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Analytical Modeling of an Elastic Metamaterial System with Simultaneously Negative Modulus and Mass

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Abstract:- Interest in metamaterials arises because they produce novel properties like negative mass and modulus which cannot be realized in conventional materials. As the field of metamaterials develops, and research efforts are directed towards fabricating and realizing them physically, it is necessary to exhaustively understand their dynamics. In this study, the detailed analytical modeling of the design of a chiral mass-spring metamaterial system is presented. The conditions and frequency interval(s) where the effective dynamic parameters are negative are clearly established. Simulation result indicates that when both the translational and rotational resonance frequencies of the material coincide, a perfectly single negative parameter metamaterial is realized, which alternate from purely (single) negative modulus to purely (single) negative mass material at the point of resonance. It is shown that the frequency band in which the system exhibit simultaneously double negative (modulus and mass) behavior can be widened by reducing the stiffness of the horizontal spring, and maximum when the stiffness is zero. Hence, the horizontal spring is apparently a redundant member and may not necessarily be needed when fabricating the chiral system in three-dimension.

Keywords:- *Double Negativity, Effective Bulk Modulus, Effective Mass/Mass Density, Metamaterial.*

I. INTRODUCTION

Naturally occurring materials possess only positive properties such as mass, bulk modulus, electrical permittivity and magnetic permeability. This is fundamentally necessary to maintain their innate constitution and structural stability. In static load applications for example, typical structural materials have positive mass and bulk modulus. In contrast, under dynamic loading at a certain frequency or range of frequencies, apparent negative mass and negative modulus can be observed in some properly structured man-made materials. Such engineered structures with the capacity for negative constitutive parameters are termed metamaterials. The emerging field of metamaterials leads to the design of structures which possess these anomalous but attractive characteristics for application in areas such as elimination of noise, acoustic cloaking for waves and reduction of vibration.

The concept of a metamaterial originated from the field of electromagnetics in 1967, after Vaselego theoretically investigated a material with potentially negative permittivity ε and negative permeability μ [1]. Originally referred to as left-handed material (LHM), it was found to respond to electromagnetic wave propagation with both negative permittivity and permeability [2, 3]. When the permittivity ε and permeability μ of a material are both positive, the refractive index n ($n = \pm\sqrt{\varepsilon * \mu}$) is positive real; hence, refraction of wave through it is conventionally in agreement with Snell's law. If either of the two parameters has a negative sign, a complex refractive index results, and the medium will partially inhibit the propagation of incident waves through it. However, for a LHM the refractive index is negative real, and the phase and group velocities are anti-parallel to each other. Incident waves from a positive-index medium to such a material would result in refracted waves which, though is in accordance with Snell's law, but would lie on the same side of the boundary normal as the incident wave.

LHMs are fabricated by combining a material that has the potential to provide negative permittivity with another that can provide negative permeability, usually at low resonant frequencies [3]. Since Vaselego's discovery, extensive research attention is being focused on metamaterials. The concept has been extended to the closely-related fields of mechanical metamaterials where propagation of acoustic and elastic waves are of interest.

By coating heavy spheres with soft silicone and embedding them in epoxy, resulting in apparent negative elastic constant at certain loading frequencies, the general concept of elastic metamaterial with local mechanical resonance was introduced [4]. Subsequently, a good number of elastic/acoustic (EA) metamaterial that exhibit negative modulus has been reported in literature. Fang *et. al.* [5] experimentally demonstrated a one-dimensional (1D) ultrasonic metamaterial with negative bulk modulus (NBM) consisting of a periodic array of shunted sub-wavelength Helmholtz resonators connected to the fluid transmission channel. By coupling a number of the Helmholtz resonators sideways and face-to-face, Cheng *et. al.* [6] recorded a broadening of the sound-forbidding negative frequency band earlier reported in [5]. Ding and Zhao [7] showed, both experimentally and in simulation, that multiple and broad

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negative frequency bands were developed in an acoustic transmission system comprising of three layers each of seven units of split-hollow-spheres arranged hexagonally and immersed in a sponge matrix, due to NBM developed in the system.

Analytical models of three component phononic crystals of coated spheres and cylinders embedded in a host matrix were reported to exhibit negative mass densities (NMDs) near local resonances [8]. A composite of solid objects dispersed in a fluid medium was observed to exhibit NMD behavior both numerically and experimentally [9]. A design of negative (mass) density metamaterial whose unit cell combines two cylindrical sub-structures and a flexible membrane that act as an acoustic resonator, was reported to show complete opacity to sound waves at low frequencies [10]. A one-dimensional mass-spring structure with a smaller mass inside a bigger mass was observed to develop negative effective mass near the local resonant frequency of the internal mass [11]. It is now well established that negative mass/mass density can be realized in structured materials by including well-designed local mechanical resonators in their units.

When both NBM and NMD are combined in a material, a metamaterial with double negative (DN) parameters can be realized. Li and Chan [12] theoretically demonstrated an acoustic metamaterial, of soft rubber in water, in which both the effective bulk modulus and density are simultaneously negative. The effective density and modulus of a 1D metamaterial with repeated units of shunted Helmholtz resonators was shown to exhibit strong dispersive characteristics with both parameters being simultaneously negative in certain frequency interval [13]. A design of DN elastic metamaterial was proposed and fabricated by a 3D printer [14]. Pope reported a DN elastic metamaterial design using analogies between electrical and mechanical circuits [15]. A composite structure consisting of an array of interspaced thin membranes and side-holes was fabricated and experimentally shown to have widened frequency range in which both the modulus and mass density are negative [16]. A detailed theoretical study of the dynamic behavior of a metamaterial system with both single negative (mass or modulus) and double negative (mass and modulus) in different loading frequencies due to rotational and translational motions built into the structure was reported [17]. A multi-layer active elastic metamaterial with simultaneous DN behavior in some frequency band, where the effective parameters can be tuned independently by incorporating feedback control forces to each layer of the system, have been reported extensively [18-21]. A DN property was reported in an elastic metamaterial of chiral microstructure which produced simultaneously translational and rotational resonances [22].

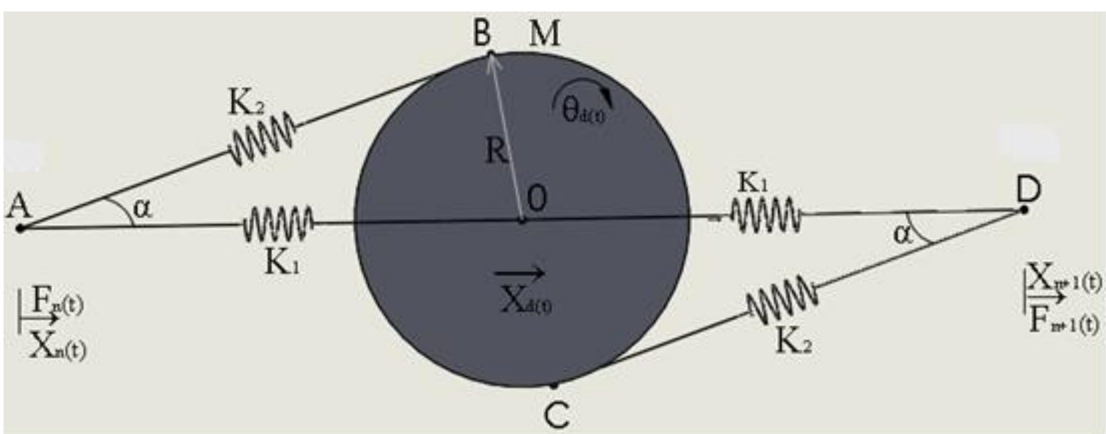

Fig 1:- 1D representative mass-spring system

II. BACKGROUND OF THE STUDY

The present work reports the analytical modeling of the 1D metamaterial system whose representative cell is shown in Fig. 1. The double negative EA metamaterial was earlier proposed, but achieved in 2D with solid media by Liu *et. al.* [22]. For the 1D mass-spring system, it was shown that the effective modulus of the unit cell can be negative (i.e. NBM) in some frequency interval due to rotational resonance introduced by the chiral structure. It was also posited that NMD can be generated by the cell through the translational resonance of the disc. However, this was not shown by the authors, yet the system was conceived to exhibit simultaneously NBM and NMD behavior. Instead of demonstrating DN behavior for the 1D system, an analogous 2D model was adopted. Moreover, citing geometric complexity, it was opined that it is not practicable to use analytical methods to solve the dynamic response of the 2D model, hence numerical method was adopted. The report notwithstanding, it apparently leaves out the attractive results realizable from an analytical study of the dynamical motions and interaction of the rotational and translational resonances of the 1D mass-spring system. Moreover, the numerical analysis approach adopted shrouds the rich results derivable from an in-depth analysis of the effect of the parameters and the response of the system. These make it imperative to revisit the 1D mass-spring system, with the aim to model its dynamical motion interactions and observe the inherently rich and attractive behavior.

III. FORMULATION OF THE PROBLEM

The problem considered is the modeling and harmonic dynamical response of the 1D elastic metamaterial system shown in Fig. 1. It is a chiral mass-spring system with a central disc core of mass and moment of inertia M and I respectively. Two pairs each of horizontal and tangential mass-less springs connect the disc to two end-pins, one on each side. The horizontal springs have constants k_1, while the tangential springs have constants k_2 and inclined at angle α to the horizontal. The representative cell, therefore, consists of one central rigid body and four linearly elastic mass-less springs that connect the disc to mass-less end-pins. The characteristic length of the cell is taken to be L, and is determined by the lengths of the springs. The central disc undergoes general plane motion with both rotation about the central axis O defined by $\theta_d(t)$ and translation in the longitudinal direction defined by $x_d(t)$. Thus, the overall linear motion of the elastodynamic system is limited only to the longitudinal direction while motion in the transverse direction is neglected.

IV. DYNAMIC ANALYSIS OF THE SYSTEM

A. Dynamic Motions and Natural Frequencies

Since harmonic response is considered, Laplace transformation is applied to the equations of motion (EOMs) of the system, assuming zero initial conditions, and changing to the frequency variable with $s = j\omega$. First, pure rotational motion of the disc about its center is analyzed. From the free rotation of the disc, (1) is obtained; and from this, the rotational resonance frequency is obtained in (2), where R is the disc radius and G is the radius of gyration ($I = MG^2$).

$$I\ddot{\theta}_d + 2k_2R^2\theta_d = 0 \tag{1}$$

$$\omega_o = \sqrt{\frac{2k_2R^2}{I}} = \sqrt{\frac{2k_2R^2}{MG^2}} \tag{2}$$

When the rotation is due to displacement by elastic forces of the springs, (3) is obtained. From this and the resonance frequency ω_o, the rotational displacement of the disc is obtained in (4).

$$I\ddot{\theta}_d = k_2(x_{n+1}\cos\alpha - R\theta_d) \times R - k_2(R\theta_d - (-x_n\cos\alpha)) \times R \tag{3}$$

$$\theta_d(\omega) = \left(\frac{\omega_0^2}{\omega_0^2 - \omega^2}\right)\frac{(X_{n+1} - X_n)\cos\alpha}{2R} \tag{4}$$

As seen from (2), it is clear that the rotational resonance is induced wholly by the inclined springs k_2; hence, the horizontal springs k_1 have no effect on it.

Next, pure translational motion of the disc is considered. The free translation of the disc is governed by (5). From this the translational resonance frequency ω_l in (6) is obtained. Similarly, by analyzing the linear displacements of the disc due to spring forces, (7) is obtained and, next to this, the linear displacement of the disc $x_d(t)$ is derived and shown in (8). Equation (6) shows that both the horizontal and inclined springs, k_1 and k_2 respectively, contribute to the translational resonance of the disc.

$$M\ddot{x}_d + 2k_1x_d + 2k_2(x_d\cos\alpha)\cos\alpha = 0 \tag{5}$$

$$\omega_l = \sqrt{\frac{2(k_1 + k_2\cos^2\alpha)}{M}} \tag{6}$$

$$M\ddot{x}_d = k_1(x_{n+1} - x_d) + k_2([x_{n+1} - x_d]\cos\alpha)\cos\alpha - k_1(x_d - x_n) - k_2([x_d - x_n]\cos\alpha)\cos\alpha \tag{7}$$

$$X_d(\omega) = \left(\frac{\omega_l^2}{\omega_l^2 - \omega^2}\right)\frac{(X_{n+1} + X_n)}{2} \tag{8}$$

Equations (4) and (8) clearly show the resonance behavior of both circular and linear motions with magnification factors of $\frac{\omega_0^2}{\omega_0^2 - \omega^2}$ and $\frac{\omega_l^2}{\omega_l^2 - \omega^2}$ respectively. Moreover, it is observed that both motions are frequency-dependent. These would affect the apparent bulk modulus and mass density respectively of the system. It is worth mentioning that the rotational resonance frequency obtained in (2) is the same as that stated by [22]. However, the translational motion, and the associated resonance, was neither modeled nor reported.

B. Effective Bulk Modulus and Effective Mass

By solving the dynamic problem using (4) and (8), the behavior of the elastodynamic system can be established. The dynamic response of the cell is defined by the forces acting on it and the resulting displacements. The forces and displacements at both end-pins are denoted by f_n and f_{n+1}, and x_n and x_{n+1} respectively as Figure 1 shows. The direct relationship between forces and displacements at both ends of the cell is used because it accurately describes the displacements through which the wave propagation in the metamaterial system can be evaluated.

The average force applied to the system is established from the relationship between the average force applied to the cell and the deformation produced. The forces at the end-pins are given by:

$$f_{n+1} = k_1(x_{n+1} - x_d) + k_2([x_{n+1} - x_d]\cos\alpha - R\theta_d)\cos\alpha \quad (9a)$$

$$f_n = k_1(x_d - x_n) + k_2([x_d - x_n]\cos\alpha - R\theta_d)\cos\alpha \quad (9b)$$

From these, the average force in (10) is derived using the expressions for $\theta_d(\omega)$ and $X_d(\omega)$ above.

$$\frac{F_{n+1}+F_n}{2} = \frac{1}{2}\left[k_1 + k_2\left(\frac{\omega^2}{\omega^2-\omega_0^2}\cos^2\alpha\right)\right](X_{n+1} - X_n) \quad (10)$$

Hence, from (10), the effective dynamic bulk modulus $k_{eff}(\omega)$ of the system is expressed as:

$$k_{eff}(\omega) = \frac{1}{2}\left[k_1 + k_2\left(\frac{\omega^2}{\omega^2-\omega_0^2}\cos^2\alpha\right)\right] \quad (11)$$

It can be observed that the effective dynamic bulk modulus $k_{eff}(\omega)$ is frequency-dependent and can become negative in certain frequency intervals due to the resonance behavior.

Similarly, the net force applied and average acceleration of the system are used to derive the expression for the effective dynamic mass. Since the springs and pins are assumed mass-less, the mass of the system is same as the mass M of the core disc. The net force applied to the system is given by (12). From this, the relationship between the net force and the average acceleration is obtained and presented in (13).

$$f_{n+1} - f_n = M\ddot{x}_d \quad (12)$$

$$F_{n+1} - F_n = -\omega^2 M\left(\frac{\omega_l^2}{\omega_l^2-\omega^2}\right)\left(\frac{X_{n+1}+X_n}{2}\right) \quad (13)$$

From (13), it can be observed that the effective dynamic mass of the representative cell $m_{eff}(\omega)$ is given by the expression

$$m_{eff}(\omega) = M\left(\frac{\omega_l^2}{\omega_l^2-\omega^2}\right) = \frac{M}{\left(1-\omega^2/\omega_l^2\right)} \quad (14)$$

Hence the effective dynamic mass $m_{eff}(\omega)$ is frequency-dependent with a magnification factor of $\frac{\omega_l^2}{\omega_l^2-\omega^2}$, and can become negative in certain frequency interval due to the resonance behavior. It is worth noting that apart from using the direct relationship between force and displacement to define the effective dynamic bulk modulus and mass, other approaches can be used. These include for example the mean motion and energy of the unit cell [23, 24].

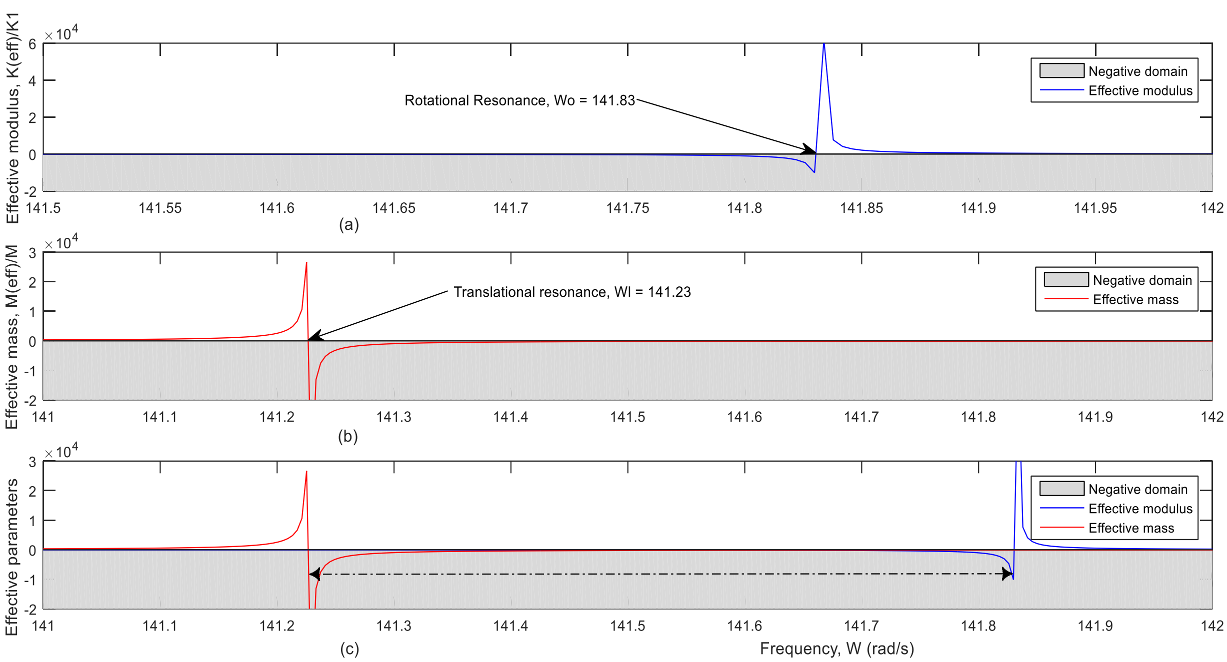

Fig 2:- Response of dynamic effective parameters (for $k_1 = k_2 = 25{,}000\ N/m$), with negative domains shown by shaded (ash color) regions. (a) (blue color plot) shows response of (normalized) effective bulk modulus, with negative behavior at frequencies below rotational resonance ($\omega_o = 141.83\ rad/s$); (b) (red color plot) shows response of (normalized) effective mass, with negative behavior at frequencies above translational resonance ($\omega_l = 141.23\ rad/s$); (c) combined plots of (a) and (b), shows frequency interval where both bulk modulus and mass density are negative (i.e. DN band) shown by dashed-dotted arrow.

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V. RESULTS AND DISCUSSION

A. Behavior of Negative Bulk Modulus

The effective dynamic modulus in (11) is a real term and comprises of two parts. The first part (k_1) is a positive constant and non-dispersive. The second part is frequency-dependent and dispersive, and can be negative. For the dynamic modulus to be negative, the second and dispersive part must not only be negative but also numerically greater than the first part (i.e. k_1). This gives rise to two conditions – a necessary condition and a sufficient condition.

Necessary condition:

$$k_2\left(\frac{\omega^2}{\omega^2-\omega_0^2}\cos^2\alpha\right) < 0 \quad (15a)$$

$$\omega^2-\omega_0^2 < 0; \qquad \omega < \omega_o \quad (15b)$$

Sufficient condition:

$$\left|k_2\left(\frac{\omega^2}{\omega^2-\omega_0^2}\cos^2\alpha\right)\right| > k_1 \quad (16)$$

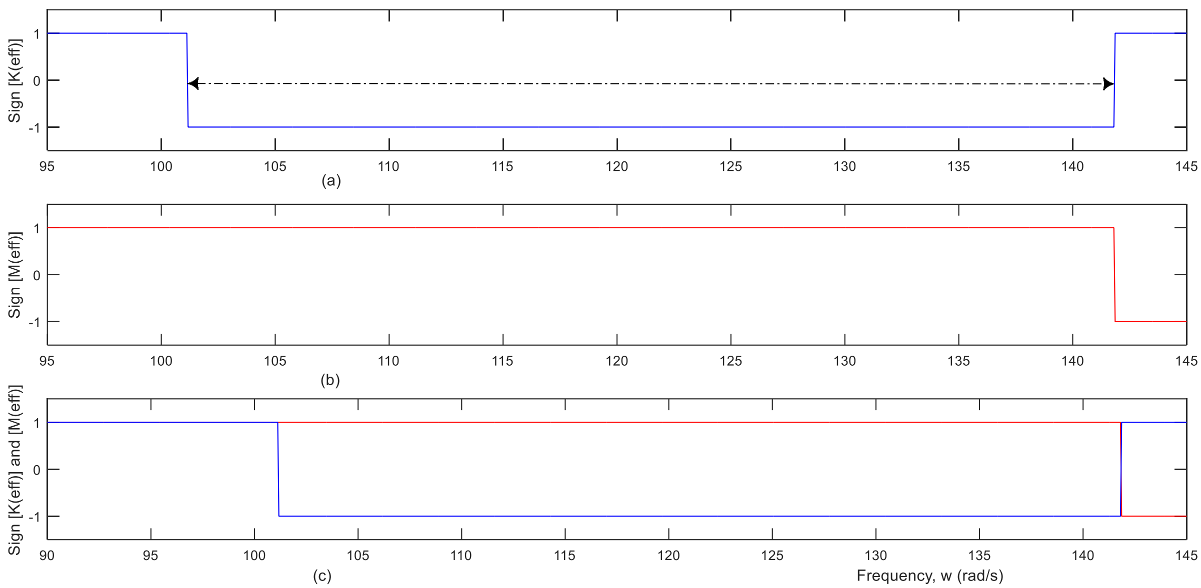

Fig 3:- Plots of the signs of the effective parameters against frequency ω (when $\omega_l = \omega_o \approx 142\ Hz$), at $k_1 = 25426\ N/m$ and $k_2 = 25000\ N/m$. (a) shows effective bulk modulus is negative in the frequency interval specified by (15b) $(i.e.\ \omega < \omega_o)$ and (16) $(i.e.\ |k_2\omega^2\cos^2\alpha/(\omega^2-\omega_0^2)| > k_1)$, as shown by the dashed-dotted arrow; (b) shows effective mass is negative when $\omega > \omega_l$; (c) combines plots (a) and (b), shows the threshold for simultaneously DN behavior when frequency $\omega_l = \omega_o \approx 142\ Hz$.

It is clear that the terms in the numerator of (15a) (i.e. k_2, ω^2 and $\cos^2\alpha$) are positive definite. Hence the necessary condition is governed by the denominator, and defined by (15b). This shows that the effective dynamic modulus would become negative only at frequencies below the rotational resonance frequency ω_o of the core disc. This is shown in Fig. 2(a). Additionally, the sufficient condition in (16) fixes the frequency point at which the behavior of negative effective bulk modulus (NBM) initiates. Hence both conditions specify and fix the frequency interval within which the NBM behavior occurs; while (15b) defines the upper bound, (16) defines the lower bound, as shown in Fig. 3(a) (where dashed-dotted arrow shows the frequency interval for NBM). Therefore the rotational resonance of the metamaterial system is responsible for the dynamic response of the effective bulk modulus leading to NBM behavior at a specified frequency interval upper-bounded by the resonance frequency ω_o.

B. Behavior of Negative Mass

The effective dynamic mass equation in (14) shows that it is a real term and possess dispersive characteristics. As a function of frequency it can become negative when the associated magnification factor $\frac{\omega_l^2}{\omega_l^2-\omega^2}$, is negative. Since the numerator is a positive constant term, the only condition for the effective dynamic mass to be negative is that the denominator must be negative. This singular condition constitutes both a necessary and sufficient condition for the effective mass to be negative; and is described by (17).

$$\omega_l^2-\omega^2 < 0; \qquad \omega > \omega_l \quad (17)$$

Thus, the effective dynamic mass of the system would be negative at frequencies above the translational resonance frequency ω_l. The response of the effective dynamic mass is plotted in Figure 2(b) which shows the NBM effect above ω_l. Similarly, the sign of the effective dynamic mass is plotted in Figures 3(b), and clearly shows the effect of NMD. Therefore the translation resonance of the metamaterial system is responsible for the dynamic response of the effective mass leading to the behavior of NMD at frequencies above the resonance frequency ω_l. This demonstrates the latent idea conceived, but not proven, in [22] for the 1D mass-spring system.

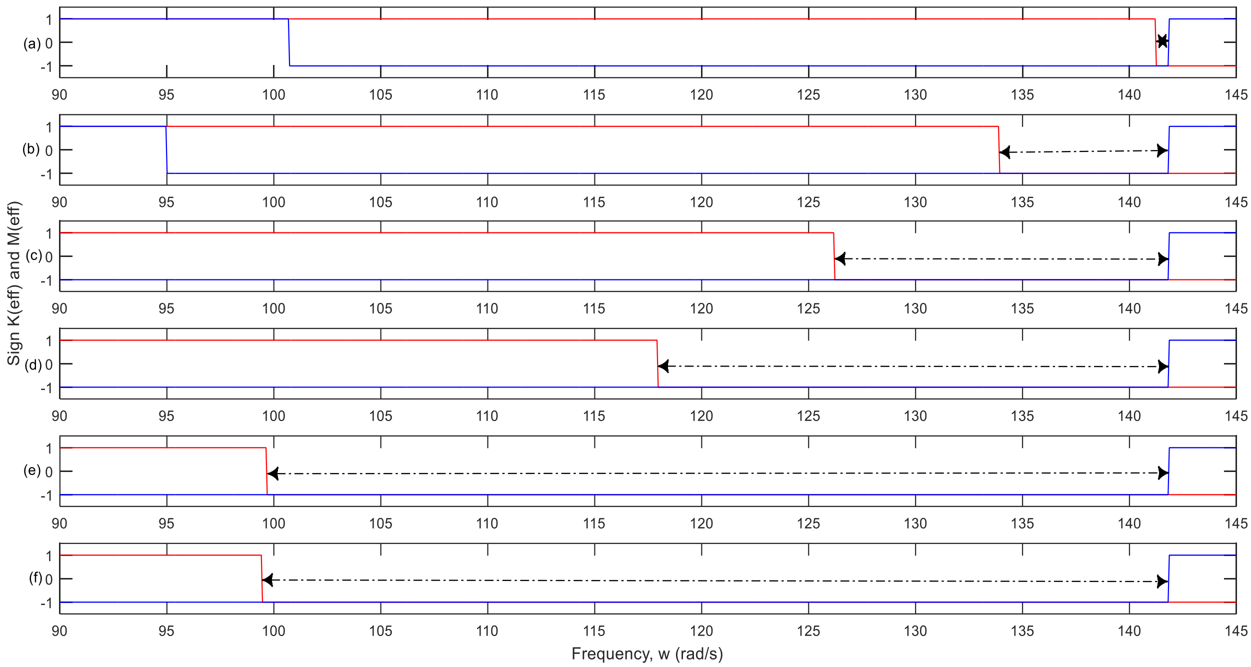

Fig 4:- Plot of signs of effective bulk modulus (blue colour) and mass (red colour) showing increase in double negative frequency bandwidth (denoted by dashed-dotted arrows) due to decrease in value of k_1; (a) shows a small DN bandwidth when $k_1 = k_2 = 25{,}000\ N/m$; (b) – (f) shows increasing DN bandwidth for constant k_2 (at $25{,}000\ N/m$) and $k_1 = 20{,}000, 15{,}000, 10000, 100, and\ 0\ N/m$ respectively

C. Simultaneous Negative Bulk Modulus and Negative Mass (Double Negative) Behavior

It is imperative to observe that when the frequency interval for both NBM and NMD overlap, the metamaterial system will exhibit simultaneously DN behavior. It is noteworthy that the potential for this DN behavior is specified by the frequency interval in which NBM occurs, as given by (15) and (16). Outside this interval, no DN behavior is achieved. Hence, for the system to exhibits DN behavior, the NMD must also occur in this interval to assure an overlap of the two intervals. This would assure an overlap of the two negative intervals. When ω_l exceeds ω_o, no DN behavior results. However, when ω_l equals and coincides with ω_o, the metamaterial system behaves as a perfectly single negative material. This occurs at frequencies above that defined by (16). As described in Fig. 3(c) this perfectly single negativity metamaterial behavior alternate from purely NBM to purely NMD at the common resonance $(i.e. at\ \omega_l = \omega = \omega_o)$. Therefore, to assure a DN frequency band and hence simultaneously NBM and NMD, the translational resonance frequency ω_l must be less than the rotational resonance frequency ω_o. Fig. 4 demonstrates this DN behavior.

It is observed that the width of the DN frequency band is the difference between the rotational resonance ω_o and the translational resonance ω_l $(i.e. w_{DN} = \omega_o - \omega_l)$. This bandwidth can be widened. Since the frequency interval for NBM is apparently fixed, the DN bandwidth can only increase as the frequency position of the translational resonance ω_l shifts leftward towards lower frequencies. Importantly, this increase in the bandwidth can be realized by reducing the stiffness of the horizontal spring k_1, as demonstrated in the plots of Fig. 4 (b) - (f). Interestingly, it is observed that maximum width of the DN band is achieved at zero k_1 (i.e. when there is no horizontal spring), as shown in Fig. 4(f) where the ω_l coincides with lower bound of the NBM interval. Zero k_1 entails removing the horizontal springs and retaining only the inclined springs. This implies that the inclined springs alone can sufficiently generate both the translational and rotational resonances (as (2) and (6) suggests), required for the simultaneously DN behavior of the system. So the horizontal springs appears to be redundant members, and not necessarily needed to achieve a simultaneously DN behavior for the elastodynamic metamaterial system. Therefore, the inclined springs alone can produce both NBM and NMD needed for DN behavior, and also assures that maximum bandwidth is attained.

VI. CONCLUSION

The modeling and analysis of the dynamical behavior of an elastic metamaterial have been demonstrated. Both the translational and rotational resonances were established, and when their frequencies coincide, the material behaves as a perfectly single negative metamaterial. The metamaterial exhibits DN property, and its width is determined by the stiffness of the horizontal spring. It widens as the stiffness reduces and becomes maximum at zero stiffness. Thus the inclined springs alone, which introduces the chiral structure, sufficiently induces DN behavior in the system. The next stages in the study include coupling a number of the cells to investigate wave propagation through a transmission channel formed by an array of unit cells, implementing a control technique that will make the passive system active and comparing the performance of both systems.

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