2.

a) Although the function \( \phi \) is lowering semi-continuous, it does not deduce that \( N(a) \) is a closed subset of \( X \).
b) A study of Caristi’s fixed point theorem contains assigning norm in \( X \).

III. CONCLUSION

Our aim is to discuss a study of Caristi’s fixed point theorem on normed space. We hope that this work will be useful for functional analysis related to normed spaces and fixed point theory.

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