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The Concept of Spacer Subspaces Associated with Complex Matrix Spaces of Order m by n, Where m≠n

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Abstract:- The article presents a formalism to define non-trivial subspaces of complex matrix spaces of order m by n, where m and n are different. This is achieved by using the Spacer matrix(as defined in Ghosh[1]) and it's Hermitian conjugate, associated with the matrix space, as the building blocks. The Mathematical formalism is presented and illustrated with suitable numerical examples.

Keywords:- *Spacer Matrices, Component Matrices of the Spacer Matrix, Embedding Dimension, Spacer Matrix based Generalized Matrix Multiplication.*

Notations

- $M_{m\times n}(C)$ denotes the Complex Matrix space of order m by n
- N denotes the set of all Natural numbers
- r denotes the Embedding dimension
- $X_{m\times n}$ denotes the Spacer Matrix of order m by n
- $P_{m\times r}, Q_{r\times n}$ are the component Matrices associated with the Spacer Matrix $X_{m\times n}$
- $|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m\times 1}$, $\langle m| = \begin{bmatrix} 1 & 1 & . & . & 1 \end{bmatrix}_{1\times m}$
- $I_{s\times s}$ denotes the Identity Matrix of order 's'
- $\max(a,b)$ denotes the maximum of the two inputs a and b, $a,b \in N$
- $\min(a,b)$ denotes the minimum of the two inputs a and b, $a,b \in N$
- $|a-b|$ denotes the absolute value of the difference of the two inputs a and b, $a,b \in N$
- A^T denotes the transpose of the Matrix A
- A^H denotes the hermitian conjugate of the Matrix A
- $|A|$ denotes the cardinality of the set A
- $A(j)$ denotes the j^th element of the set A

I. INTRODUCTION

The concept of Spacer matrices arise in context of Generalized Matrix multiplication scheme presented by Ghosh [1], it can be observed that the spacer matrix and its Hermitian conjugate are unique for a given pair of nonconforming dimensions (m,n) and (n,m). The structural form of the spacer matrix and it's constitution is determined only by 0's, 1's, and m and n(which are positive integers). This property can be utilized to define unique subspaces of the Matrix spaces $M_{m\times n}(C)$ and $M_{n\times m}(C)$, using the spacer matrix $X_{n\times m}$ and its hermitian conjugate as the building blocks. These subspaces are termed as "Spacer Subspace" of the corresponding Matrix space, these subspaces are non-trivial, and dimensionality determined by the properties of the linear dependencies of the powers of the square matrix components XX^H and X^HX that generate the spacer subsets $\hat{S}$ and $\hat{T}$, starting with the matrices $X_{n\times m}$ and $(X^H)_{m\times n}$, respectively.

II. MATHEMATICAL FRAMEWORK AND ASSOCIATED ANALYSIS

The following set of results, from Ghosh [1], forms the foundation for the framework developed in this article, here only the case m≠n is discussed:

- ➢ The Spacer Matrix associated with dimension pair (n,m) is denoted by $X_{m\times n}$, which is defined as follows:

- ➢ $X_{m\times n} = P_{m\times u}Q_{u\times n}$, where we have the following:

$u = \max(m,n) + |m-n| = \max(n,m) + |n-m|$, $m,n \in N$

$$P = \begin{bmatrix} I_{m\times m} & (\frac{1}{m})|m\rangle\langle u-m| \end{bmatrix}_{m\times u} = \left[\begin{array}{ccccc|ccccc} 1 & 0 & . & . & 0 & 1/m & 1/m & . & . & 1/m \\ 0 & 1 & . & . & 0 & 1/m & 1/m & . & . & 1/m \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 1/m & 1/m & . & . & 1/m \end{array}\right]$$

,

$$Q=\begin{bmatrix} I_{n\times n} \\ (\frac{1}{n})|u-n\rangle\langle n| \end{bmatrix}_{u\times n} = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \\ \hline 1/n & 1/n & . & . & 1/n \\ 1/n & 1/n & . & . & 1/n \\ . & . & . & . & . \\ . & . & . & . & . \\ 1/n & 1/n & . & . & 1/n \end{bmatrix}$$

- The Spacer Matrix associated with dimension pair (m,n) is denoted by $Y_{n\times m}$, which is defined as follows:
- $Y_{n\times m}=P_{n\times u}Q_{u\times m}$, where we have the following:

$$P_{n\times u}=\begin{bmatrix} I_{n\times n} & (\frac{1}{n})|n\rangle\langle u-n| \end{bmatrix},$$

$$Q_{u\times m}=\begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})|u-m\rangle\langle m| \end{bmatrix}$$

1. The following set of results, relating the spacer matrices $X_{m\times n}$ and $Y_{n\times m}$, can be easily observed :

- $Y\neq 0_{n\times m}$, $X_{m\times n}=(Y_{n\times m})^{H}=(Y_{n\times m})^{T}$

Construction of the Spacer Subsets and associated Spacer Subspaces

$\hat{S}\subset M_{m\times n}(C)$, We define $h=\min(m,n)$, We define the Spacer Subset of the complex matrix space $M_{m\times n}(C)$ as follows:

$$\hat{S}=\{X_{m\times n},(XX^{H})_{m\times m}X_{m\times n},......,(XX^{H})_{m\times m}{}^{h-1}X_{m\times n}\}$$

, where we have:

$(XX^{H})_{m\times m}{}^{h-1}=(XX^{H})_{m\times m}.........(XX^{H})_{m\times m}$ .... (h-1 times)

In an analogous manner, we define the Spacer Subset associated with the complex matrix space $M_{n\times m}(C)$:

$\hat{T}\subset M_{n\times m}(C)$, we define: $\hat{T}=\{Y_{n\times m},(YY^{H})_{n\times n}Y_{n\times m},......,(YY^{H})_{n\times n}{}^{h-1}Y_{n\times m}\}$, where we have:

$(YY^{H})_{n\times n}{}^{h-1}=(YY^{H})_{n\times n}.........(YY^{H})_{n\times n}$ .... (h-1 times)

We follow the convention: $(XX^{H})^{0}=I_{m\times m}$, $(YY^{H})^{0}=I_{n\times n}$

The following can easily be observed:

- $|\hat{S}|=|\hat{T}|=h$, $\because m\neq n,\ \ m,n\in N$, this implies: $h\geq 1$
- $\hat{S}(j)=\left[\hat{T}(j)\right]^{H}$, $\forall j=1,2,...,h$
- $span(\hat{S})\subseteq M_{m\times n}(C)$, $1\leq \dim.(span(\hat{S}))\leq h\leq m.n=\dim.(M_{m\times n}(C))$
- $span(\hat{T})\subseteq M_{n\times m}(C)$, $1\leq \dim.(span(\hat{S}))\leq h\leq m.n=\dim.(M_{n\times m}(C))$

- ❖ $span(\hat{S})$ is termed as the Spacer Subspace associated with the Matrix Space $M_{m\times n}(C)$
- ❖ $span(\hat{T})$ is termed as the Spacer Subspace associated with the Matrix Space $M_{n\times m}(C)$

III. NUMERICAL EXAMPLES:

1. (m,n) = (1,2) , (n,m) = (2,1)

We have: $u=3,h=1$

$$X=[(3/2)\quad (3/2)]_{1\times 2} \quad , \quad Y=\begin{bmatrix}(3/2)\\(3/2)\end{bmatrix}_{2\times 1} ,$$

$$\hat{S}=\{[(3/2)\quad (3/2)]_{1\times 2}\},\ \ \hat{T}=\{\begin{bmatrix}(3/2)\\(3/2)\end{bmatrix}_{2\times 1}\},$$

$$span(\hat{S})=\{v\in M_{1\times 2}(C)\,|\,v=\begin{bmatrix}c & c\end{bmatrix}_{1\times 2},c\in C\}$$

$$span(\hat{T})=\{w\in M_{2\times 1}(C)\,|\,w=\begin{bmatrix}c\\c\end{bmatrix}_{2\times 1},c\in C\}$$

$$\dim.(span(\hat{S}))=\dim.(span(\hat{T}))=1,$$
$$\dim.(M_{1\times 2}(C))=\dim.(M_{2\times 1}(C))=2$$

2. (m,n) = (1,3) , (n,m) = (3,1)

We have: $u=5,h=1$

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$$X=\begin{bmatrix}(5/3) & (5/3) & (5/3)\end{bmatrix}_{1\times 3} \quad , \quad Y=\begin{bmatrix}(5/3)\\(5/3)\\(5/3)\end{bmatrix}_{3\times 1} ,$$

$$\hat{S}=\{\begin{bmatrix}(5/3) & (5/3) & (5/3)\end{bmatrix}_{1\times 3}\} \ , \ \hat{T}=\{\begin{bmatrix}(5/3)\\(5/3)\\(5/3)\end{bmatrix}_{3\times 1}\}$$

$$, \ span(\hat{S})=\{v\in M_{1\times 3}(C)\,|\,v=\begin{bmatrix}c & c & c\end{bmatrix}_{1\times 3}, c\in C\} ,$$

$$span(\hat{T})=\{w\in M_{3\times 1}(C)\,|\,w=\begin{bmatrix}c\\c\\c\end{bmatrix}_{3\times 1}, c\in C\} ,$$

$$\dim.(span(\hat{S}))=\dim.(span(\hat{T}))=1$$

$$\dim.(M_{1\times 3}(C))=\dim.(M_{3\times 1}(C))=3$$

3. <u>(m,n) = (2,3) , (n,m) = (3,2)</u>

We have: $u=4, h=2$

$$X=(\frac{1}{6})\begin{bmatrix}7 & 1 & 4\\1 & 7 & 4\end{bmatrix}_{2\times 3} ,$$

$$(XX^H)X=(\frac{1}{18})\begin{bmatrix}41 & 23 & 32\\23 & 41 & 32\end{bmatrix}_{2\times 3} ,$$

$$Y=(\frac{1}{6})\begin{bmatrix}7 & 1\\1 & 7\\4 & 4\end{bmatrix}_{3\times 2} , \ (YY^H)Y=(\frac{1}{18})\begin{bmatrix}41 & 23\\23 & 41\\32 & 32\end{bmatrix}_{3\times 2} ,$$

$$span(\hat{S})=\{v\in M_{2\times 3}(C)\,|\,v=a\begin{bmatrix}7 & 1 & 4\\1 & 7 & 4\end{bmatrix}_{2\times 3}+b\begin{bmatrix}41 & 23 & 32\\23 & 41 & 32\end{bmatrix}_{2\times 3}, a,b\in C\}$$

$$span(\hat{T})=\{w\in M_{3\times 2}(C)\,|\,w=\gamma\begin{bmatrix}7 & 1\\1 & 7\\4 & 4\end{bmatrix}_{3\times 2}+\delta\begin{bmatrix}41 & 23\\23 & 41\\32 & 32\end{bmatrix}_{3\times 2}, \gamma,\delta\in C\}$$

$$\dim.(span(\hat{S}))=\dim.(span(\hat{T}))=2,$$
$$\dim.(M_{2\times 3}(C))=\dim.(M_{3\times 2}(C)=6$$

4. <u>(m,n) = (2,4) , (n,m) = (4,2)</u>

We have: $u=6, h=2$

$$X=(\frac{1}{4})\begin{bmatrix}5 & 1 & 3 & 3\\1 & 5 & 3 & 3\end{bmatrix}_{2\times 4} ,$$

$$(XX^H)X=(\frac{1}{8})\begin{bmatrix}31 & 23 & 27 & 27\\23 & 31 & 27 & 27\end{bmatrix}_{2\times 4} ,$$

$$Y=(\frac{1}{4})\begin{bmatrix}5 & 1\\1 & 5\\3 & 3\\3 & 3\end{bmatrix}_{4\times 2} , \ (YY^H)Y=(\frac{1}{8})\begin{bmatrix}31 & 23\\23 & 31\\27 & 27\\27 & 27\end{bmatrix}_{4\times 2} ,$$

$$span(\hat{S})=\{v\in M_{2\times 4}(C)\,|\,v=a\begin{bmatrix}5 & 1 & 3 & 3\\1 & 5 & 3 & 3\end{bmatrix}_{2\times 4}+b\begin{bmatrix}31 & 23 & 27 & 27\\23 & 31 & 27 & 27\end{bmatrix}_{2\times 4}, a,b\in C\}$$

$$span(\hat{T})=\{w\in M_{4\times 2}(C)\,|\,w=\gamma\begin{bmatrix}5 & 1\\1 & 5\\3 & 3\\3 & 3\end{bmatrix}_{4\times 2}+\delta\begin{bmatrix}31 & 23\\23 & 31\\27 & 27\\27 & 27\end{bmatrix}_{4\times 2}, \gamma,\delta\in C\}$$

$$\dim.(span(\hat{S}))=\dim.(span(\hat{T}))=2, \dim.(M_{2\times 4}(C))=\dim.(M_{4\times 2}(C)=8$$

IV. DISCUSSION AND CONCLUSION

The article presents a mathematical framework to construct a unique subspace for any strictly rectangular complex matrix space, this subspace is nontrivial(since the spacer matrix is non-zero and non-negative by construction, which is thus also reflected in its hermitian conjugate counterpart, which is the spacer matrix for the permutated ordered pair representing the nonconforming dimensions), using these matrices as initiators, finite subsets in the corresponding Matrix spaces are formed, which act as the spanning sets of the Spacer subspaces of the corresponding matrix spaces.

The issue of determining the precise dimension of the spacer subspaces can possibly be answered by analysis of the characteristic polynomial and minimal polynomial associated with the square, hermitian matrices XX^H and YY^H (it can be easily inferred that they are diagonalizable, hence, it is the number of distinct roots of the characteristic polynomial of these matrices that dictate the dimensionality of the associated Spacer subspace). A follow up study will address this issue with more depth and clarity, and also focus on applicability of these subspaces in solving theoretical/applied problems.

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