Numerical Solutions of the Work Done on Finite Order-Preserving Injective Partial Transformation Semigroup

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Abstract:- In this research work, we have used an effective methodology to obtain formulas for calculating the total work done and consequently the average work done and the power of the transformation by elements of finite order-preserving injective partial transformation semigroup. The generalized formulas was applied to obtain the numerical solutions and results tabulated. We equally plot graphs to illustrate the nature of the total work done and the average work done by elements of the finite order-preserving partial transformation semigroup. Results obtained showed that moving the elements from domain to co-domain involve allot of work to be done on a numerical scale.

Keywords:- *Partial Transformation, Semigroup, Total Work Done, Average Work Done, Power, Injective.*

I. INTRODUCTION

Semigroup of finite partial transformation on a fixed set n has properties of being injective, surjective or both (i.e bijective), hence there is a need for us to specifically study all subsets of the finite partial transformation semigroup, in this research work we are going to present the finite order-preserving injective partial transformation semigroup denoted by POI_n which is a well known subset of the partial transformation semigroup (PT_n). Semigroup is simply the generalization of group without requiring the existence of identity and the existence of inverses. Semigroup theory has formed a great branch of research interest in mathematics because of the fascinating properties of semigroups. Studies in Semigroup are of much interest to mathematicians especially those in the field of abstract algebra and this is because of the associative property of the semigroup operation which is usually composition of functions.

Transformation Semigroup is an important and a useful tool in semigroup theory as every semigroup has been shown to be isomorphic to a transformation semigroup. Transformation semigroups are mainly functions from a given set to itself and one of the important transformation in semigroup theory is the finite partial transformations semigroups. Many researchers have published several works in the transformation semigroup theory, researchers like Laradji.A, Umar.A and Wilson have put together allot of useful contents on different concepts of the transformation semigroup. In fact this work is totally an advancement of the research work put together by Umar.A in his work "**Some combinatorial problems in the theory of partial transformation semigroups"** as published in 2014, we have mainly explored the cardinality given for each subset of the partial transformation semigroup alongside a lemma on finite summation of series and this has proved very effective and reliable.

In this research work, we have made attempt to analyze functions on the finite order-Preserving injective partial transformation semigroup using some physics concept in line with valid mathematical facts. The main goal is to assume elements of POI_n as set of points that are equally spaced on a line, then we made a claim that the work done by an arbitrary transformation τ in moving element x of the domain of the given fixed set n to element y of the co-domain is given by a metric distance defined by $|x-y|$. The total work done is given by the sum of these distances as the point x varies along the domain of the given set n.

For convenience we shall be denoting this total work done by $W_\tau(POI_n)$, consequently the average work done denoted by $\overline{W}_\tau(POI_n)$ and the power of the transformation which we shall denote by $P_t(POI_n)$ are respectively given by the following relations: $\overline{W}_\tau(POI_n) = \frac{W_\tau(POI_n)}{|POI_n|}$ and $P_t(POI_n) = \frac{W_\tau(POI_n)}{t}, t > 0$. Where $|POI_n|$ is the order of POI_n and t is the fixed given time in space. We will now use a lemma and some combinatorial results to obtain generalized formulas for each of the relations above.

II. MATERIALS AND METHOD

➢ *Lemma 1.10*
Let $S \subseteq PT_n$, then $W(S) = \sum_{x,y\in n}|x-y|.n_{xy}(S)$.

In this research work we are considering the case where $S = POI_n$, Now let $\tau \in PT_n$ and $x \in n$, we define the work done by τ in moving x as:

$W_x(\tau) = |x-\tau(x)|$ if $x \in dom(\tau)$ and $W_x(\tau) = 0$ if $x \notin dom(\tau)$. The total work done by τ will be given by
$W(\tau) = \sum_{x\in n} W_x(\tau)$.

Now for the subset POI_n of PT_n , we define the total work done by a transformation τ on POI_n by:
$W_\tau(POI_n) = \sum_{\tau\in S} W(\tau)$.

Also for each $x, y \in n$, let $N_{xy}(POI_n) = \{\tau \in POI_n \mid \tau(x) = y\}$ be the set of all elements of POI_n which moves x to y and write $n_{xy}(POI_n) = |N_{xy}(POI_n)|$, which is the cardinality. Note that; $W_x(\tau) = |x - \tau(x)|$ for all $\tau \in N_{xy}(POI_n)$

➢ *Lemma1.11*

Let $x, y \in n$. Then $n_{xy}(POI_n) = \binom{x+y-2}{x-1} \cdot \binom{2n-x-y}{n-x}$, furthermore for $0 \le a$ and $b \le n$. We let $POI_{a,b}$ denote the set of all order-preserving partial maps from a to b, then $|POI_{a,b}|$= $\binom{a+b}{a}$=$\binom{a+b}{b}$.Now if $a = n$ and $b = n$, then we obtain that $|POI_n|$=$\binom{2n}{n}$. Substituting the result of Lemma 2 into Lemma 1 we obtain that the total work done by elements of POI_n is given by

$$W_\tau(POI_n) = \sum_{x,y\in n}|x - y|.\binom{x+y-2}{x-1} \cdot \binom{2n-x-y}{n-x},$$

The average work done and the power of the transformation are respectively given by:

$$\bar{W}_\tau(POI_n) = \frac{\sum_{x,y\in n}|x-y|.\binom{x+y-2}{x-1}\cdot\binom{2n-x-y}{n-x}}{|POI_n}$$ and $$P_t(POI_n) = \frac{\sum_{x,y\in n}|x-y|.\binom{x+y-2}{x-1}\cdot\binom{2n-x-y}{n-x}}{t}, t > 0.$$

It can be shown that
$W_\tau(POI_n) = \sum_{x,y\in n}|x - y|.\binom{x+y-2}{x-1} \cdot \binom{2n-x-y}{n-x}$=$(n-1) \times 2^{2n-3}$ for all n and this help to reduce the computational procedures.

III. III.RESULTS

Since we have obtained a generalized formula for $W_\tau(POI_n), \bar{W}_\tau(POI_n)$ *and* $P_t(POI_n)$ that is true for each value of non-zero and positive $n (n \le 10)$, we now test the formulas for small values of n and a time in space of 5unit (arbitrarily chosen). This is demonstrated below:
When $n = 1$
$W_\tau(POI_1) = (1-1) \times 2^{2-3}$
$W_\tau(POI_1) = 0$, as a consequence of this we have that
$\bar{W}_\tau(POI_1) = 0$ and $P_5(POI_1) = 0$
When $n = 2$
$W_\tau(POI_2) = (2-1) \times 2^{4-3}$
$= 2.00 \times 10^0$

$\bar{W}_\tau(POI_2) = \frac{2.00\times10^0}{|POI_n}$
$= \frac{2.00\times10^0}{\binom{2\times2}{2}}$
$= \frac{2.00\times10^0}{6}$
$\therefore \bar{W}_\tau(POI_2) = 3.33 \times 10^{-1}$
And $P_5(POI_2) = \frac{W_\tau(POI_2)}{5}$
$= 4.00 \times 10^{-1}$

Above is a demonstration of how we have tested the general formulas, note that results are written in scientific notation for consistency, continuing like this for other values of $n = 3,4,5,6,7,8,9,10$, the corresponding solutions are as shown in the tables below:

n	$W_\tau(POI_n)$	$\bar{W}_\tau(POI_n)$
1	0.00×10^0	0.00×10^0
2	2.00×10^0	3.33×10^{-1}
3	1.60×10^1	8.00×10^{-1}
4	9.6×10^1	1.47×10^0
5	5.12×10^2	2.03×10^0
6	2.56×10^3	2.77×10^0
7	1.23×10^4	3.58×10^0
8	5.73×10^4	4.46×10^0
9	2.62×10^5	5.39×10^0
10	1.18×10^6	6.39×10^0

Fig1:- showing the results obtained for the total work done and the average work done by elements of POI_n.

$P_5(POI_1)$	0.00×10^0
$P_5(POI_2)$	4.00×10^{-1}
$P_5(POI_3)$	3.20×10^0
$P_5(POI_4)$	1.92×10^1
$P_5(POI_5)$	1.02×10^2
$P_5(POI_6)$	5.12×10^2
$P_5(POI_7)$	2.46×10^3
$P_5(POI_8)$	1.15×10^4
$P_5(POI_9)$	5.25×10^5
$P_5(POI_{10})$	2.36×10^5

Fig 2:- showing the corresponding powers of POI_n for the given time in space.

The results obtained for the total work done and the average work done is shown pictorially with the aid of graphs as shown in the figures below:

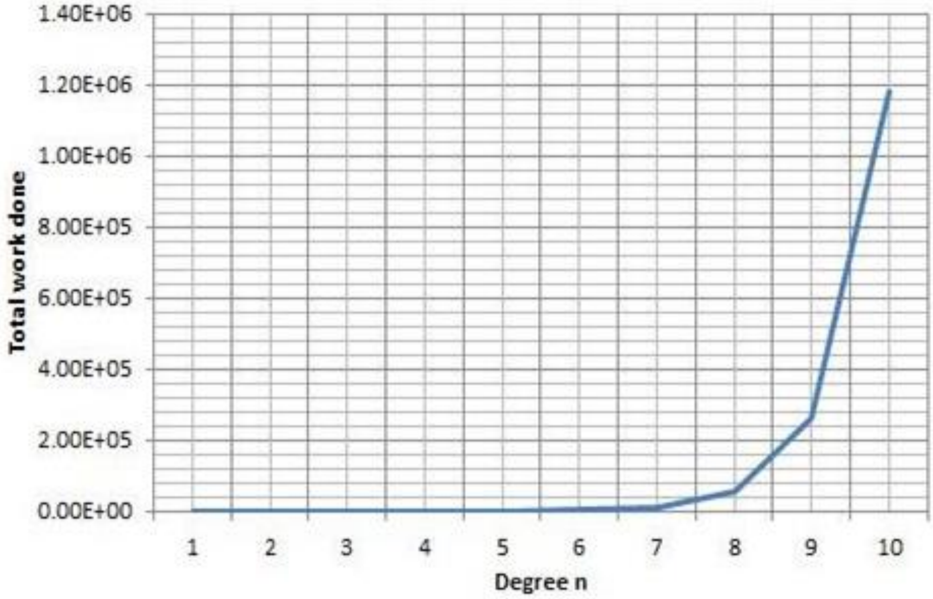

Fig 3:- graph of the total work done on elements of POI_n

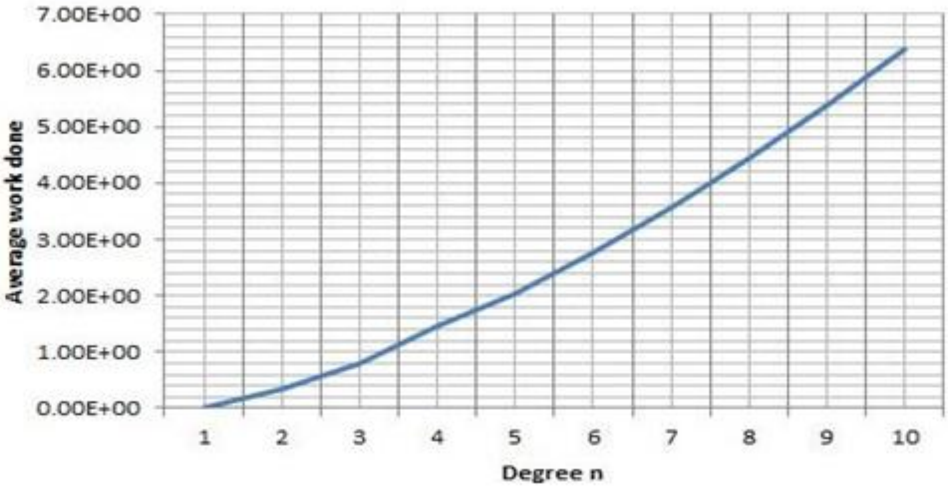

Fig 4:- graph of the average work done on elements of POI_n

IV. IV.CONCLUSION

We have been able to formulate a generalized formula to represent the numerical solutions of the work done on POI_n which is clearly one of the subsets of the finite partial transformation semigroup POI_n.We consequently obtain generalized formulas for the average work done and the power of the transformation, the reliability of these formulas were tested by substituting small values of n and the illustrative graphs shows strong consistency and effectiveness of these formulas. We shall be explicitly looking at the physical significance of the slopes of the graphs obtained in this work in our subsequent research.

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