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Countably Generated Commutative Ring in the Norm Over the Field

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Abstract:- This paper is based on Norms over finitely generated commutative ring over field F.It will give the connection between the graph theory and ring theory under finite norms respectively.

Keywords:- *Semigroup, Norm, Ring, Field.*

I. INTRODUCTION

We introduce the notion of norms over finitely generated commutative ring over field F.Also we give the concept of right and left invariant of the ring over field F.

II. PRELIMINARIES

A. Definition

A group is a set of together with an operation * that combines any two elements a and b from another elements denotes a*b or ab. The set and operation (a,*) must satisfies four requirements

Closure: For all a, b ∈G a * b∈ G
Associative: a, b, c∈G (a*b)*c=a*(b*c)
Identity element: e*a=a*e=a
Inverse element: For each *a* in *G*, there exists an element *b* in *G*, commonly denoted a^{-1}

B. Definition

A semigroup is a set S together with a binary operation F:S×S→S that satisfied associative property for all a,b,c ∈S (a*b)*c = a* (b*c).

C. Definition:

If ‖·‖: G → [0, +∞) is a group-norm if it satisfy the following condition

1.$\| g_1 * g_2 \| \leq \| g_1 \| + \| g_2 \|$
2.$\| g_1 \| > 0$ $with$ $\| g_1 \|=0$ iff g_1=e
3$\| g_1^{-1} \| = \| g_1 \|$ for all $g_1, g_2 \in G$

D. Definition:

If ‖·‖: G →[0,+∞) is said to be an abelian norm if $\| g_1 g_2 \| = \| g_2 g_1 \|$ $for\ all$ $g_1, g_2 \in G$

E. Note:

Let G= (A, e,.) be a finite abelian group. Then ||a||=log ord(a) is a group norm.

F. Definition

If (G,‖·‖, e,*) is said to be right variant it satisfy d:G×G→R by $d_R(g_1 . g_2)= \| g_1^{-1} * g_2 \|$ $for\ all$ $g_1, g_2 \in G$ where d denotes the distance function .

G. Remark:

The product of finite sequence of normed groups is a normed group.

III. MAIN RESULT

A. Definition:

A vector space V with a ring structure and a vector norm such that for all v,w∈V,
1.‖ vw‖ |≤‖v‖‖w‖
2. ‖v *w‖≤‖v‖+‖ w‖
3.If V has an additive identity 0 such that ‖ 0 ‖=0
4.If V has a multiplicative identity 1 such that ‖1‖=1, also‖v‖ ≥ 0 with ‖v‖=0 iff v=e, $\| v^{-1} \|$=‖v‖

B. Note:

The field of real number ℝ is a normed ring with respect to the absolute value.
The field of complex number ℂ is a normed ring with respect to the modulus.

C. Definition:

A ring norm is said to be commutative norm if its satisfy $\|r_1 r_2\|=\|r_1 . r_2\|$ for all $r_1 r_2 \epsilon$R

D. Lemma:

If $\|\cdot\|: R_n \to (-\infty, +\infty)$ by $\| a \|_{max}$= max{a, n-a}, a∈ R_n then $(R_n, \|\cdot\|, e, +, *)$is a normed ring.

➢ ***Proof;***

(i) Let a,b∈ R_n since $\| a + b \|_{max}$= max{a+b, n-(a+b)} then,
max {a+b, n-(a+b)} ≤max {a, n-a} +max {b, n-b}
$\| a + b \|_{max} \leq \| a \|_{max} + \| b \|_{max}$
$\therefore \| a + b \|_{max} \leq \| a \|_{max} + \| b \|_{max}$
$\| ab \|_{max}$= max {ab, n-ab} ≤max {a, n-a}. max {b, n-b}≤||a|| ||b||

(ii) If max {a, n-a} ≥0 for all a∈ R_n and max {a, n-a} =0 if and only if a=e=0

(iii) Let a∈ R_n we have $\| a^{-1} \|_{max} = \| n - a \|_{max}$
$\| n - a \|_{max}$=max {n-a, n-(n-a)}
=max {n-a, a}
$=\| a \|_{max}$

Hence $\| a^{-1} \|_{max}=\| a \|_{max}$
$(R_n, \|\cdot\|, e, +)$ is a normed ring
Obviously,
$(R_n, \|\cdot\|, e, *)$ is a normed ring
$\therefore$ $(R_n, \|\cdot\|, e, +, *)$ is a normed ring.

E. Definition:

If (R,‖·‖,e,*) is said to be right invariant if d:R×R→ ℝ by $d_R(r_1, r_2)=\|r_1 * r_2{}^{-1}\|$ where $r_1, r_2 \in \mathbb{R}$ where d denotes the distance function.

F. Definition:

If (R,‖·‖,e,*) is said to be left invariant if d:R×R→ ℝ by $d_L(r_1, r_2)=\|r_1 * r_2{}^{-1}\|$ where $r_{1,} r_2 \in \mathbb{R}$ where d denotes the distance function.

G. Lemma:

If R= (B,‖·‖, e,.) is right and left invariant with respect to [x,y] and [y,x] then
(i)d_R(xy. yx) =0 iffxy=yx
(ii)d_R(xy, yx) =d_R(yx, xy)
(iii)d_R((xy)a, (yx)a) =d_R(xy, yx) for any aϵℝ
(iv)Let a=[x,y] b=[p,q] c=[s,t] then d(a,c)≤d(a,b)+d(b,c)

➢ ***Proof:***

By 3.4 definition,
$d_R(r_1, r_2)=\| r_1 * r_2{}^{-1} \|$
Let d_R(xy. yx) =0
To prove that xy=yx
d_R(xy. yx) $=\| xy * (yx)^{-1} \|$
$0=\|xy * x^{-1}y^{-1} \|$
$0 =\|xx^{-1}. yx^{-1}. xy^{-1}. yy^{-1} \|$
0 =‖ e‖
e = 0 then
xy=yx
conversely
xy = yx
to prove that
d_R(xy. yx) =0
d_R(xy. yx)= d_R(yx. yx)
$=\| yx * (yx)^{-1} \|$
$=\|yx * x^{-1}y^{-1} \|$
$=\|y (x * x^{-1})y^{-1} \|$
$=\| yy^{-1} \|$
=‖ e‖
=0
(ii) d_R(xy. yx) $=\|xy * (yx)^{-1} \|$
$= \|xy * x^{-1}y^{-1} \|$
$=\|yx * y^{-1}x^{-1} \|$
$=\|yx * (xy)^{-1} \|$
=d (yx.xy)
(iii)d_R((xy)a, (yx)a) $=\|(xy)a*[(yx)a]^{-1}\|$
$=\|(xy)a*(x^{-1}y^{-1})\, a^{-1}\|$
$=\|(xy)a* a^{-1} (x^{-1}y^{-1}) \|$
$=\|(xy) (a* a^{-1}) (x^{-1}y^{-1}) \|$
$=\|(xy). (x^{-1}y^{-1}) \|$
$=\|(xy). (yx)^{-1}\|$
$= d_R$(xy. yx)

iv)let a=[x,y] b=[p,q] c=[s,t]
$d(a,c)= \|a*c^{-1}\|$
$=\| a.e*c^{-1}\|$
$=\|a.(b*b^{-1}) *c^{-1}\|$
$=\|a*b^{-1}. b*c^{-1}\|$
$\leq\|a*b^{-1}\|+\|b*c^{-1}\|$
=d (a.b) +d (b.c)
d(a,c)≤d(a,b)+d(b,c)
Hence proved

➢ ***Remark:***

If R is a commutative normed ring if and only if d_R(xy , yx) =0=d_L(yx , yx) for all x,y∈ ℝ

H. Lemma:

Direct product of two commutative normed ring over F $(R_1, \|\cdot\|_{1}, e_{1}, *)$ and $(R, \|\cdot\|_2, e_2, \bullet)$ is a commutative normed ring over F with respect to a norm define an Rings $(R_1, *)$and $(R_2, \bullet)$respectively.

➢ ***Proof:***

Let $R_1 \times R_2=\{(u_1.u_2) \ / u_1\in R_1, u_2\in R_2.\}$ define a norm on $R_1 \times R_2$ by norm $\|u\|=\|(u_1.u_2)\|=\|u_1\|_1+\|u_2\|_2$, $u=(u_1.u_2) \in R_1 \times R_2$ (by 3.1(2)definition)

i)let u. v$\in (R_1 \times R_1)$ then u= $(u_1.u_2)$, v= $(v_1.v_2)$ with $u_1, v_1 \in R_1 u_2, v_2 \in R_2$
$\|uv\|=\|(u_1.u_2)(v_1.v_2) \|$
$=\|(u_1 * v_1), (u_2.v_2) \|$
$=\| (u_1 * v_1 \|_1 +\| (u_2.v_2) \|_2$
$\leq\|(u_1) \|_1+\|(v_1) \|_1+\|(u_2) \|_2+\|(v_2) \|_2)$
$= (\|(u_1) \|_1+\|(u_2) \|_2) +(\|(v_1) \|_1+\|(v_2) \|_2)$
$=\| (u_1.u_2)\|\|\| (v_1.v_2) \|$
=‖u‖ + ‖v‖
Hence ‖uv‖≤‖u‖+‖v‖ for all u, v∈ $R_1 \times R_2$
Obviously‖ uv‖ ≤‖ u‖‖ v‖
ii) Let u ∈ $R_1 \times R_2$ *then* u= $(u_1.u_2)$ with $u_1 \in R_1 u_2 \in R_2$
Since R_1 *and* R_2 are normed ring over F
We know that ‖u‖≥0 and ‖u‖=0 if and only if a= $(e_1.e_2)$
iii)Let u = $(u_1, u_2)\in R_1\times R_2$, then
$\|u^{-1}\|=\|(u_1^{-1}. u_2^{-1}) \|$
$=\|u_1^{-1}\|_1+\|u_2^{-1}\|_2$
$=\|u_1\|_1+\|u_2\|_2$
$\|u^{-1}\|=\|u\|$

Hence direct product $R_1\times R_2$ is a normed rings over F.
Now we have to prove that if it is finite and commutative.
Let R_1 and R_2 be a normed ring over F and $(R,\|\cdot\|_i, e_i)$where i=1,2,3…. n can be written as
$\prod_{i=1}^{n} R_i = R_1\times R_2\times\ldots. R_1$
$\prod_{i=1}^{n} R_i= \{(u_1, u_2\ldots. U_n)$ where $u_i\in R_i$ for each i}
The algebraic operations $\prod_{i=1}^{n} R_i$ for u,v∈ $\prod_{i=1}^{n} R_i$ is defined component $uv=(u_1,u_2,\ldots u_n)(v_1,v_2,\ldots.v_n)=(u_1v_1,u_2v_2,\ldots u_n,v_n)$ define a norm for the finite product of normed ring over F $\prod_{i=1}^{n} R_i$as follows.
$\|u\| = \|(u_1, u_2\ldots u_n) \| = \|u_1\|_1+\|u_2\|_2+\ldots. \|u_n\|_n$ Where $u_i\in R_i$ for each i
$(R_1, \|\cdot\|, e_i)$ where i=1,2,3…n is also normed ring over F.

I. Analogy:

The product of finite sequence of normed ring (R_1, ‖·‖ ,e_i) is normed ring over F.

J. Theorem:

If R is a Finitely generated commutative ring over F then $R \cong Z^r \times Z_{n_1} \times Z_{n_2} \times \ldots \ldots \times Z_{n_s}$ for some positive integers r such that $n_1 \geq n_2 \geq \ldots . n_s \geq 2$ and $n_{i+1} | n_i$. Also it is unique.

K. Analogy:

Let R be a finitely generated commutative normed ring over F. then there exists norm ‖·‖on R such that (R, ‖·‖) is a normed ring over F.

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