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On Three Figurate Numbers with Same Values

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Abstract:- Explicit formulas for the ranks of Triangular numbers, Hexagonal numbers, Centered Hexagonal numbers, Centered Octagonal numbers, Centered Decagonal numbers and Centered Dodecagonal numbers satisfying the relations;
$\mathbf{t_{3,N} = t_{6,h} = ct_{6,H}}$, $\mathbf{t_{3,N} = t_{6,h} = ct_{8,M}}$, $\mathbf{t_{3,N} = t_{6,h} = ct_{10,M}}$,
$\mathbf{t_{3,N} = t_{6,h} = ct_{12,D}}$ **are obtained.**

Keywords:- *Equality of polygonal numbers, Centered Hexagonal numbers, Centered Octagonal numbers, Centered Decagonal numbers, Centered Dodecagonal numbers, Hexagonal numbers, Triangular numbers.*

Mathematics Subject Classification: 11D09, 11D99

I. INTRODUCTION

The theory of numbers has occupied a remarkable position in the world of mathematics and it is unique among the mathematical sciences in its appeal to natural human curiosity. Nearly every century has witnessed new and fascinating discoveries about the properties of numbers. They form sequences, they form patterns and so on. An enjoyable topic in number theory with little need for prerequisite knowledge is polygonal numbers which is one of the very best and interesting subjects. A polygonal number is a number representing dots that are arranged into a geometric figure. As the size of the figure increases, the number of dots used to construct it grows in a common pattern. Polygonal numbers have been meticulously studied since their very beginnings in ancient Greece. Numerous discoveries arise from these peculiar polygonal numbers and have become a popular field of research for mathematicians. In [1-5], one polygonal number simultaneously equal to an another polygonal number has been studied.

The main thrust of this paper is to obtain ranks of special polygonal and Centered polygonal numbers with the same value.

❖ ***Notations***

- Centered Hexagonal Number of rank H
  $ct_{6,H} = 3H(H-1)+1$
- Centered Octagonal Number of rank M
  $ct_{8,M} = 4M(M-1)+1$
- Centered Decagonal Number of rank M
  $ct_{10,M} = 5M(M-1)+1$
- Centered Dodecagonal Number of rank D
  $ct_{12,D} = 6D(D-1)+1$
- Triangular Number of rank N
  $t_{3,N} = \dfrac{N(N+1)}{2}$
- Hexagonal Number of rank h
  $t_{6,h} = 2h^2 - h$

II. METHOD OF ANALYSIS

1. Equality of $\mathbf{t_{3,N} = t_{6,h} = ct_{6,H}}$

Let N, h, H be the ranks of Triangular, Hexagonal and Centered Hexagonal numbers respectively.

The relation
$t_{3,N} = t_{6,h}$

leads to

$$N = 2h - 1 \tag{1}$$

The assumption $t_{6,h} = ct_{6,H}$ gives

$$2h^2 - h = 3H^2 - 3H + 1$$

which is written as

$$Y^2 = 6X^2 + 3 \tag{2}$$

where

$$Y = 4h - 1,\ X = 2H - 1 \tag{3}$$

To obtain the other solutions of (2), consider the pell equation

$$Y^2 = 6X^2 + 1$$

whose general solution is given by

$$\tilde{Y}_n = \frac{1}{2} f_n \quad , \quad \tilde{X}_n = \frac{1}{2\sqrt{6}} g_n$$

where
$$f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1},$$
$$g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1} \quad , n = -1, 0, 1, \ldots$$

Applying Brahmagupta Lemma between (X_0, Y_0) and $(\tilde{X}_n , \tilde{Y}_n)$, the other integer solutions of (2) are given by

$$X_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{6}} g_n$$
$$Y_{n+1} = \frac{3}{2} f_n + \frac{3}{\sqrt{6}} g_n$$

In view of (1) and (3), we get

$$h_{n+1} = \frac{1}{4}\left(\frac{3}{2} f_n + \frac{3}{\sqrt{6}} g_n + 1\right)$$

$$N_{n+1} = \frac{3}{4} f_n + \frac{3}{2\sqrt{6}} g_n + \frac{1}{2}$$

$$H_{n+1} = \frac{1}{2}\left(\frac{1}{2} f_n + \frac{3}{2\sqrt{6}} g_n + 1\right)$$

Note that the values of $t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{6,H_{n+1}}$, $n = 0, 1, 2, \ldots$

A few numerical examples satisfying the relations are given in the Table: 1 below:

n	N_{n+1}	h_{n+1}	H_{n+1}	$t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{6,H_{n+1}}$
0	13	7	6	91
1	133	67	55	8911
2	1321	661	540	873181
3	13081	6541	5341	85562821
4	129493	64747	52866	8384283271

Table: 1:- Numerical Examples

2. *Equality of* $t_{3,N} = t_{6,h} = ct_{8,M}$

Let N, h, M be the ranks of Triangular, Hexagonal and Centered Octagonal numbers respectively.

The relation

$$t_{3,N} = t_{6,h}$$

leads to

$$N = 2h - 1 \tag{4}$$

The assumption $t_{6,h} = ct_{8,M}$ gives

$$2h^2 - h = 4M^2 - 4M + 1$$

which is written as

$$Y^2 = 8X^2 + 1 \tag{5}$$

where

$$Y = 4h - 1 \, , \, X = 2M - 1 \tag{6}$$

The general solution of (5) is given by

$$Y_n = \frac{1}{2} f_n \quad , \quad X_n = \frac{1}{4\sqrt{2}} g_n$$

where
$$f_n = \left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1},$$
$$g_n = \left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1} , n = -1, 0, 1, \ldots$$

In view of (4) and (6), we get

$$h_n = \frac{1}{8}(f_n + 2)$$

$$N_n = \frac{1}{4} f_n - \frac{1}{2}$$

$$M_n = \frac{1}{8}\left(4 + \frac{1}{\sqrt{2}} g_n\right)$$

Note that the values of $t_{3,N_n} = t_{6,h_n} = ct_{8,M_n}$, $n = 0, 2, 4, \ldots$

A few numerical examples satisfying the relations are given in the Table: 2 below:

n	N_n	h_n	M_n	$t_{3,N_n} = t_{6,h_n} = ct_{8,M_n}$
0	1	1	1	1
2	49	25	18	1225
4	1681	841	595	1413721

Table 2:- Numerical Examples

3. *Equality of* $t_{3,N} = t_{6,h} = ct_{10,M}$

Let N, h, M be the ranks of Triangular, Hexagonal and Centered Decagonal numbers respectively.

The relation
$t_{3,N} = t_{6,h}$

leads to

$$N = 2h - 1 \quad (7)$$

The assumption $t_{6,h} = ct_{10,M}$ gives

$$2h^2 - h = 5M^2 - 5M + 1$$

which is written as

$$Y^2 = 10X^2 - 1 \quad (8)$$

where
$$Y = 4h - 1,\ X = 2M - 1 \quad (9)$$

To obtain the other solutions of (8), consider the pell equation
$$Y^2 = 10X^2 + 1$$

whose general solution is given by
$$\tilde{Y}_s = \frac{1}{2} f_s \ ,\ \tilde{X}_s = \frac{1}{2\sqrt{10}} g_s$$

where
$$f_s = \left(19 + 6\sqrt{10}\right)^{s+1} + \left(19 - 6\sqrt{10}\right)^{s+1},$$
$$g_s = \left(19 + 6\sqrt{10}\right)^{s+1} - \left(19 - 6\sqrt{10}\right)^{s+1}, s = -1, 0, 1, ...$$

Applying Brahmagupta Lemma between (X_0, Y_0) and $(\tilde{X}_s, \tilde{Y}_s)$, the other integer solutions of (8) are given by

$$X_{s+1} = \frac{1}{2} f_s + \frac{3}{2\sqrt{10}} g_s$$

$$Y_{s+1} = \frac{3}{2} f_s + \frac{5}{\sqrt{10}} g_s$$

In view of (7) and (9), we get

$$h_{s+1} = \frac{1}{4}\left(\frac{3}{2} f_s + \frac{5}{\sqrt{10}} g_s + 1\right)$$

$$N_{s+1} = \frac{3}{4} f_s + \frac{5}{2\sqrt{10}} g_s - \frac{1}{2}$$

$$M_{s+1} = \frac{1}{2}\left(\frac{1}{2} f_s + \frac{3}{2\sqrt{10}} g_s + 1\right)$$

Note that the values of $t_{3,N_{s+1}} = t_{6,h_{s+1}} = ct_{10,M_{s+1}}$, $s = -1, 1, 3, ...$

A few numerical examples satisfying the relations are given in the Table: 3 below:

s	N_{s+1}	h_{s+1}	M_{s+1}	$t_{3,N_{s+1}} = t_{6,h_{s+1}} = ct_{10,M_{s+1}}$
-1	1	1	1	1
1	2221	1111	703	2467531
3	3203401	1601701	1013005	5.13089E+12

Table 3:- Numerical Examples

4. Equality of $\mathbf{t_{3,N} = t_{6,h} = ct_{12,D}}$

Let N, h, D be the ranks of Triangular, Hexagonal and Centered DoDecagonal numbers respectively.

The relation
$t_{3,N} = t_{6,h}$

leads to
$$N = 2h - 1 \quad (10)$$

The assumption $t_{6,h} = ct_{12,D}$ gives
$$2h^2 - h = 6D^2 - 6D + 1$$

which is written as
$$Y^2 = 12X^2 - 3 \quad (11)$$

where
$$Y = 4h - 1,\ X = 2D - 1 \quad (12)$$

To obtain the other solutions of (11), consider the pell equation
$$Y^2 = 12X^2 + 1$$

whose general solution is given by

$$\tilde{Y}_n = \frac{1}{2} f_n \ ,\ \tilde{X}_n = \frac{1}{4\sqrt{3}} g_n$$

where
$$f_n = \left(7 + 4\sqrt{3}\right)^{n+1} + \left(7 - 4\sqrt{3}\right)^{n+1},$$
$$g_n = \left(7 + 4\sqrt{3}\right)^{n+1} - \left(7 - 4\sqrt{3}\right)^{n+1} , n = -1, 0, 1, ...$$

Applying Brahmagupta Lemma between (X_0, Y_0) and $(\tilde{X}_n, \tilde{Y}_n)$, the other integer solutions of (11) are given by
$$X_{n+1} = \frac{1}{2} f_n + \frac{\sqrt{3}}{4} g_n$$

$$Y_{n+1} = \frac{3}{2} f_n + \sqrt{3} g_n$$

In view of (10) and (12), we get

$$h_{n+1} = \frac{1}{4}\left(\frac{3}{2}f_n + \sqrt{3}g_n + 1\right)$$

$$N_{n+1} = \frac{3}{4}f_n + \frac{\sqrt{3}}{2}g_n - \frac{1}{2}$$

$$D_{n+1} = \frac{1}{2}\left(\frac{1}{2}f_n + \frac{\sqrt{3}}{4}g_n + 1\right)$$

Note that the values of $t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{12,D_{n+1}}$, $n = -1,1,3,...$

A few numerical examples satisfying the relations are given in the Table: 4 below:

n	N_{n+1}	h_{n+1}	D_{n+1}	$t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{12,D_{n+1}}$
-1	1	1	1	1
1	313	157	91	49141
3	60817	30409	17557	1849384153
5	11798281	5899141	3405871	6.95997E+13

Table 4:- Numerical Examples

III. CONCLUSION

This paper exhibits the ranks of three figurate numbers having the same values. As these numbers are rich in variety, one may attempt for other figurate numbers whose values are equal.

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