ISSN No:-2456-2165

The Solution of Two Phase Spherical Stefan Problem in Mathematical Modeling of Electrical Contact Phenomena

Stanislav Nikolaevich Kharin
Department of Mathematical and Computer Modeling
Institute of Mathematics and Mathematical Modeling
Kazakh-British Technical University
Almaty, Kazakhstan

Targyn Atanbekovich Nauryz
Department Mathematical and Computer Modeling
Institute of Mathematics and Mathematical Modeling
Al-Farabi Kazakh National University
Kazakh-British Technical University
Satbayev University
Almaty, Kazakhstan

Abstract:- Process of electrical contact heating can be described in spherical model of temperature distribution in electrical contacts introducing by R. Holm. In this research two phase spherical Stefan problem is considered and solution represented in series for integral error function and heat polynomials with undetermined coefficients. All coefficients are founded. Series convergence is proved by condition of constant temperature on the free boundary.

Keywords:- *Stefan Problem, Heat Polynomials, Integral Error Function, Free Boundaries, Faa-Di Bruno.*

I. INTRODUCTION

Heat flux entering into electrical arc with radius smaller than $<10^{-3}$ distributed radially in spherical field and at initial time liquid zone starts boiling at electrical spot and on free boundary liquid zone and solid zone has melting temperature. The free boundary can be calculated by Stefan's condition, you can see more detail in [1], [2]. Such one problem is calculating temperature distribution in liquid and solid zones in modeling electrical contact phenomena [3]. Solution of such like problems can be found by separable variable method based on radius and time. In this research, Integral Error Function series is representing the solution of spherical problem which satisfy the spherical heat equation. To calculate the temperature of zones, two calculation algorithms will be developed. The first of them is based on the integral method of heat balance, where Integral Error function act as basic functions. It will be used in the calculation of heat of zones. The second method, based on the asymptotic approach, will make it possible to analyze the beginning of the process of softening and melting of the contact material. The similar approaches considered in researches [4]-[7]. The mathematical model, which will be used to study the problems of heat and mass transfer with phase transitions, is based on the representation of the solution in the form of linear combinations or series of Integral Error function and heat polynomials that a priori satisfy the differential equation.

II. PROBLEM FORMULATION

The heat distribution in liquid and solid zones can be modeled spherically as follows:

$$\frac{\partial\theta_1}{\partial t}=a_1^2\left(\frac{\partial^2\theta_1}{\partial r^2}+\frac{2}{r}\frac{\partial\theta_1}{\partial r}\right),\quad \alpha(t)<r<\beta(t),\quad t>0,\quad (1)$$

$$\frac{\partial\theta_2}{\partial t}=a_2^2\left(\frac{\partial^2\theta_2}{\partial r^2}+\frac{2}{r}\frac{\partial\theta_2}{\partial r}\right),\quad \beta(t)<r<\infty,\quad t>0,\quad (2)$$

at initial time we have

$$\theta_1\big|_{t=0}=0, \quad (3)$$

$$\theta_2\big|_{t=0}=f(r) \quad (4)$$

subjected to free boundary conditions

$$\theta_1\big|_{r=\alpha(t)}=\theta_m, \quad (5)$$

$$-\lambda_1\frac{\partial\theta_1}{\partial r}\bigg|_{r=\alpha(t)}=Q, \quad (6)$$

where $Q=\dfrac{P(t)}{2\pi\alpha^2(t)}$ and

$$\theta_1\big|_{r=\beta(t)}=\theta_2\big|_{r=\beta(t)}=\theta_m, \quad (7)$$

the Stefan's condition

$$-\lambda_1\frac{\partial\theta_1}{\partial r}\bigg|_{r=\beta(t)}=-\lambda_2\frac{\partial\theta_2}{\partial r}\bigg|_{r=\beta(t)}+L\gamma\frac{d\beta}{dt}, \quad (8)$$

at infinity condition we get

$$\theta_2\big|_{r=\infty}=0 \quad (9)$$

Where $P(t)$ – a function of heat flux and θ_m is a melting temperature of electric contact material. The heat spread in the solid is negligible because of the physical properties of contact material. This condition is valid for refractory metals like wolfram.

ISSN No:-2456-2165

III. PROBLEM SOLUTION

To solve this problem, we use $\theta = \frac{u}{r} + \theta_m$ and $r = x$ to reduce linear simple heat equation problem (1)-(9) as the following

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial r^2}, \quad \alpha(t) < r < \beta(t), \quad t > 0, \qquad (10)$$

$$\frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2}, \qquad \beta(t) < x < \infty, \; t > 0, \qquad (11)$$

$$x = \alpha(t): \qquad u_1(\alpha(t), t) = U_m, \qquad (12)$$

$$-\lambda_1 \left[\alpha(t) \frac{\partial u_1}{\partial x} - u_1 \right]_{x=\alpha(t)} = \frac{P(t)}{2\pi}, \qquad (13)$$

$$x = \beta(t): \qquad u_1(\beta(t), t) = u_2(\beta(t), t) = 0, \qquad (14)$$

the Stefan's condition

$$-\lambda_1 \left[\beta(t) \frac{\partial u_1}{\partial x} - u_1 \right]_{x=\beta(t)} = -\lambda_2 \left[\beta(t) \frac{\partial u_2}{\partial x} - u_2 \right]_{x=\beta(t)} + \beta^2(t) L\gamma \frac{d\beta}{dt}, \qquad (15)$$

and at infinity

$$u_2(\infty, t) = 0, \qquad (16)$$

at initial condition

$$t = 0: \quad u_1(0, 0) = 0, \qquad (17)$$

$$u_2(x, 0) = f(x). \qquad (18)$$

Let analytic functions at initial time and at boundary condition can be written as expanded form of Maclaurin series

$$P(t) = \sum_{n=0}^{\infty} \frac{P^{(n)}(0)}{n!} t^{\frac{n}{2}}, \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (19)$$

and

$$\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^{\frac{n}{2}}, \quad \beta(t) = \sum_{n=0}^{\infty} \beta_n t^{\frac{n}{2}}.$$

The solution of (10)-(18) can introduced as the following form

$$u_1(x, t) = \sum_{n=0}^{\infty} A_{2n} (2a_1\sqrt{t})^{2n} \left[i^{2n} erfc \frac{-x}{2a_1\sqrt{t}} + i^{2n} erfc \frac{x}{2a_1\sqrt{t}} \right] + \sum_{n=0}^{\infty} A_{2n+1} (2a_1\sqrt{t})^{2n+1} \left[i^{2n+1} erfc \frac{-x}{2a_1\sqrt{t}} - i^{2n+1} erfc \frac{x}{2a_1\sqrt{t}} \right] \qquad (20)$$

$$u_2(x, t) = \sum_{n=0}^{\infty} (2a_2\sqrt{t})^n \left[B_n i^n erfc \frac{-x}{2a_2\sqrt{t}} + C_n i^n erfc \frac{x}{2a_2\sqrt{t}} \right] \qquad (21)$$

Where A_{2n}, A_{2n+1}, B_n and C_n coefficients must be founded. Using Hermit's Polynomials, we represent (20) in the form of Heat polynomials:

$$u_1(x, t) = \sum_{n=0}^{\infty} A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m B_{2n,m} + \sum_{n=0}^{\infty} A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m B_{2n+1,m} \qquad (22)$$

From condition (12) we get expression by making substitution $\sqrt{t} = \tau$

$$\sum_{n=0}^{\infty} A_{2n} \sum_{m=0}^{n} [\alpha(\tau)]^{2n-2m} t^m B_{2n,m} + \sum_{n=0}^{\infty} A_{2n+1} \sum_{m=0}^{n} [\alpha(\tau)]^{2n-2m+1} t^m B_{2n+1,m} = U_m$$

By using multinomial coefficients of Newton's Polynomial we have

$$\sum_{n=0}^{\infty} A_{2n} \sum_{s_1+s_2+...+s_k=2n-2m}^{n} \binom{2n-2m}{s_1, s_2, ...s_k} \alpha_1^{s_1}\alpha_2^{s_2}...\alpha_k^{s_k} \tau^{2(s_1+2s_2+...+ks_k+m)} + \sum_{n=0}^{\infty} A_{2n+1} \sum_{s_1+s_2+...+s_k=2n-2m+1}^{n} \binom{2n-2m+1}{s_1, s_2, ...s_k} \alpha_1^{s_1}\alpha_2^{s_2}...\alpha_k^{s_k} \tau^{2(s_1+2s_2+...+ks_k+m)} = U_m \qquad (23)$$

To get recurrent formula for finding A_{2n+1}, we take both sides of (23), 2k-times derivatives by taking as $\tau = 0$, by using multinomial coefficients we can obtain the following expressions.

$$\sum_{n=1}^{l} A_{2n} \sum_{m=0}^{n-1} C_{2n,m}[4l]\beta_{2n,m} + \sum_{n=l+1}^{2l-1} A_{2n} \sum_{m=0}^{2l-n-1} C_{2n,m+2(n-l)}[4l]\beta_{2n,m+2(n-l)} + A_{4l}\beta_{4l,2l}[4l] + \sum_{n=1}^{l} A_{2n+1} \sum_{m=0}^{n-1} C_{2n+1,m}[4l]\beta_{2n+1,m} + \sum_{n=l+1}^{2l} A_{2n+1} \sum_{m=0}^{2l-n} C_{2n+1,m+2(n-l)-1}[4l]\beta_{2n+1,m+2(n-l)-1} = 0 \qquad (24)$$

if $k = 2l$ and for $k = 2l+1$

$$\sum_{n=1}^{l+1} A_{2n} \sum_{m=0}^{n-1} C_{2n,m}[2(2l+1)]\beta_{2n,m} + \sum_{n=l+1}^{2l+1} A_{2n} \sum_{m=0}^{2l-n+1} C_{2n,m+2(n-l)}[2(2l+1)]\beta_{2n,m+2(n-l)} + A_{2(2l+1)}\beta_{2(2l+1),2l+1}[2(2l+1)] + \sum_{n=1}^{l} A_{2n+1} \sum_{m=0}^{n-1} C_{2n+1,m}[2(2l+1)]\beta_{2n+1,m} + \sum_{n=l+1}^{2l} A_{2n+1} \sum_{m=0}^{2l-n} C_{2n+1,m+2(n-l)-1}[2(2l+1)]\beta_{2n+1,m+2(n-l)-1} = 0 \qquad (25)$$

Thus A_{2n+1} coefficient are found explicitly and can be expressed from (24) and (25) where $C_{i,j}[4l]$ or $C_{i,j}[4l+2]$ are multinomial coefficients.

From condition (13) with making substitution $\sqrt{t} = \tau$

$$-\lambda_1 \left[\alpha(\tau) \left\{ \sum_{n=0}^{\infty} \left[A_{2n} \sum_{m=0}^{n} (2n-2m) \sum_{s_1+s_2+...+s_k=2n-2m-1} \binom{2n-2m-1}{s_1, s_2, ..., s_k} \alpha_1^{s_1}\alpha_2^{s_2}...\alpha_k^{s_k} \tau^{2(s_1+2s_2+...+ks_k+m)} \beta_{2n,m} \right.\right.\right.$$

$$+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+...+p_k=2n-2m}\binom{2n-2m}{p_1,p_2,...,p_k}\alpha_1^{p_1}\alpha_2^{p_2}...\alpha_k^{p_k}\tau^{2(p_1+2p_2+...+kp_k+m)}\beta_{2n+1,m}\Big]\Big\}$$

$$-\left\{\sum_{n=0}^{\infty}A_{2n}\sum_{s_1+s_2+...+s_k=2n-2m}^{n}\binom{2n-2m}{s_1,s_2,...s_k}\alpha_1^{s_1}\alpha_2^{s_2}...\alpha_k^{s_k}\tau^{2(s_1+2s_2+...+ks_k+m)}\beta_{2n,m}\right.$$

(26)

$$\left.+\sum_{n=0}^{\infty}A_{2n+1}\sum_{s_1+s_2+...+s_k=2n-2m+1}^{n}\binom{2n-2m+1}{s_1,s_2,...s_k}\alpha_1^{s_1}\alpha_2^{s_2}...\alpha_k^{s_k}\tau^{2(s_1+2s_2+...+ks_k+m)}\beta_{2n+1,m}\right\}\Big]$$

$$=\sum_{n=0}^{\infty}\frac{P^{(n)}(0)}{2\pi n!}\tau^{2n}$$

If we substitute expression (23) into (26) then we get

$$-\lambda_1\left[\alpha(\tau)\left\{\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)\sum_{s_1+s_2+...+s_k=2n-2m-1}\binom{2n-2m-1}{s_1,s_2,...,s_k}\alpha_1^{s_1}\alpha_2^{s_2}...\alpha_k^{s_k}\tau^{2(s_1+2s_2+...+ks_k+m)}\beta_{2n,m}\right.\right.\right.$$

$$+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+...+p_k=2n-2m}\binom{2n-2m}{p_1,p_2,...,p_k}\alpha_1^{p_1}\alpha_2^{p_2}...\alpha_k^{p_k}\tau^{2(p_1+2p_2+...+kp_k+m)}\beta_{2n+1,m}\Big]\Big\}$$

$$-U_m\Big]=\sum_{n=0}^{\infty}\frac{P^{(n)}(0)}{2\pi n!}\tau^{2n}$$

(27)

Since $P(t)$ function is analytic and can be expanded into Maclaurin series we can easily derive recurrent formula for A_{2n} by taking both sides of expression (27) 2k and 2k+1 times derivatives and equate coefficients of both sides

$$\frac{P^{(2k)}(0)}{2\pi\lambda_1(2k)!\alpha_{2k}}=-\left(\sum_{n=1}^{k}A_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k]\cdot(2n-2m)\cdot\beta_{2n,m}\right)$$
$$-\left(\sum_{n=k+1}^{2k}A_{2n}\sum_{m=0}^{2k-n}c_{2n,m+2(n-k)-1}[2k]\cdot2(2k-n-m+1)\cdot\beta_{2n,m+2(n-k)-1}\right)$$
$$-\left(\sum_{n=1}^{k}A_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k]\cdot(2n+1-2m)\cdot\beta_{2n+1,m}\right)+$$
$$-\left(\sum_{n=k+1}^{2k-1}A_{2n+1}\sum_{m=0}^{2k-n-1}c_{2n+1,m+2(n-k)}[2k]\cdot(2(2k-n-m)+1)\cdot\beta_{2n+1,m+2(n-k)}\right)$$
$$-A_{4k+1}\cdot c_{4k+1,2k}[2k]\cdot\beta_{4k+1,2k}$$

(28)

$$\frac{P^{(2k+1)}(0)}{2\pi(2k+1)!\alpha_{2k+1}}=-\left(\sum_{n=1}^{k+1}A_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k+1]\cdot(2n-2m)\cdot\beta_{2n,m}\right)$$
$$-\left(\sum_{n=k+2}^{2k+1}A_{2n}\sum_{m=0}^{2k-n+1}c_{2n,m+2(n-k-1)}[2k+1]\cdot2(2k-n-m+2)\cdot\beta_{2n,m+2(n-k-1)}\right)$$
$$-\left(\sum_{n=1}^{k}A_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k+1]\cdot(2n+1-2m)\cdot\beta_{2n+1,m}\right)+$$
$$-\left(\sum_{n=k+1}^{2k}A_{2n+1}\sum_{m=0}^{2k-n}c_{2n+1,m+2(n-k)-1}[2k+1]\cdot(2(2k-n-m)+3)\cdot\beta_{2n+1,m+2(n-k)-1}\right)$$
$$-A_{4k+3}\cdot c_{4k+3,2k+1}[2k+1]\cdot\beta_{4k+3,2k+1}$$

(29)

$$A_{4k}=\left\{\frac{P^{(2k)}(0)}{2\pi(2k)!\alpha_{2k}}+\left(\sum_{n=1}^{k}A_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k]\cdot(2n-2m)\cdot\beta_{2n,m}\right)\right.$$
$$+\left(\sum_{n=k+1}^{2k-1}A_{2n}\sum_{m=0}^{2k-n}c_{2n,m+2(n-k)-1}[2k]\cdot2(2k-n-m+1)\cdot\beta_{2n,m+2(n-k)-1}\right)$$
$$+\left(\sum_{n=1}^{k}A_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k]\cdot(2n+1-2m)\cdot\beta_{2n+1,m}\right)-$$
$$+\left(\sum_{n=k+1}^{2k-1}A_{2n+1}\sum_{m=0}^{2k-n-1}c_{2n+1,m+2(n-k)}[2k]\cdot(2(2k-n-m)+1)\cdot\beta_{2n+1,m+2(n-k)}\right)$$
$$\left.+A_{4k+1}\cdot c_{4k+1,2k}[2k]\cdot\beta_{4k+1,2k}\right\}/2c_{4k,2k-1}[2k]\cdot\beta_{4k,2k-1}$$

(30)

$$A_{4k+2}=\left\{\frac{P^{(2k+1)}(0)}{2\pi(2k+1)!\alpha_{2k+1}}+\left(\sum_{n=1}^{k+1}A_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2k+1]\cdot(2n-2m)\cdot\beta_{2n,m}\right)\right.$$
$$+\left(\sum_{n=k+2}^{2k}A_{2n}\sum_{m=0}^{2k-n+1}c_{2n,m+2(n-k-1)}[2k+1]\cdot2(2k-n-m+2)\cdot\beta_{2n,m+2(n-k-1)}\right)$$
$$+\left(\sum_{n=1}^{k}A_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2k+1]\cdot(2n+1-2m)\cdot\beta_{2n+1,m}\right)-$$
$$+\left(\sum_{n=k+1}^{2k}A_{2n+1}\sum_{m=0}^{2k-n}c_{2n+1,m+2(n-k)-1}[2k+1]\cdot(2(2k-n-m)+3)\cdot\beta_{2n+1,m+2(n-k)-1}\right)$$
$$\left.+A_{4k+3}\cdot c_{4k+3,2k+1}[2k+1]\cdot\beta_{4k+3,2k+1}\right\}/2c_{4k+2,2k}[2k+1]\cdot\beta_{4k+2,2k}$$

(31)

And from condition (14) for $x=\beta(t)$ and $u_1(x,t)$ we can determine the coefficients of $\alpha(\tau)$ in non-linear equation.

By using L'Hopital's rule for (14) we can introduce the property of Integral Error Function

$$\lim_{x\to\infty}\frac{i^n erfc(-x)}{x^n}=\frac{2}{n!}$$

yields

$$\lim_{t\to0}(2a\sqrt{t})i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right)=\frac{2}{n!}x^n$$

Then we get

$$B_n=\frac{1}{2}f^{(n)}(0)$$

To calculate coefficient C_n, at first, we use Leibniz rule. Then we have

$$\left.\frac{\partial^k[2^{n/2}\tau^n i^n erfc\beta]}{\partial\tau^k}\right|_{\tau=0}=\frac{2^{n/2}k!}{(k-n)!}[i^n erfc\beta]^{(k-n)} \qquad (32)$$

Faa-di Bruno's formula and Bell polynomials for a derivative of a composite function gives us the following form

ISSN No:-2456-2165

$$\left.\frac{\partial^{(k-n)}[i^n erfc(\pm\beta)]}{\partial\tau^{k-n}}\right|_{\tau=0} = \sum_{m=1}^{k-n}(i^n erfc(\pm\beta))^{(m)}\Big|_{\beta=0}$$
$$B_{k-n,m}(\beta'(\tau), \beta''(\tau), ..., \beta^{(k-n-m+1)}(\tau))|_{\tau=0}$$
(33)

where $B_{k-n,m} = \sum \frac{(k-n)!}{j_1!\, j_2! ... j_{k-n-m+1}!}\cdot\beta_1^{j_1}\beta_2^{j_2}\beta_3^{j_3}...\beta_{k-n-m+1}^{j_{k-n-m+1}}$;

and $j_1, j_2, ...$ satisfy the following equations

$$\begin{cases} j_1 + j_2 + ... + j_{k-n-m+1} = m \\ j_1 + 2j_2 + ... + (k-n-m+1)j_{k-n-m+1} = k-n \end{cases}$$

and for $m \geq n$ we have

$$[i^n erfc(\pm\beta)]^{(m)}|_{\beta=0} = (-1)^m i^{n-m} erfc 0 = (\mp1)^m \frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m!)\sqrt{\pi}}$$

(34)

From (14) for $x = \beta(t)$ and $u_2(x, t)$ we have

$$\sum_{n=0}^{k}\frac{f^{(n)}(0)}{2}\cdot(2)^{1/2}\frac{k!}{(k-n)!}\cdot\sum_{m=1}^{k-n}(-1)^m\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{\pi}}\cdot$$
$$\cdot\sum\frac{(k-n)!}{j_1!\, j_2! ... j_{k-n-m+1}!}\beta_1^{j_1}\beta_2^{j_2}...\beta_{k-n-m+1}^{j_{k-n-m+1}} + \sum_{n=0}^{k}C_n\cdot(2)^{1/2}\frac{k!}{(k-n)!}\cdot\sum_{m=1}^{k-n}(-1)^m\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{\pi}}\cdot$$
$$\cdot\sum\frac{(k-n)!}{j_1!\, j_2! ... j_{k-n-m+1}!}\beta_1^{j_1}\beta_2^{j_2}...\beta_{k-n-m+1}^{j_{k-n-m+1}} = 0$$
(35)

From this expression we get coefficient

$$C_n = -\frac{f^{(n)}(0)(2)^{1/2}\frac{k!}{(k-n)!}\cdot\sum_{m=1}^{k-n}(-1)^m\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{\pi}}\cdot\sum\frac{(k-n)!}{j_1!\, j_2!...j_{k-n-m+1}!}\beta_1^{j_1}\beta_2^{j_2}...\beta_{k-n-m+1}^{j_{k-n-m+1}}}{(2)^{1/2+1}\frac{k!}{(k-n)!}\cdot\sum_{m=1}^{k-n}(-1)^m\frac{\Gamma\left(\frac{n-m+1}{2}\right)}{(n-m)!\sqrt{\pi}}\cdot\sum\frac{(k-n)!}{j_1!\, j_2!...j_{k-n-m+1}!}\beta_1^{j_1}\beta_2^{j_2}...\beta_{k-n-m+1}^{j_{k-n-m+1}}}$$

where

Let $a_1 = a_2 = 1$ (it is always possible by substitution $\tau = a^2 t$)
To calculate coefficient of $\beta(\tau)$, we use Stefan's condition (15), making substitution $\sqrt{t} = \tau$ and using condition (14) and above expression (15) we reduce to following

$$\lambda_2\frac{1}{\beta(\tau)}\cdot\frac{\partial u_2}{\partial x}\Big|_{x=\beta(\tau)} - \lambda_1\frac{1}{\beta(\tau)}\cdot\frac{\partial u_1}{\partial x}\Big|_{x=\beta(\tau)} = \beta(\tau)\cdot L\gamma$$
(36)

By making both sides of (36) k-times derivatives at $\tau = 0$ we have

$$\lambda_2\left[\frac{\partial^k[u_{2x}(\beta(\tau),\tau)\beta^{-1}(\tau)]}{\partial\tau^k}\right]\Bigg|_{\tau=0} - \lambda_1\left[\frac{\partial^k[u_{1x}(\beta(\tau),\tau)\beta^{-1}(\tau)]}{\partial\tau^k}\right]\Bigg|_{\tau=0} = \beta_k L\gamma \quad (37)$$

where
$u_{ix}(\beta(\tau), \tau) = \frac{\partial u_{ix}}{\partial x}\Big|_{x=\beta(\tau)}$ and $i = 1, 2$

Finally we have following recurrent formula from (15)

$$\beta_k = \frac{\lambda_2}{L\gamma}\sum_{p=0}^{k}\binom{k}{p}\Bigg\{\sum_{n=0}^{\infty}\frac{2^{n/2}(k-p)!}{(k-p-n)!}\Big[(-1)^n B_n\cdot$$
$$\cdot\left(\sum_{m=1}^{k-p-n} i^{n-m} erfc\beta_1\sum\frac{(k-p)!\,\beta_1^{j_1}\beta_2^{j_2}...\beta_{k-p-n-m+1}^{j_{k-p-n-m+1}}}{j_1!\, j_2!...j_{k-p-n-m+1}!}\right)+$$
$$+C_n\sum_{m=1}^{k-p-n} i^{n-m} erfc(-\beta_1)\sum\frac{(k-p)!\,\beta_1^{j_1}\beta_2^{j_2}...\beta_{k-p-n-m+1}^{j_{k-p-n-m+1}}}{j_1!\, j_2!...j_{k-p-n-m+1}!}\Big]\Bigg\}$$
(38)
$$\left\{\sum_{m=1}^{k-p-n}(-1)^m\frac{1}{\beta^{m+1}}\sum\frac{n!\,\beta_1^{j_1}\beta_2^{j_2}...\beta_{n-m+1}^{j_{n-m+1}}}{j_1!\, j_2!...j_{n-m+1}!}\right\} -$$
$$\frac{\lambda_1}{L\gamma}\left[\frac{\partial^k[u_{1x}(\beta(\tau),\tau)]\beta^{-1}(\tau)}{\partial\tau^k}\right]\Bigg|_{\tau=0}$$
where

$$\left[\frac{\partial^k[u_{1x}(\beta(\tau),\tau)]\beta^{-1}(\tau)}{\partial\tau^k}\right] = \left(\sum_{n=1}^{l}A_{2n}\sum_{m=0}^{n-1}c_{2n,m}[2l]\cdot(2n-2m)\cdot\beta_{2n,m}\right)$$
$$+\left(\sum_{n=l+1}^{2l}A_{2n}\sum_{m=0}^{2k-n}c_{2n,m+2(n-l)-1}[2l]\cdot 2(2l-n-m+1)\cdot\beta_{2n,m+2(n-l)-1}\right)$$
$$+\left(\sum_{n=1}^{l}A_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}[2l]\cdot(2n+1-2m)\cdot\beta_{2n+1,m}\right)+$$
$$+\left(\sum_{n=l+1}^{2l-1}A_{2n+1}\sum_{m=0}^{2l-n-1}c_{2n+1,m+2(n-l)}[2l]\cdot(2(2k-n-m)+1)\cdot\beta_{2n+1,m+2(n-l)}\right)$$
$$+A_{4l+1}\cdot c_{4l+1,2l}[2l]\cdot\beta_{4l+1,2l}$$

if $k = 2l$ and for $k = 2l+1$

$$\left[\frac{\partial^k[u_{1x}(\beta(\tau),\tau)]\beta^{-1}(\tau)}{\partial\tau^k}\right]=\left(\sum_{n=1}^{l+1}A_{2n}\sum_{m=0}^{n-1}c_{2n,m}\left[2l+1\right]\cdot\left(2n-2m\right)\cdot\beta_{2n,m}\right)$$
$$+\left(\sum_{n=l+2}^{2l+1}A_{2n}\sum_{m=0}^{2l-n+1}c_{2n,m+2(n-l-1)}\left[2l+1\right]\cdot2\left(2l-n-m+2\right)\cdot\beta_{2n,m+2(n-l-1)}\right)$$
$$+\left(\sum_{n=1}^{l}A_{2n+1}\sum_{m=0}^{n-1}c_{2n+1,m}\left[2l+1\right]\cdot\left(2n+1-2m\right)\cdot\beta_{2n+1,m}\right)$$
$$+\left(\sum_{n=l+1}^{2l}A_{2n+1}\sum_{m=0}^{2l-n}c_{2n+1,m+2(n-l)-1}\left[2l+1\right]\cdot\left(2\left(2l-n-m\right)+3\right)\cdot\beta_{2n+1,m+2(n-l)-1}\right)$$
$$+A_{4l+3}\cdot c_{4l+3,2l+1}\left[2l+1\right]\cdot\beta_{4l+3,2l+1}$$

Finally, by making summary we can see that coefficients A_{2n}, A_{2n+1}, C_n and coefficients of free boundaries $\alpha(t)$, $\beta(t)$ can be described from (12)-(15).

IV. CONVERGENCE OF SERIES

Convergence of (20) and (21) can be proved as following:

Let $\beta(t_0)=\mu_0$ for any time $t=t_0$. Then the series

$$U_1(x,t)=\sum_{n=0}^{\infty}A_{2n}(2t)^n\left[i^{2n}erfc\frac{-\mu_0}{2\sqrt{t}}+i^{2n}erfc\frac{\mu_0}{2\sqrt{t}}\right]$$
$$+\sum_{n=0}^{\infty}A_{2n+1}\left(2t\right)^{(2n+1)/2}\left[i^{2n+1}erfc\frac{-\mu_0}{2\sqrt{t}}-i^{2n}erfc\frac{\mu_0}{2\sqrt{t}}\right],$$
$$U_2(x,t)=\sum_{n=1}^{\infty}B_{2n}(2t)^{n/2}\left[i^{n}erfc\frac{-\mu_0}{2\sqrt{t}}\right]+\sum_{n=0}^{\infty}C_n\left(2a_2t\right)^{n/2}\left[i^{n}erfc\frac{\mu_0}{2\sqrt{t}}\right],$$

should be convergent, because $u_1=u_2=0$ on the interface. Therefore there exist some constants C_1, C_2, independent of n, such that

$$|A_{2n}|<C_1/(2t_0)^n\left[i^{2n}erfc\frac{-\mu_0}{2\sqrt{t_0}}+i^{2n}erfc\frac{\mu_0}{2\sqrt{t_0}}\right]. \qquad (39)$$

Since A_{2n} are bounded and expressed in terms of A_{2n+1}, then A_{2n+1} is also bounded. Multiplying both sides of (39) by $(2t)^n\left[i^{2n}erfc\frac{(-\beta(t))}{2\sqrt{t}}+i^{2n}erfc\frac{\beta(t)}{2\sqrt{t}}\right]$ and taking sum we obtain

$$\sum_{n=0}^{\infty}A_{2n}(2t)^n\left[i^{2n}erfc\frac{(-\beta(t))}{2\sqrt{t}}+i^{2n}erfc\frac{\beta(t)}{2\sqrt{t}}\right]<$$
$$C_1\sum_{n=0}^{\infty}\frac{(2t)^n\left[i^{2n}erfc\frac{(-\beta(t))}{2\sqrt{t}}+i^{2n}erfc\frac{\beta(t)}{2\sqrt{t}}\right]}{(2t_0)^n\left[i^{2n}erfc\frac{-\mu_0}{2\sqrt{t_0}}+i^{2n}erfc\frac{\mu_0}{2\sqrt{t_0}}\right]}<C_1\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n.$$

In the same manner

$$|C_n|<C_2/(2t_0)^n i^{2n}erfc\frac{\mu_0}{2\sqrt{t_0}},$$

$$\sum_{n=0}^{\infty}C_n(2t)^n\left[i^{2n}erfc\frac{\beta(t)}{2\sqrt{t}}\right]<$$
$$C_2\sum_{n=0}^{\infty}\frac{(2t)^n\left[i^{2n}erfc\frac{\beta(t)}{2\sqrt{t}}\right]}{(2t_0)^n\left[i^{2n}erfc\frac{\mu_0}{2\sqrt{t_0}}\right]}<C_2\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n.$$

These are geometric series and the series for $u_1(x,t)$ converges for all $x<\mu_0$, while the series for $u_2(x,t)$ converges for all $x>\mu_0$ and $t<t_0$. The series $\beta(t)$ can be estimated similarly from the equation (38).

ACKNOWLEDGMENT

The research work is supported by the grant project AP05133919 (2018-2020) from the Ministry of Science and Education of the Republic of Kazakhstan.

REFERENCES

[1]. A Fasano, M. Primicerio, “Free boundary problems: theory and applications”, Research Notes in Mathematics, vol. 78, (pt I), vol. 79, (pt II) Boston: Pitman, 1983.

[2]. J. Stefan, “Uber die Theorie der Eisbildung”, Monatshefte Mat. Phys., 1: pp.1-6, 1890.

[3]. S. N. Kharin, H. Nouri, T. Davies, “The mathematical model of welding dynamics in closed and switching electrical contacts”, Proc, 49th IEEE Holm Conf. on Electrical Contacts, Washington, USA, pp. 107-123, 2003.

[4]. S. N. Kharin, “The analytical solution of the two-phase Stefan problem with boundary flux condition”, Mathematical journal, vol. 14, № 1 (51), pp. 55-75, 2014.

[5]. Alexey A. Kavokin, Targyn A. Nauryz, Nazerke T. Bizhigitova, “Exact solution of two phase spherical Stefan problem with free boundaries”, AIP Conference Proceedings 1759, 020117, 2016. https://doi.org/10.1063/1.495931.

[6]. M. M. Sarsengeldin, A. S. Erdogan, T.A. Nauryz, H. Nouri, “An approach for solving an inverse spherical Stefan problem arising in modeling of electrical contact phenomena ”, Mathematical Methods in the Applied Sciences, vol. 41, issue 2, 2017. https://doi.org/10.1007/978-3-319-67053-937.

[7]. M. Sarsengeldin, S. N. Kharin, “Method of the Integral Error Functions for the solution of one- and two-phase Stefan problems and its application”, Filomat 31:4, 1017-1029. 2017. DOI 10.2298/FIL1704017S. Available at: http://www.pmf.ni.ac.rs/filomat.